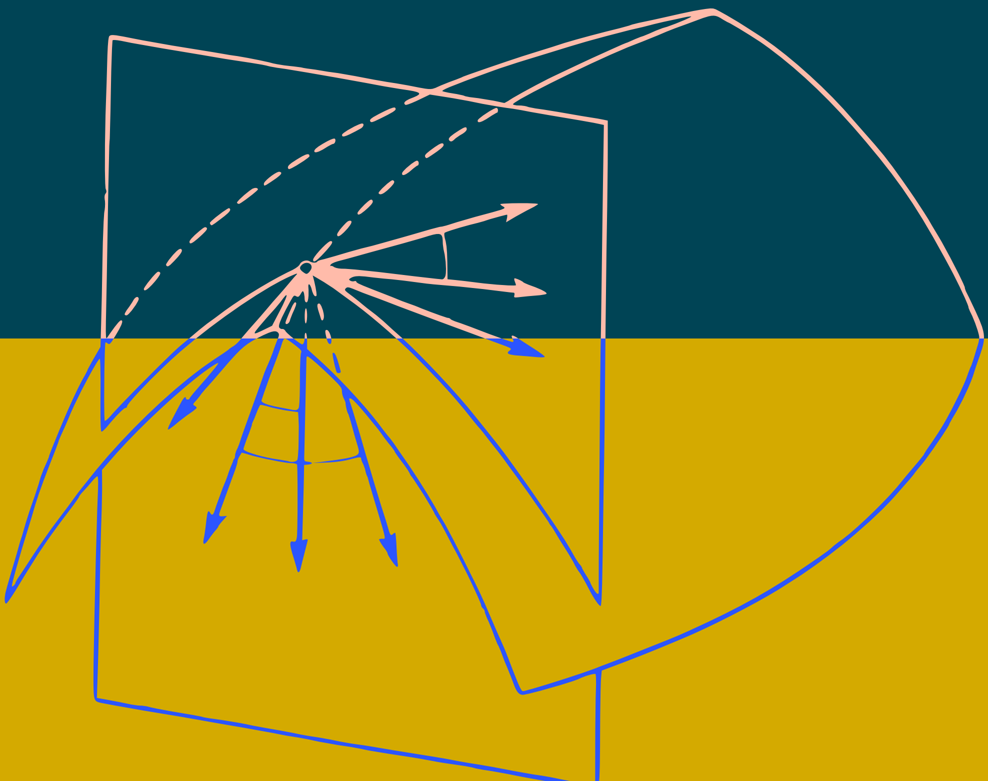
I. VEKUA

GENERALIZED ANALYTIC FUNCTIONS



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И. Н. ВЕКУА

ОБОБЩЕННЫЕ АНАЛИТИЧЕСКИЕ ФУНКЦИИ

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ФИЗИКО-МАТЕМАТИЧЕСКОЙ ЛИТЕРАТУРЫ
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GENERALIZED ANALYTIC FUNCTIONS

by

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ANNOTATION

THIS book is concerned with foundations of the general theory of generalized analytic functions and some applications to problems of differential geometry and theory of shells.

The book is intended for students of advanced courses of the mechanico-mathematical faculties, postgraduates, and likewise for research workers.

FOREWORD

TRADITIONAL applications of the classical theory of analytic functions are mainly connected with the topics of analysis or its applications based either on the Cauchy–Riemann system of equations or on equations the solutions of which can comparatively simply be represented by solutions of the Cauchy–Riemann system. An example is provided by the equations of plane hydrodynamics or of the plane theory of elasticity. Recently, however, the sphere of applications of the theory of analytic functions has been considerably extended. In particular, it also enters into the general theory of elliptic equations. Naturally investigations in this direction were originally concerned with equations with analytic coefficients. In recent years, however, they have been generalized to equations with non-analytic coefficients and the results thus obtained make possible a significant development of the classical theory of analytic functions and its applications. These generalizations concern a class of functions which contain families of solutions of a very wide class of elliptic systems of differential equations of the first order with two independent variables, and even some functions not differentiable in the ordinary sense. In this class which even contains functions non-differentiable in the ordinary sense, a number of fundamental topological properties of analytic functions of one complex variable are preserved (the uniqueness theorem, principle of the argument, etc.). Moreover, such analytic facts as the Taylor and Laurent expansions, the Cauchy integral formula, etc. remain valid. In view of these circumstances the functions under consideration in this book are called generalized analytic functions.

The first part of the book is concerned with various problems of the general theory of generalized analytic

functions. The exposition includes not only the foundations of the theory but also a fairly wide range of boundary value problems. Our considerations are based on a number of relationships and formulae which connect the families of solutions of the systems of differential equations under consideration with the class of analytic functions of one complex variable. These basic relationships and formulae constitute the foundations of the entire theory; they make it possible to reduce investigations to the classical theory of analytic functions. It should be observed that the above results constitute a further natural development of the previous investigations on equations with analytic coefficients. Just as in the analytic case the integral representations of solutions contain kernels which depend on the coefficients of the equation. The constructions carried out make use of integral equations (over the complex domain) the properties of which are similar to those of equations of Volterra type employed in the analytic case.

The power and value of any mathematical theory is most clearly revealed in comparing its results with the actual object of investigation. This connection makes it possible to supply the theory with a definite content, and moreover, to determine the course of its development. If the results of a theory enable us to extend considerably the range of its applications this fact is a sign of the vitality of the theory. In this respect the possibilities of the theory of generalized analytic functions are very large. It is intimately connected with many branches of analysis, geometry and mechanics (quasi-conformal mappings, theory of surfaces, theory of shells, gas dynamics, etc.).

For instance, the new analytic structure makes possible a considerable extension and profound investigation of geometric and mechanical problems arising in connection with infinitesimal bendings of surfaces of positive curvature and equilibrium membrane states of stress of convex shells. These problems are to a large extent considered

in the second part of the book; the considerations led to a number of new results and, moreover, revealed the geometric and mechanical nature of the generalized analytic functions.

Unfortunately, it was not possible within the bounds of this book to present a sufficient exposition of many other important applications of the theory of generalized analytic functions. For instance applications to the problems of quasi-conformal mappings have been dealt with only very roughly; in this connection important results were recently obtained by Bojarski [11]. Also, some applications to non-linear problems are indicated. Notwithstanding the fact that our reasoning is mainly based on linear differential equations, the results obtained can be employed in an investigation of properties of non-linear elliptic equations.

It should be observed that the book contains many results of the author and his collaborators published here for the first time. In addition, it should be noted that the appendix to Chapter IV was written by B. V. Bojarski.

In the preparation of the manuscript great help was given to the author by V. S. Vinogradov, L. S. Klabukova, Sun Che-shen and Ten En Cher. All the figures were prepared by Y. P. Krivenkov. A. V. Bitsadse, B. V. Bojarski, I. I. Daniluk and E. G. Posnyak read the completed manuscript of the book and the author is obliged to them for many valuable suggestions. The author is sincerely grateful to all those mentioned above.

I. VEKUA

PART ONE

FOUNDATIONS OF THE GENERAL THEORY OF GENERALIZED ANALYTIC FUNCTIONS AND BOUNDARY VALUE PROBLEMS

In this part of the book the main attention will be devoted to the construction of the general theory of complex functions $w(z)$ of the point $z = x + iy$, which satisfy an equation of the form

$$\partial_{\bar{z}} w + Aw + B\overline{w} = F \qquad \left(\partial_{\bar{z}} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right). \qquad (1.1)$$

This equation constitutes the complex form of the system of real equations

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + au + bv = f, \qquad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + cu + dv = g. \quad (1.2)$$

The latter system is the canonical form of a more general elliptic system of equations (Ch. II, §7). A very wide class of partial differential equations of the second order can be reduced to a system of the form (1.2) (Ch. III, §9).

In the subsequent investigations we shall assume that the coefficients A and B and the free term F of the equation (1.1) are summable functions in a power $p > 2$, in the domain under consideration. This extension of the class of the investigated equations is expedient not only for purely theoretical reasons; it will frequently be observed below that it is also justified from the point of view of practical applicability.

A theory of such equations cannot, however, be established by the usual classical methods. To this end

we have to make use of methods requiring a knowledge of the theory of Lebesgue integral, functional analysis, etc. Accordingly Chapter I, which is of auxiliary character, deals with various classes of functions and function spaces, and contains an investigation of properties of some special operators. It should also be observed that for an understanding of the present treatment it is sufficient to possess a knowledge of the relevant topics of a university programme.

Chapter II is mainly devoted to the justification of the possibility of global reduction to the canonical form of the positive quadratic form

$$a(x, y)\,dx^2 + 2b(x, y)\,dx\,dy + c(x, y)\,dy^2 . \tag{1.3}$$

The investigation of this problem is carried out by means of the method indicated earlier by the author, [14c], and based on an application of the simplest two-dimensional singular integral equation. Let us observe that in the examination of this integral equation it is essential to base on an important theorem of Zygmund–Calderon, [36a, b], concerning the properties of a singular integral in the sense of the Cauchy principal value (Ch. I, §9.2). This chapter contains the fundamental theorem stating that if a, b and c are bounded and measurable (in the plane), and satisfy everywhere the condition of rigorous positiveness of the form $ac - b^2 \geqslant \Delta_0 > 0$, then there exists a transformation

$$x = x(\xi, \eta) , \qquad y = y(\xi, \eta) ,$$

establishing a one-to-one continuous mapping of the plane $z = x + iy$ onto the plane $\zeta = \xi + i\eta$, such that the quadratic form (1.3) is then reduced to the canonical form

$$\Lambda(d\xi^2 + d\eta^2) , \qquad \Lambda \neq 0 .$$

The latter result is of an auxiliary character. First, it is employed in the proof of the possibility of a reduction to the canonical form of an elliptic equation of the second

order, and also of a system of elliptic equations; secondly it is used in solving the geometric problems of constructing on surfaces the isometric and isometric—conjugate nets of lines. The latter results are basically made use of in the second part of the book concerned with geometric and mechanical problems of infinitesimal bendings of surfaces, and the membrane theory of shells. It should also be observed that the method of investigation employed in this problem opens new possibilities in the theory of quasi-conformal mappings [11d].

Chapters III and IV, constituting together the principal part of the book, are in the main an exposition of author's work [14a], which has been in many cases supplemented and revised. We have taken into account here recent results of Soviet and foreign mathematicians, including the important investigations of L. Bers [5a, b, c].

Chapter III contains an account of the general properties of solutions of equations of the form (1.1). In Chapter IV we investigate various boundary value problems for equations of elliptic type, the principal attention being given to a thorough investigation of the boundary value problem for the system of equations (1.2) with boundary conditions of the form

$$\alpha u+\beta v=\gamma\,. \tag{1.4}$$

We shall not give here a more detailed exposition of the contents of the chapters of this part of the book. In fact there is no need of this, since the *Contents* of the book contains a fairly complete list of almost all important problems, and secondly, with few exceptions, every chapter and every paragraph are preceded by an introduction which presents a short description of the contents of the corresponding section.

CHAPTER I

SOME CLASSES OF FUNCTIONS AND OPERATORS

§1. Classes of functions and functional spaces

In this paragraph we shall consider some classes of functions and functional spaces which will frequently be used henceforth. We shall confine ourselves to the consideration of functions of two independent variables.

1.1. Let $C(\bar{G})$ be a set of continuous functions * of the point $z = x + iy$ in a closed domain $\bar{G}$. If the norm of an element f of the set $C(\bar{G})$ be defined according to the formula

$$C(f, \bar{G}) \equiv C(f) = \max_{z \in \bar{G}} |f(z)|, \tag{1.1}$$

we have a complete normed space of the Banach type. It is important to note the following property of the space $C(\bar{G})$: if $f, g \in C(\bar{G})$, then the product $fg \in C(\bar{G})$, and

$$C(fg) \leqslant C(f)\, C(g)\,.$$

It should be observed that in the present work we shall employ a somewhat unusual notation for the norm of an element of the space of the Banach type, namely *if x is an element of the space X, then the norm of the element x is denoted by $X(x)$. Sometimes we shall also write $X(x) = \|x\|_X$.*

Let a function $f(z)$ and its partial derivatives up to the mth order be continuous in a domain G. The set of these functions will be denoted by $C^m(G)$. It constitutes a *linear*

* Henceforth, unless otherwise stated, we shall consider functions assuming, in general, complex values.

manifold over the field of complex numbers. If f and its partial derivatives up to the mth order are continuous in a closed domain $\bar{G}$, then we shall write $f \in C^m(\bar{G})$. It is necessary to note that at a boundary point z_0 the derivatives are defined as the limits of the corresponding derivatives inside the domain, i.e.

$$\left(\frac{\partial^{k+l} f}{\partial x^k \partial y^l}\right)_{z_0} = \lim_{z \to z_0} \frac{\partial^{k+l} f}{\partial x^k \partial y^l} \quad (k, l = 0, 1, \ldots).$$

If the norm of an element f of the set $C^m(\bar{G})$ is defined according to the formula

$$C^m(f, \bar{G}) \equiv C^m(f) = \sum_{k=0}^{m} \sum_{l=0}^{k} C\left(\frac{\partial^k f}{\partial x^{k-l} \partial y^l}, \bar{G}\right), \tag{1.2}$$

we have a complete Banach type space.

Obviously, if f and $g \in C^m(\bar{G})$, then $fg \in C^m(\bar{G})$, and

$$C^m(fg) \leqslant C^m(f)\, C^m(g).$$

We shall set $C^0 \equiv C$.

Let $f(z)$ satisfy the following condition on the closed set $\bar{G}$

$$|f(z_1) - f(z_2)| \leqslant H|z_1 - z_2|^\alpha, \quad 0 < \alpha \leqslant 1, \tag{1.3}$$

where z_1 and z_2 are arbitrary points of $\bar{G}$; H and α are positive constants independent of the choice of the points z_1 and z_2. The lower bound of the numbers H satisfying the inequality (1.3) will be denoted by $H(f)$ [or $H(f, \alpha)$, or else by $H(f, \alpha, \bar{G})$] and called the Hölder constant of the function f. Evidently,

$$H(f) \equiv H(f, \alpha, \bar{G}) = \sup_{z_1, z_2 \in \bar{G}} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^\alpha},$$

$$|f(z_1) - f(z_2)| \leqslant H(f)|z_1 - z_2|^\alpha, \quad 0 < \alpha \leqslant 1. \tag{1.4}$$

Obviously, in the last inequality, the constant $H(f)$ cannot be replaced by a constant H' smaller than $H(f)$.

Henceforth, by $H_\alpha(\bar{G})$ we shall denote a set of functions satisfying an inequality of the form (1.4) where α is the

same for all functions of the set $H_\alpha(\bar{G})$ and satisfies the inequality $0 < \alpha \leqslant 1$.

The inequality (1.4) is called the Hölder condition. For the set $H_\alpha(\bar{G})$ the notation $\mathrm{Lip}(\alpha, \bar{G})$ is frequently employed.

Let us denote by $C_\alpha(\bar{G})$ the set of all bounded functions $f(z)$ satisfying the inequality (1.4); α is the same for all elements of $C_\alpha(\bar{G})$ and is called the Hölder index of the function f.

If G is a bounded domain, then, evidently, the sets $C_\alpha(\bar{G})$ and $H_\alpha(\bar{G})$ coincide, i.e. $C_\alpha(\bar{G}) \equiv H_\alpha(\bar{G})$. The case is different if G is unbounded. Then $C_\alpha(\bar{G}) \subset H_\alpha(\bar{G})$ and there exist functions belonging to $H_\alpha(\bar{G})$ and not belonging to $C_\alpha(\bar{G})$. As an example we have the function $r^\alpha = |z|^\alpha$.

If the norm of an element f of the set $C_\alpha(\bar{G})$ is defined according to the formula

$$C_\alpha(f, \bar{G}) \equiv C_\alpha(f) = C(f, \bar{G}) + H(f, \alpha, \bar{G}), \qquad (1.5)$$

then we have a complete space of Banach type. If f and $g \in C_\alpha(\bar{G})$, then $fg \in C_\alpha(\bar{G})$ and

$$C_\alpha(fg) \leqslant C_\alpha(f)\, C_\alpha(g).$$

Assume that $f \in C_\alpha(\bar{G})$, $g \in C_\beta(\bar{G})$. If the values of the function $g(z)$ belong to the domain of definition of the function $f(z)$, then

$$f_*(z) \equiv f[g(z)] \in C_{\alpha\beta}(\bar{G}).$$

In fact,

$$\begin{aligned}|f_*(z') - f_*(z'')| &\leqslant \left|f\big(g(z')\big) - f\big(g(z'')\big)\right| \\ &\leqslant H_\alpha(f)|g(z') - g(z'')|^\alpha \leqslant H_\alpha(f)[H_\beta(g)]^\alpha |z' - z''|^{\beta\alpha}.\end{aligned}$$

We shall also consider the Banach type space $C_\alpha^m(\bar{G})$, the elements of which are the elements of $C^m(\bar{G})$ satisfying the condition

$$\frac{\partial^m f}{\partial x^{m-k}\partial y^k} \in C_\alpha(\bar{G}) \quad (k = 0, 1, \ldots, m), \qquad 0 < \alpha \leqslant 1.$$

The norm of an element of this space is defined by the formula

$$C_\alpha^m(f) \equiv C_\alpha^m(f, \bar{G}) = C^m(f) + \sum_{k=0}^{m} H\left(\frac{\partial^m f}{\partial x^{m-k}\partial y^k}, \alpha, \bar{G}\right). \quad (1.6)$$

If f and $g \in C_\alpha^m(\bar{G})$, then the product $fg \in C_\alpha^m(\bar{G})$, and

$$C_\alpha^m(fg, \bar{G}) \leqslant C_\alpha^m(f)\, C_\alpha^m(g). \quad (1.6a)$$

Let us note one more inequality which can easily be derived; if $f, g \in C_\alpha(\bar{G})$ then

$$C_\alpha(fg, \bar{G}) \leqslant C_\alpha(f, \bar{G})\, C(g, \bar{G}) + (Cf, \bar{G})\, C_\alpha(g, \bar{G}). \quad (1.7)$$

If $C_\alpha(f) \leqslant M$, $C_\alpha(g) \leqslant M$, $C(f) < \varepsilon$, $C(g) < \varepsilon$, then (1.7) implies

$$C_\alpha(fg, \bar{G}) \leqslant 2M\varepsilon. \quad (1.8)$$

In other words, *if the elements f and g of a bounded set belonging to $C_\alpha(\bar{G})$ are small in the norm of the space $C(\bar{G})$ then their product is small in the norm of the space $C_\alpha(\bar{G})$.* We shall later make use of this assertion (Ch. II, §4.1).

The foregoing definitions may be extended to the case of G covering the entire plane z which will hereafter be denoted by E.

We shall understand by $C^m(E)$ $\left(C_\alpha^m(E)\right)$ a set of functions $f(z)$ satisfying the condition $f(z), f(1/z) \in C^m(E_1)$ $\left(C_\alpha^m(E_1)\right)$ where E_1 is the circle $|z| \leqslant 1$. Therefore we can speak of the Banach type spaces $C^m(E)$ and $C_\alpha^m(E)$.

1.2. Let a function $f(z)$ given in the domain G satisfy the inequality

$$\iint_{G'} |f(z)|^p\, dx\, dy < M_{G'}, \qquad p \geqslant 1, \quad (1.9)$$

where G' is an arbitrary closed (bounded) subset of the domain G and $M_{G'}$ is a constant in general depending on G', p being the same for all $G' \in G$. The set of such

functions will be denoted by $L_p(G)$. Thus, L_p contains functions summable to the power p on every closed subset of the domain G.

Let us now consider the set of functions satisfying the condition

$$L_p(f) \equiv L_p(f, \bar{G}) = \Big(\iint\limits_G |f(z)|^p dx\,dy\Big)^{1/p} < \infty . \quad (1.10)$$

It will be denoted by $L_p(\bar{G})$. The non-negative number $L_p(f)$ is said to be the norm of the element f of the set $L_p(\bar{G})$. As is known, $L_p(\bar{G})$ is a complete Banach type space. This assertion may be proved by means of the inequalities of Hölder and Minkowski. We shall give their definitions, since they will frequently be used below.

The *Hölder inequality.* If

$$f_k \in L_{p_k}(\bar{G})\,(k = 1, \ldots, n)\,, \quad \frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_n} \leqslant 1\,,$$

then

$$f_1 f_2 \ldots f_n \in L_p(G)$$

and

$$L_p(f_1 f_2 \ldots f_n) \leqslant L_{p_1}(f_1) L_{p_2}(f_2) \ldots L_{p_n}(f_n)\,, \qquad p \geqslant 1\,. \quad (1.11)$$

The *Minkowski inequality.* If

$$f_1, \ldots, f_n \in L_p(\bar{G})\,,$$

then

$$f_1 + \ldots + f_n \in L_p(\bar{G})$$

and

$$L_p(f_1 + \ldots + f_n) \leqslant L_p(f_1) + \ldots + L_p(f_n)\,, \qquad p \geqslant 1\,. \quad (1.12)$$

We now note, without proof, a number of properties of the functions of the class $L_p(\bar{G})$, (78), [79a].

THEOREM 1.1. *If $f \in L_p(\bar{G})$ and $f = 0$ outside G, then for every $\varepsilon > 0$ a number $\delta(\varepsilon) > 0$ can be found, such that*

$$\Big(\iint\limits_G |f(z + \Delta z) - f(z)|^p dx\,dy\Big)^{1/p} < \varepsilon\,, \quad \text{if} \quad |\Delta z| < \delta(\varepsilon)\,. \quad (1.13)$$

We shall call this property *the continuity of the function* $f(z) \in L_p(\bar{G})$ *in the metric of* $L_p(\bar{G})$, *or the continuity in the mean of order* p. *

A set of functions of the class $L_p(\bar{G})$ is uniformly continuous in the metric L_p if $\delta(\varepsilon)$ appearing in (1.13) is independent of the choice of the element of this set.

THEOREM 1.2. *Let a sequence* f_n *of functions belonging to the class* $L_p(\bar{G})$, $p > 1$, *be strongly convergent to the function* $f \in L_p(\bar{G})$:

$$L_p(f - f_n) \to 0 \quad \text{when} \quad n \to \infty .$$

Then we have

(1) *the sequence* f_n *is convergent to* f *in measure, i.e. for every fixed* $a > 0$

$$\operatorname{mes} \mathcal{E}(|f - f_n| \geqslant a) \to 0;$$

(2) *the sequence* f_n *is weakly convergent to* f, *i.e.*

$$\lim_{n\to\infty} \iint_G f_n g \, dx \, dy = \iint_G f g \, dx \, dy , \tag{1.14}$$

where g *is an arbitrary function belonging to the conjugate class* $L_q(\bar{G})$, $q = \dfrac{p}{p-1}$;

(3) *from the sequence* f_n *a subsequence* f_{n_k} *can be extracted, which converges to* $f(z)$ *almost everywhere in* G.

A set of functions of the class $L_p(\bar{G})$ is said to be *compact* if every infinite sequence of its elements contains a subsequence strongly convergent (in the metric of L_p) to an element of the set.

THEOREM 1.3. *A necessary and sufficient condition for the compactness of a set of functions of the class* $L_p(\bar{G})$ *is the uniform boundedness and uniform continuity of the set (in the metric of* L_p*).*

* In Sobolev's monograph [79a] this property of functions of the class L_p is called "continuity in the whole" (page 16). He also gives a proof of the inequality (1.13).

It is also useful to introduce the notion of the *weak compactness* of a set in $L_p(\bar{G})$. A set of elements of $L_p(\bar{G})$ is said to be weakly compact in $L_p(\bar{G})$ if every infinite sequence of its elements contains a subsequence weakly convergent to an element of $L_p(\bar{G})$.

THEOREM 1.3′. *A necessary and sufficient condition for the weak compactness of a set of functions from $L_p(\bar{G})$ is the uniform boundedness of the set in the metric of L_p.*

Proofs of the theorems 1.3 and 1.3′ can be found, for example, in [79a].

We observe that generally strong convergence of a sequence of elements of $L_p(\bar{G})$ does not follow from the weak convergence. Nevertheless, there exist subspaces of $L_p(\bar{G})$ in which the weak and strong convergence are equivalent (see below, §1.7).

1.3. Let $f \in L_p(\bar{G})$, $f = 0$ outside G, and

$$\Big(\iint\limits_G |f(z+\Delta z)-f(z)|^p\,dx\,dy\Big)^{1/p} \leqslant B|\Delta z|^\alpha, \quad 0<\alpha\leqslant 1, \tag{1.15}$$

where Δz is an arbitrary complex number and B is a constant independent of Δz. The smallest of the constants B satisfying the inequality (1.15) will be denoted by $B(f)$ or $B(f, G, \alpha, p)$. Obviously,

$$B(f) = \sup \frac{\big(\iint\limits_G |f(z+\Delta z)-f(z)|^p\,dx\,dy\big)^{1/p}}{|\Delta z|^\alpha}, \tag{1.16}$$

where α, p, G are fixed and Δz assumes an arbitrary value. Also, in the inequality (1.15), $B(f)$ can be taken for B, but one cannot use a constant smaller than $B(f)$.

The set $L_p^\alpha(\bar{G})$, which contains functions satisfying the inequality (1.15), becomes a complete Banach type space if the norm of an element of $L_p^\alpha(\bar{G})$ be given by the relation

$$L_p^\alpha(f) \equiv L_p^\alpha(f, \bar{G}) = L_p(f, \bar{G}) + B(f, \bar{G}, \alpha, p)\,. \tag{1.17}$$

1.4. If G is a bounded domain, then the following relations hold:

In the case of an unbounded domain, however, the last two relations are in general not true.

In the case of an unbounded domain, therefore, it is expedient to consider the following sets: (1) $L_pC_\alpha^m(\bar{G})$—the intersection of the sets $L_p(\bar{G})$ and $C_\alpha^m(\bar{G})$; (2) $L_pL_{p'}(\bar{G})$—the intersection of the sets $L_p(\bar{G})$ and $L_{p'}(\bar{G})$. These sets become Banach spaces if the norms of their elements are defined as follows:

$$\begin{aligned} &1)\ \text{if } f \in L_pC_\alpha^m(\bar{G})\,, \quad \text{then} \quad L_pC_\alpha^m(f) = L_p(f) + C_\alpha^m(f);\\ &2)\ \text{if } f \in L_pL_{p'}(\bar{G})\,, \quad \text{then} \quad L_pL_{p'}(f) = L_p(f) + L_{p'}(f)\,. \end{aligned} \tag{11.8}$$

In general, if we have two Banach type spaces X and Y, and their intersection is a non-empty set, then by defining the norm of an element x of the set XY according to the formula $XY(x) = X(x) + Y(x)$ we shall obtain a new Banach type space, which will be denoted by XY.

1.5. It is important to consider also the following spaces of functions given on the entire plane E.

Let $f(z)$ be given on the entire plane E and satisfy the conditions

$$f(z) \in L_p(E_1)\,, \qquad f_\nu(z) = |z|^{-\nu} f\left(\frac{1}{z}\right) \in L_p(E_1)\,, \qquad p \geqslant 1\,, \tag{1.19}$$

where E_1 is the circle $|z| \leqslant 1$ and ν is a real number. The set of such functions will be denoted by $L_{p,\nu}(E)$, or simply $L_{p,\nu}$. If $f \in L_p(E)$, $p \geqslant 1$, then

$$\iint\limits_{|\zeta|\geqslant 1} |f(\zeta)|^p\,dx\,dy = \iint\limits_{|\zeta|<1} |\zeta|^{-4}\left|f\left(\frac{1}{\zeta}\right)\right|^p dx\,dy$$
$$= \iint\limits_{|\zeta|\leqslant 1} \left(|\zeta|^{-\frac{4}{p}}\left|f\left(\frac{1}{\zeta}\right)\right|\right)^p dx\,dy\,.$$

Hence, $L_p(E) \equiv L_{p,\frac{4}{p}}(E)$. If $\mu \leqslant \dfrac{4}{p} \leqslant \nu$, then $|\zeta|^{-\mu p} \geqslant |\zeta|^{-4} \geqslant$ $\geqslant |\zeta|^{-\nu p}$ when $|\zeta| \leqslant 1$. Therefore, it is obvious that

$$\iint\limits_{|\zeta|\leqslant 1} |\zeta|^{-\mu p}\left|f\left(\frac{1}{\zeta}\right)\right|^p dx\,dy \geqslant \iint\limits_{|\zeta|\leqslant 1} |\zeta|^{-4}\left|f\left(\frac{1}{\zeta}\right)\right|^p dx\,dy$$
$$\geqslant \iint\limits_{|\zeta|\leqslant 1} |\zeta|^{-\nu p}\left|f\left(\frac{1}{\zeta}\right)\right|^p dx\,dy\,.$$

This implies the relations

$$L_{p,\mu}(E) \supset L_p(E) \equiv L_{p,\frac{4}{p}}(E) \supset L_{p,\nu}(E) , \qquad (1.19a)$$

$$p\mu \leqslant 4 \leqslant p\nu .$$

If the norm of an element of $L_{p,\nu}$ be defined by the formula

$$L_{p,\nu}(f) = L_p(f, E_1) + L_p(f_\nu, E_1) , \quad p \geqslant 1 , \quad (1.19b)$$

then it can easily be verified that we have a complete linear Banach type space. If g is a bounded, measurable function on the plane, then, evidently, $fg \in L_{p,\nu}$ when $f \in L_{p,\nu}$. If G is a bounded set, then $L_p(\bar{G}) \subset L_{p,\nu}(E)$, where ν is an arbitrary number; we assume that the elements of $L_p(\bar{G})$ are continued outside $\bar{G}$ by setting them equal to zero. It is readily observed that if $f \in L_p(\bar{G})$, then

$$L_{p,\nu}(f, E) \leqslant M_0 L_p(f, \bar{G}) , \quad M_0 = \text{const.}\ * \quad (1.20)$$

Let $C_{\alpha,\nu}(E)$ $(C_{\alpha,\nu})$ be a set of functions continuous on the entire plane E and satisfying the conditions

$$f(z) \in C_\alpha(E_1) , \quad f_\nu(z) = |z|^{-\nu} f\left(\frac{1}{z}\right) \in C_\alpha(E_1) , \quad (1.21)$$

$$0 \leqslant \alpha \leqslant 1.\ \dagger$$

The set $C_{\alpha,\nu}$ is a complete Banach type space if the norm is defined by the formula

$$C_{\alpha,\nu}(f) = C_\alpha(f, E_1) + C_\alpha(f_\nu, E_1) . \quad (1.22)$$

1.6. Let $f \in C^m(G)$, and let there exist a closed subset G_f of the set G, such that $f = 0$ outside G_f. The set of such

* Below, the letter M (sometimes with various indices) will always denote constant numbers, the indices being used for stressing the dependence of M on a parameter. For example, if M depends only on p we shall write M_p. If it also depends on the domain we shall write $M_p(G)$, etc.

† By $C_0(E_1)$ we understand $C(E_1)$.

functions f will be denoted by $D_m^0(G)$, bearing in mind that the subset G_f may be distinct for every element f of the set $D_m^0(G)$. Obviously, $D_m^0(G)$ is a linear manifold.

The subset of $D_m^0(G)$ consisting of functions possessing derivatives of an arbitrary order will be denoted by $D_\infty^0(G)$, and this also is a linear manifold. *An important property of the linear manifold $D_\infty^0(G)$ (hence, also, of an arbitrary $G_m^0(G)$) is its density in the spaces C, C_α^m, L_p and L_p^α.* Let us rigourously state the appropriate theorems.

THEOREM 1.4. *The linear manifold $D_\infty^0(G)$ is dense in every space $L_p^\alpha(\bar{G})$, $p \geqslant 1$, $0 \leqslant \alpha \leqslant 1$.* *

In other words, if $f \in L_p^\alpha(\bar{G})$, a sequence f_n of elements of $D_\infty^0(G)$ can be found which converges to f in the metric of the space $L_p^\alpha(\bar{G})$, i.e.

$$L_p^\alpha(f-f_n, \bar{G}) \to 0 \quad \text{when} \quad n \to \infty .$$

THEOREM 1.5. *If $f \in C_\alpha^m(\bar{G})$ and G_* is an open set such that $\bar{G} \subset G_*$, then there exists a sequence f_n of elements of $D_\infty^0(G_*)$ which converges to f in the metric of $C_\alpha^m(\bar{G})$, i.e. $C_\alpha^m(f-f_n, \bar{G}) \to 0$ when $n \to \infty$.*

The proofs of the above theorems will not be considered here; they can be carried out by using properties of so-called mean functions (cf. [79a], Ch. I, §2).

1.7. Let $\Phi(z)$ be a single-valued function analytic in z in the domain G. Inside G it may have a discrete set of isolated singular points—poles and essential singularities. The set of such functions will be denoted by $\mathfrak{A}_0^*(G)$. If f and $g \in \mathfrak{A}_0^*$, then

$$f \pm g, \quad fg, \quad \frac{f}{g}, \quad f(g(z)) \in \mathfrak{A}_0^* .$$

It is of course understood that the values of $g(z)$ in the last relation belong to the domain of definition of $f(z)$.

We shall also deal with the sets $\mathfrak{A}_0^* C(G) \equiv \mathfrak{A}_0(G)$, $\mathfrak{A}_0 L_p(G)$, etc.

* By L_p^0 we understand L_p.

The sets $\mathfrak{A}_0(G)C_a^m(\bar{G})$ and $\mathfrak{A}_0(G)L_p^a(\bar{G})$ form closed subspaces of the corresponding spaces $C_a^m(\bar{G})$ and $L_p^a(\bar{G})$.

The validity of this assertion relating to $\mathfrak{A}_0(G)C_a^m(\bar{G})$ is obvious. The proof will now be given for other sets; it is sufficient to carry out the proof for the set $\mathfrak{A}_0(G)L_p(\bar{G})$, $p \geqslant 1$. We have to prove that $\mathfrak{A}_0(G)L_p(\bar{G})$ is a closed linear manifold of elements of $L_p(\bar{G})$. Let a sequence Φ_n of elements of $\mathfrak{A}_0(G)L_p(\bar{G})$ satisfy the condition

$$L_p(\Phi_n - \Phi_m, \bar{G}) \to 0 \quad \text{when} \quad m, n \to \infty .$$

It follows that the sequence Φ_n is convergent in the mean to a function Φ of the class $L_p(\bar{G})$. We have to prove that Φ is holomorphic inside G, i.e. that $\Phi \in \mathfrak{A}_0(G)L_p(\bar{G})$.

Let G_{δ_0} be a set of points of the domain G whose distances from the boundary of G are greater or equal to δ_0. Then, in accordance with the formula for the mean value of a holomorphic function, we have the relation

$$\Phi_n(z) - \Phi_m(z) = \frac{1}{\pi\delta^2} \iint\limits_{|\zeta - z| \leqslant \delta} [\Phi_n(\zeta) - \Phi_m(\zeta)]\, d\xi\, d\eta ,$$

$$0 < \delta < \delta_0 , \qquad z \in G_{\delta_0}$$

valid for the points of the set G_{δ_0}. Hence, making use of the Hölder inequality, we obtain

$$\begin{aligned} |\Phi_n(z) - \Phi_m(z)| &\leqslant \frac{1}{\pi\delta^2} \iint\limits_{|\zeta - z| \leqslant \delta} |\Phi_n(\zeta) - \Phi_m(\zeta)|\, d\xi\, d\eta \\ &\leqslant (\pi\delta^2)^{-\frac{1}{p}} \Big(\iint\limits_{|\zeta - z| \leqslant \delta} |\Phi_n(\zeta) - \Phi_m(\zeta)|^p\, d\xi\, d\eta \Big)^{\frac{1}{p}} \\ &\leqslant (\pi\delta^2)^{-\frac{1}{p}} L_p(\Phi_n - \Phi_m, \bar{G}) . \end{aligned}$$

This inequality implies that the sequence Φ_n is uniformly convergent in every set G_δ. Consequently, the limiting function Φ_δ is holomorphic inside G_δ. We shall prove that $\Phi_\delta = \Phi$; this fact follows from the inequality

$$L_p(\Phi_\delta - \Phi, G_\delta) \leqslant L_p(\Phi_\delta - \Phi_n, G_\delta) + L_p(\Phi_n - \Phi, \bar{G}) ,$$

the right-hand side of which, evidently, tends to zero when $n \to \infty$ for every positive δ. This completes the proof.

We shall now prove that weak convergence in $L_p(\bar{G})$ of a sequence of elements Φ_n belonging to $\mathfrak{A}_0 L_p(\bar{G})$ implies strong convergence. Let $\Phi_n \xrightarrow{\text{weak}} \Phi_0 \in L_p(\bar{G})$. Then, according to Theorem 1.3′ the set $\{\Phi\}$ is uniformly bounded, i.e. $L_p(\Phi_n, \bar{G}) \leqslant M$. Then the formula for the mean value immediately implies that $\{\Phi_n\}$ is uniformly bounded inside G in the metric C. Consequently, in view of a well known theorem of Montel, [57], from $\{\Phi_n\}$ a sequence Φ_{n_k} can be extracted which converges uniformly inside G to an element Φ of the class $\mathfrak{A}_0 L_p(\bar{G})$; moreover, we have clearly $\Phi_{n_k} \xrightarrow{\text{weak}} \Phi$. But, by assumption, $\Phi_{n_k} \xrightarrow{\text{weak}} \Phi_0$. Therefore $\Phi_0 \equiv \Phi \in \mathfrak{A}_0 L_p(\bar{G})$. It is now easy to discover that Φ_n converges uniformly and strongly inside G to Φ. This completes the proof.

It should be noted that we did not asssume beforehand that Φ_0 is holomorphic in G.

Consequently, we have

THEOREM 1.6. *If a sequence of functions Φ_n holomorphic in the domain G is weakly convergent in $L_p(\bar{G})$, $p \geqslant 1$, then it is convergent uniformly inside G, and therefore the limiting function is holomorphic in G.*

We also observe that $\mathfrak{A}_0 C(E)$ is a set of constants and $\mathfrak{A}_0 L_p(E)$ contains only a function identically equal to zero. This follows immediately from Liouville's theorem.

1.8. We shall also consider some classes of functions which are (in general) not summable. Let $\mathfrak{A}_0^* \times L_p(G)$ be a set of functions of the form $f = \Phi g$, where $\Phi \in \mathfrak{A}_0^*(G)$ and $g \in L_p(G)$, $p \geqslant 1$. It is obvious that $\mathfrak{A}_0^* \times L_p(G)$ is wider than $L_p(G)$.

In other words, it will be understood that $f \in \mathfrak{A}_0^* \times L_p(G)$ if an analytic function $\Phi \in \mathfrak{A}_0^*(G)$ can be found such that Φf belongs to $L_p(G)$. In this case Φ is said to be *the analytic summability factor of the function f*. The latter

function is called *quasi-summable*. A large class of quasi-summable functions exists; it contains, for example, measurable functions possessing concentrated singularities of the pole type $|z-z_0|^{-\alpha}$ where α is an arbitrary real number. Then $(z-z_0)^{[\alpha]}$, where $[\alpha]$ is the greatest integer contained in α, is the summability factor in the vicinity of the point z_0.

If $f \in \mathfrak{A}_0^* \times L_p(G)$, $g \in \mathfrak{A}_0^* \times L_q(G)$, $\dfrac{1}{p}+\dfrac{1}{q}=1$, then $fg \in \mathfrak{A}_0^* \times \times L_1(G)$. But if f and $g \in \mathfrak{A}_0^* \times L_p(G)$, then $f+g$, in general, does not belong to $\mathfrak{A}_0^* \times L_p(G)$. Consequently, $\mathfrak{A}_0^* \times L_p(G)$ is not a linear manifold. We shall consider, therefore, the set $\Sigma\mathfrak{A}_0^* \times L_p(G)$ which contains all linear combinations of quasi-summable functions of the class $\mathfrak{A}_0^* \times L_p(G)$.

Let $f = \Phi g$ where $\Phi \in \mathfrak{A}_0^*(G)$, $g \in C(G)$. The set of these functions will be denoted by $\mathfrak{A}_0^* \times C(G)$ and their linear combinations by $\Sigma\mathfrak{A}_0^* \times C(G)$.

In Chapter 3 we shall also encounter functions of the form $f = \Phi e^g$ where $\Phi \in \mathfrak{A}_0^*(G)$ and $g \in \Sigma\mathfrak{A}_0^* \times C_\alpha(E)$. In general, the function f is not quasi-summable. This class of functions will be denoted by $\mathfrak{A}_0^* \times e^{\Sigma\mathfrak{A}_0^* \times C_\alpha(E)}$.

§2. Classes of curves and domains. Some properties of conformal mapping

2.1. Let Γ be a simple, closed or open, rectifiable Jordan curve. Then its equation may be written in the form

$$z(s) = x(s) + iy(s)\,, \tag{2.1}$$

where $z(s)$ is the coordinate of a point of the curve Γ corresponding to the length of arc s counted from an arbitrary fixed point on Γ. Let l be the length of the curve. The origin of the length s can always be chosen such that the condition $0 \leqslant s \leqslant l$ is satisfied. The function $z(s)$ is continuous in the range $0 \leqslant s \leqslant l$ and $z(0) = z(l)$ if Γ is closed. In the case of a closed curve, therefore, $z(s)$ is a periodic function, the period being equal to the length of the curve.

A curve Γ is said to belong to the class C^m if all the derivatives of the function $z(s)$ up to the mth order are continuous on the arc $0 \leqslant s \leqslant l$. Furthermore, if the derivative $z^{(m)}(s)$ of the mth order satisfies on this arc the Hölder condition with an index α, $0 < \alpha \leqslant 1$, then it will be said that $\Gamma \in C_\alpha^m$.

Let Γ be a piecewise smooth, simple, closed curve consisting of a finite number of arcs of the class C_α^m. Let $\nu_1\pi, \ldots, \nu_k\pi$ be the interior angles at the vertices of the curve. It will be assumed that $0 < \nu_j \leqslant 2 (j = 1, 2, \ldots, k)$. The set of these curves will be denoted by $C^m_{\alpha,\nu_1,\ldots,\nu_k}$.

If $z(s)$ is a function analytic in the argument s, then Γ will be called an analytic curve. The class of these curves will be denoted by $\mathfrak{A}$. We can also consider piecewise analytic curves of the class $\mathfrak{A}_{\nu_1,\ldots,\nu_k}$.

If the boundary of the domain G consists of a finite number of simple, closed or open, rectifiable Jordan curves possessing no points in common, then G will be said to belong to the class C. If, now, all these curves are closed and belong to the class $C^m (C_\alpha^m, C^m_{\alpha,\nu_1,\ldots,\nu_k}, \mathfrak{A}, \mathfrak{A}_{\nu_1,\ldots,\nu_k})$ the domain G will be said to belong to the class $C^m (C_\alpha^m, C^m_{\alpha,\nu_1,\ldots,\nu_k}, \mathfrak{A}, \mathfrak{A}_{\nu_1,\ldots,\nu_k})$.

2.2. Let there be given a function $f(z)$ of the point $z \in \Gamma$ on a rectifiable simple Jordan curve Γ. This function may be regarded as a function of the length of the arc s, i.e. $f(z(s)) = f(s)$. If $f(s)$ and all its derivatives up to the mth order are continuous on the arc $0 \leqslant s \leqslant l$, f will be said to belong to the class $C^m(\Gamma)$. If, moreover, $f^{(m)}(s)$ satisfies the Hölder condition with an index α, $0 < \alpha \leqslant 1$, then it will be said that $f \in C_\alpha^m(\Gamma)$.

The sets $C^m(\Gamma)$ and $C_\alpha^m(\Gamma)$ will become Banach type spaces if the norm of their elements be defined as follows.

$$C^m(f, \Gamma) = \sum_{k=0}^{m} C\left(\frac{d^k f}{ds^k}, \Gamma\right), \quad \text{if} \quad f \in C^m(\Gamma),$$

$$C_\alpha^m(f, \Gamma) = C^m(f, \Gamma) + H\left(\frac{d^m f}{ds^m}, \Gamma, \alpha\right), \quad \text{if} \quad f \in C_\alpha^m(\Gamma),$$

where $C(f, \Gamma)$ and $H(f, \Gamma, \alpha)$ denote

$$C(f, \Gamma) \equiv \max_{t \in \Gamma} |f(t)|, \quad H(f, \Gamma, \alpha) \equiv \sup_{t_1, t_2 \in \Gamma} \frac{|f(t_1)-f(t_2)|}{|t_1-t_2|^\alpha}.$$

2.3. Let G_z be a domain of the z-plane, the complement of which consists of $m+1$ continua $G_0, \ldots, G_m$. We shall assume that each continuum contains at least two points. In this case, as is well known (cf. e.g. [40a]), the domain G_z can be mapped conformally onto the canonical domain G_ζ of a ζ-plane bounded by the circles $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$, Γ_0 being the unit circle $|\zeta| = 1$ with centre $\zeta = 0$ belonging to G_ζ and containing the circles $\Gamma_1, \ldots, \Gamma_m$. The function $z = \varphi(\zeta)$ giving the above mapping of the domain G_ζ onto G_z may always be subjected to the additional requirement

$$\varphi(0) = z_0, \quad \varphi'(0) > 0, \tag{2.2}$$

where z_0 is an arbitrary fixed point of the domain G_z. It is known that the conditions (2.2) determine the function $\varphi(\zeta)$ uniquely.

Evidently, the limiting properties of the function $\varphi(\zeta)$ and of the inverse function $\psi(z)$ depend on the smoothness properties of the boundary of the domain G_z. We shall present below, without proof but with a reference to appropriate sources, some theorems dealing with the character of the continuity of these functions in the closed domains $\bar{G}_\zeta$ and $\bar{G}_z$, under various assumptions with respect to the smoothness properties of the boundary of the domain G_z.

THEOREM 1.7. *If G_z is bounded by simple closed Jordan curves $L_0, \ldots, L_m$, L_0 containing the remaining curves, then $\varphi(\zeta)$ and $\psi(z)$ are continuous in the closed domains $G_\zeta + \Gamma$ and $G_z + L$, respectively, $L = L_0 + \ldots + L_m$, $\Gamma = \Gamma_0 + \ldots + \Gamma_m$ and the circle Γ_j is a homeomorphic image of the curve L_j, i.e. $\Gamma_j = \psi(L_j)$ $(j = 0, 1, \ldots, m)$.*

THEOREM 1.8. *If the curves $L_0, L_1, \ldots, L_m \in C_\alpha^k (k \geq 0$, $0 < \alpha < 1)$, then*

$$\varphi(\zeta) \in C_\alpha^k(G_\zeta + \Gamma), \quad \psi(z) \in C_\alpha^k(G_z + L).$$

THEOREM 1.9. *If* $L \in C^1_{\alpha,\nu_1,\dots,\nu_k}$ $(0<\alpha<1,\ 0<\nu_j\leqslant 2)$, *then*

$$\psi(z) \in C_{\nu'}(G_z+L)\ , \qquad \varphi(\zeta) \in C_{\nu''}(G_\zeta+\Gamma)\ ,$$

where

$$\nu' = \min\left(1\ , \frac{1}{\nu_1}\ , \dots , \frac{1}{\nu_k}\right), \qquad \nu'' = \min(1, \nu_1, \dots, \nu_k)\ .$$

In the neighbourhood of a boundary corner point z_j *with interior angle* $\nu_j\pi$ *the function*

$$\psi_j(z) = \frac{\psi(z)-\psi(z_j)}{(z-z_j)^{1/\nu_j}} \tag{2.3}$$

tends to a definite limit $\psi_j(z_j)\neq 0$ *when* $z\to z_j$. *Moreover, in the neighbourhood of* z_j *the derivative of* $\psi(z)$ *has the form*

$$\psi'(z) = (z-z_j)^{\frac{1}{\nu_j}-1}\psi_0(z)\ , \tag{2.4}$$

where $\psi_0(z)$ *is a continuous function and* $\psi_0(z_j)\neq 0$.

If ζ_j *is the image of* z_j, *then in the neighbourhood of* ζ_j *the function*

$$\varphi_j(\zeta) = \frac{\varphi(\zeta)-\varphi(\zeta_j)}{(\zeta-\zeta)^{\nu_j}} \tag{2.5}$$

tends to a definite limit $\varphi_j(\zeta_j)$ *when* $\zeta\to\zeta_j$ *and* $\varphi_j(\zeta_j)\neq 0$. *Besides, in the neighbourhood of* ζ_j

$$\varphi'(\zeta) = (\zeta-\zeta)^{\nu_j-1}\varphi_0(\zeta)\ , \tag{2.6}$$

where $\varphi_0(\zeta)$ *is a continuous function and* $\varphi_0(\zeta_j)\neq 0$.

Proofs of these theorems were given by various authors (see for instance [13]). A fairly complete list of references on the subject may be found in [22].

We shall now indicate a way of proving continuity in the Hölder sense of the functions $\varphi(\zeta)$ and $\psi(\zeta)$ in a closed domain, by means of the formulae (2.3) and (2.4).

The relation (2.3) implies that in the vicinity of a corner point z_j the function $\psi(z)$ has the form

$$\psi(z) = \psi(z_j)+(z-z_j)^{\frac{1}{\nu_j}}\psi_j(z)\ . \tag{2.7}$$

On any closed arc of the contour L which does not contain a corner point, $\psi(z)$ has a continuous derivative. Therefore the bounded function $\psi_j(z)$ has a derivative everywhere in the neighbourhood of z_j with the exception of this point; in view of (2.4) the following inequality holds in the vicinity of z_j:

$$\left|\frac{d\psi_j}{dz}\right| \leqslant \frac{M_0}{|z-z_j|} \qquad (M_0 = \text{const}) .$$

On the other hand, it was proved by Muskhelishvili, [60a] (Ch. I, §7), that in the vicinity of a point z_j on L a function of the form (2.7) belongs to the class $C_{\nu_j'}$ where $\nu_j' = \min\left(1, \frac{1}{\nu_j}\right)$. On closed arcs of the contour L which contain no corner points, $\psi(z) \in C_1$; we therefore observe that the following is true: on the entire contour L the function $\psi \in C_{\nu'}(L)$ where $\nu' = \min(1, \nu_1', \ldots, \nu_k')$. It follows that $\psi(z) \in C_{\nu'}(G_z+L)$ (cf. e.g. [60a], Ch. I, §22). In an analogous way we can prove that $\varphi(\zeta) \in C_{\nu''}(G_\zeta+\Gamma)$ where $\nu'' = \min(1, \nu_1, \ldots, \nu_k)$.

§3. Some properties of Cauchy type integrals

In this paragraph an important property of the Cauchy type integral will be proved; it will frequently be used henceforth.

THEOREM 1.10. *Let $G \in C_\alpha^{m+1}$ and $f \in C_\alpha^m(\Gamma)$ where Γ is the boundary of the domain G, $\Gamma \in C_\alpha^{m+1}$, $0 < \alpha < 1$, $m \geqslant 0$. Then the Cauchy type integral*

$$\Phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f_m(t)\,dt}{t-z} \tag{3.1}$$

belongs to the class $C_\alpha^m(G+\Gamma)$.

PROOF. By integration by parts it can easily be shown that

$$\Phi^{(m)}(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f_m(t)\,dt}{t-z}, \tag{3.2}$$

where

$$f_m(t) \equiv \frac{d^m f}{dt^m} = \overline{t'(s)}\,\frac{d}{ds}\left(\overline{t'(s)}\,\frac{d}{ds}\left(\ldots\,\overline{t'(s)}\,\frac{df}{ds}\right)\right), \qquad t'(s) = \frac{dt}{ds},$$

the operation $\overline{t'(s)}\,\dfrac{d}{ds}$ on the right-hand side of the relation being repeated m times. Since $t'(s)$ and $f \in C^m_\alpha(\Gamma)$, then $f_m(t) \in C_\alpha(\Gamma)$. Therefore, in view of a well known property of the Cauchy type integral ([60a] Ch. I, §22), the function $\Phi^{(m)}(z) \in C_\alpha(\bar{G})$ when $0 < \alpha < 1$, i.e. $\Phi \in C^m_\alpha(\bar{G})$. This completes the proof.

The following relation also holds

$$C^m_\alpha(\Phi, \bar{G}) \leqslant M C^m_\alpha(f, \Gamma), \qquad (M = \text{const}). \tag{3.3}$$

When $m = 0$ the proof of this assertion is easily obtained from Muskhelishvili's results, [60a] (Ch. I, §§ 19, 20, 22); it can readily be generalized by means of the formula (3.2) to the case $m > 0$.

Remark. When $m = 0$ the requirement in respect of the domain may be somewhat weakened. We have the following theorem ([60a] Ch. I, §21).

THEOREM 1.11. *If $G \in C^1$ and $f \in C_\alpha(\Gamma)$, $0 < \alpha < 1$, then $\Phi(z) \in C_\alpha(G+\Gamma)$. The derivative $\Phi'(z)$ has the estimate*

$$|\Phi'(z)| < C_\alpha(f)\,\delta^{\alpha-1}, \qquad 0 < \alpha < 1, \tag{3.4}$$

where δ is the distance of the point z from the boundary of the domain G.

The last relation implies that $\Phi'(z) \in L_p(\bar{G})$ where p is an arbitrary number satisfying the condition

$$1 < p < \frac{1}{1-\alpha}. \tag{3.5}$$

Also, obviously,

$$L_p(\Phi', \bar{G}) \leqslant M C_\alpha(f, \Gamma) \qquad (M = \text{const}). \tag{3.6}$$

According to this remark, under the conditions of Theorem 1.10, we have

$$L_p(\Phi^{(m+1)}, \bar{G}) \leqslant M^m_\alpha C(f, \Gamma). \tag{3.7}$$

It follows from (3.5) that $p > 2$, when $\frac{1}{2} < \alpha < 1$.

§4. Non-homogeneous Cauchy–Riemann system

4.1. Let us consider the non-homogeneous Cauchy–Riemann system of equations

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=g(x,y)\,,\qquad \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=h(x,y)\,,\tag{4.1}$$

where g and h are known real functions of the real variables x and y. This system can be written in the form

$$\frac{\partial w}{\partial \bar{z}}=f\,,\qquad f=\frac{g+ih}{2}\,,\qquad w=u+iv\,,\tag{4.2}$$

where

$$\frac{\partial w}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial w}{\partial x}+i\frac{\partial w}{\partial y}\right)\equiv\partial_{\bar{z}}w\equiv w_{\bar{z}}\,.\tag{4.3}$$

We shall also consider the operation

$$\frac{\partial w}{\partial z}\equiv\partial_z w\equiv w_z=\frac{1}{2}\left(\frac{\partial w}{\partial x}-i\frac{\partial w}{\partial y}\right).\tag{4.4}$$

We shall agree to call the quantities $\partial_{\bar{z}}w$ and $\partial_z w$ the partial derivatives with respect to $\bar{z}$ and z, respectively. It is readily observed that the derivatives with respect to x and y are related to $\partial_{\bar{z}}w$ and $\partial_z w$, namely we have

$$\frac{\partial w}{\partial x}=\frac{\partial w}{\partial z}+\frac{\partial w}{\partial \bar{z}}\,,\qquad \frac{\partial w}{\partial y}=i\frac{\partial w}{\partial z}-i\frac{\partial w}{\partial \bar{z}}\,.$$

We shall further see that $\partial_{\bar{z}}$ and ∂_z may be regarded as the original differential operations which we shall define directly, with no reference to the partial derivatives with respect to x and y.

Applying the operations $\partial_{\bar{z}}$ and ∂_z to an analytic function $\Phi(z)$ we obtain

$$\frac{\partial \Phi}{\partial \bar{z}}=0\,,\qquad \frac{\partial \Phi}{\partial z}=\Phi'(z)\,.\tag{4.5}$$

The first relation is a complex notation of the Cauchy–Riemann system, while the second represents the de-

rivative of an analytic function with respect to the complex argument. If $w \in C^1(G)$ and $\Phi \in \mathfrak{A}_0(G)$, then, obviously,

$$\partial_{\bar z}(\Phi w) = \Phi \partial_{\bar z} w\,, \qquad \partial_z(\bar\Phi w) = \bar\Phi \partial_z w\,. \tag{4.6}$$

Let $G \in C$ and $w \in C^1(\bar G)$. Then by means of the well-known Green formula it is easy to derive

$$\left.\begin{aligned} \iint_G \frac{\partial w}{\partial \bar z}\, dx\, dy &= \frac{1}{2i} \int_\Gamma w(z)\, dz\,, \\ \iint_G \frac{\partial w}{\partial z}\, dx\, dy &= -\frac{1}{2i} \int_\Gamma w(z)\, d\bar z\,. \end{aligned}\right\} \tag{4.7}$$

It can easily be seen that these formulae remain valid if $w \in C^1(G)$ and it is continuous in the closed domain $\bar G$. *

If ζ is a fixed point of the domain G, then in view of the formula (4.7) and the relation (4.6) we have

$$\frac{1}{2i} \int_\Gamma \frac{w(z)\, dz}{z-\zeta} - \frac{1}{2i} \int_{|z-\zeta|=\varepsilon} \frac{w(z)\, dz}{z-\zeta} = \iint_{G_\varepsilon} \frac{\partial w(z)}{\partial \bar z} \frac{dx\, dy}{z-\zeta}\,, \tag{4.8}$$

where G_ε is the intersection of the domain G and the domain $|z-\zeta| > \varepsilon$; obviously $G_\varepsilon \subset G$. Passing in the above relation to the limit $(\varepsilon \to 0)$ we obtain

$$w(\zeta) = \frac{1}{2\pi i} \int_\Gamma \frac{w(z)\, dz}{z-\zeta} - \frac{1}{\pi} \iint_G \frac{\partial w(z)}{\partial \bar z} \frac{dx\, dy}{z-\zeta}\,. \tag{4.9}$$

In an analogous way the following formula is derived:

$$w(\zeta) = -\frac{1}{2\pi i} \int_\Gamma \frac{w(z)\, d\bar z}{\bar z-\bar\zeta} - \frac{1}{\pi} \iint_G \frac{\partial w(z)}{\partial z} \frac{dx\, dy}{\bar z-\bar\zeta}\,. \tag{4.10}$$

The above identities have been proved under the condition that $w(z)$ belongs simultaneously to $C(\bar G)$ and $C^1(G)$. We shall discover later on that they remain valid for a wider class of functions (§6.1, page 38).

* Wider generalizations of these formulae will be indicated below (§7).

The formulae (4.9) and (4.10) are encountered in the papers of many authors. Probably, they were first derived in the paper of Pompeiu [71] (1912); on this basis he generalized the concept of the derivatives ∂_z and $\partial_{\bar{z}}$ (see below, §7). They represent a complex form of some well known integral relations encountered in the theory of logarithmic potential. We shall find later, however, that they are particularly useful for further applications exactly in the complex form given above.

4.2. Let us return to the equation (4.2). If $f \in C^1(G)$ it is easy to obtain a formula yielding all solutions of the equation (4.2).

If w is a solution of the equation (4.2) we have

$$w(z) = \Phi(z) - \frac{1}{\pi} \iint_G \frac{f(\zeta)\,d\xi\,d\eta}{\zeta - z} \equiv \Phi(z) + Tf\,, \quad (4.11)$$

where

$$\Phi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{w(\zeta)\,d\zeta}{\zeta - z}\,, \qquad Tf = -\frac{1}{\pi} \iint_G \frac{f(\xi)\,d\xi\,d\eta}{\zeta - z}\,. \quad (4.12)$$

4.3. The formula (4.11), in general, has no sense if f has discontinuities in G. Nevertheless, it can easily be generalized to the equations (4.2) with right-hand sides belonging to $\mathfrak{A}_0^* \times C(\bar{G})$ (see also §5.7).

Let $f \in \mathfrak{A}_0^* \times C(\bar{G})$. Then an analytic function $\Phi_f(z)$ of the class $\mathfrak{A}_0^*(G)$ can be found, such that $f\Phi_f \in C(\bar{G})$. Multiplying both sides of the equation (4.2) by Φ_f we have

$$\frac{\partial \Phi_f w}{\partial \bar{z}} = \Phi_f(z) f(z)\,.$$

Hence, in view of the formula (4.11) we obtain

$$w(z) = \Phi(z) - \frac{1}{\pi \Phi_j(z)} \iint_G \frac{f(\zeta)\Phi_f(\zeta)\,d\xi\,d\eta}{\zeta - z}\,, \quad (4.13)$$

where $\Phi(z)$ is an arbitrary analytic function of the class $\mathfrak{A}_0^*(G)$.

The formula (4.13) is easily generalized to the case in which $f = f_1 + f_2 + \ldots + f_n$, where

$$f_k = \frac{g_k}{\Phi_k}, \quad g_k \in C(\bar{G}), \quad \Phi_k \in \mathfrak{A}_0^*(G), \quad k = 1, \ldots, n.$$

Then the formula

$$w(z) = \Phi(z) - \sum_{k=1}^{n} \frac{1}{\pi \Phi_k(z)} \iint_G \frac{\Phi_k(\zeta) f_k(\zeta)\, d\xi\, d\eta}{\zeta - z}, \quad (4.14)$$

where Φ is an arbitrary analytic function of the class $\mathfrak{A}_0^*(G)$, yields the solution of the equation (4.2).

In the following sections of this chapter we shall examine in more detail various properties of functions represented by formulae of the form (4.12) and (4.13).

4.4. If $f = f(x, y)$ is an analytic function in the variables x and y we can derive a formula which simplifies the computation of the integral Tf. Replacing the arguments x and y by $\frac{1}{2}(z+\bar{z})$ and $\frac{1}{2i}(z-\bar{z})$, respectively, and computing the indefinite integral (with respect to $\bar{z}$)

$$F(z, \bar{z}) = \int f\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) d\bar{z},$$

we shall have according to the formulae (4.9)

$$\begin{aligned} Tf &\equiv -\frac{1}{\pi} \iint_G \frac{f(\xi, \eta)}{\zeta - z}\, d\xi\, d\eta \\ &= F(z, \bar{z}) - \frac{1}{2\pi i} \int_\Gamma \frac{F(\zeta, \bar{\zeta})}{\zeta - z}\, d\zeta, \quad z \in G. \quad (4.15) \end{aligned}$$

If z lies outside $G + \Gamma$

$$Tf \equiv -\frac{1}{\pi} \iint_G \frac{f(\xi, \eta)}{\zeta - z}\, d\xi\, d\eta = -\frac{1}{2\pi i} \int_\Gamma \frac{F(\zeta, \bar{\zeta})}{\zeta - z}\, d\zeta. \quad (4.16)$$

The last statement follows from the fact that Tf is continuous on the entire plane, holomorphic outside $G + \Gamma$

and vanishes at infinity. Making use of some familiar limiting properties of the Cauchy type integral ([60a], Ch. I, §17) we readily observe that the right-hand sides of the relations (4.15) and (4.16) are identical on the boundary Γ of the domain G.

It should be noted that the formulae (4.15) and (4.16) are valid for both simply-connected and multiply-connected domains. It is only necessary to demand that $f(x, y)$ be an analytic function in the variables x and y in a simply-connected domain G_0 which contains the whole domain G. Assume for instance that $f = z^n \bar{z}^m$ where n and m are non-negative integers. If G is the circle $|z| < 1$ we have, (when $z \in G$),

$$-\frac{1}{\pi}\iint_G \frac{\zeta^n \bar{\zeta}^m}{\zeta - z} d\xi\, d\eta = \begin{cases} \dfrac{z^n \bar{z}^{m+1}}{m+1} - \dfrac{z^{n-m-1}}{m+1}, & \text{for} \quad n \geqslant m-1, \\ \dfrac{z^n \bar{z}^{m+1}}{m+1}, & \text{for} \quad n < m-1. \end{cases}$$

When z lies outside $G + \Gamma$

$$-\frac{1}{\pi}\iint_G \frac{\zeta^n \bar{\zeta}^m}{\zeta - z} d\xi\, d\eta = \begin{cases} 0, & \text{for} \quad n \geqslant m-1, \\ \dfrac{z^{n-m-1}}{m+1}, & \text{for} \quad n < m-1. \end{cases}$$

§5. Generalized derivatives in the Sobolev sense and their properties

In this paragraph we shall investigate classes of functions which possess derivatives in a generalized sense.

5.1. LEMMA 1. *Let $f \in L_p(\bar{G})$, and $f = 0$ outside G. Then the function*

$$g(z) = \iint_G \frac{f(\zeta)\, d\xi\, d\eta}{|\zeta - z|^\lambda} = \iint_E \frac{f(\zeta + z)\, d\xi\, d\eta}{|\zeta|^\lambda}, \quad \lambda < 2, \quad (5.1)$$

is continuous on the entire plane, provided $p > \dfrac{2}{2-\lambda}$.

PROOF. The second integral in (5.1) may be carried out over a circle G_R with the centre at the point $z = 0$ and with a fixed radius R. Since

$$g(z_1)-g(z_2) = \iint_{G_R} \frac{f(\zeta+z_1)-f(\zeta+z_2)}{|\zeta|^\lambda} d\xi\, d\eta\,,$$

we have in view of the Hölder inequality (1.11)

$$\leqslant \left(\iint_{G_R} |f(\zeta+z_1)-f(\zeta+z_2)|^p d\xi\, d\eta\right)^{1/p} \left(\iint_{G_R} |\zeta|^{-q\lambda} d\xi\, d\eta\right)^{1/q},$$

$$\frac{1}{p}+\frac{1}{q} = 1\,. \tag{5.2}$$

Since $q\lambda = \dfrac{p\lambda}{p-1} < 2$ the second factor in the right-hand side of the inequality (5.2) is bounded. The first factor, according to Theorem 1.1, tends to zero when $|z_1-z_2| \to 0$. This implies the continuity of $g(z)$, which was to be proved.

THEOREM 1.12. *Let G be a bounded domain. If $f \in L_1(\bar{G})$, then the integrals*

$$\begin{aligned} \mathrm{T}f &\equiv \mathrm{T}_G f = -\frac{1}{\pi} \iint_G \frac{f(\zeta)\, d\xi\, d\eta}{\zeta - z}\,, \\ \bar{\mathrm{T}}f &\equiv \bar{\mathrm{T}}_G f = -\frac{1}{\pi} \iint_G \frac{f(\zeta)\, d\xi\, d\eta}{\bar{\zeta} - \bar{z}} \end{aligned} \tag{5.3}$$

exist for all points z outside $\bar{G}$, $\mathrm{T}f$ and $\bar{\mathrm{T}}f$ are holomorphic outside $\bar{G}$ with respect to z and $\bar{z}$, respectively, and vanish at infinity.

The theorem is obvious and no proof is required.

THEOREM 1.13. *Let G be a bounded domain. If $f \in L_1(\bar{G})$, then $\mathrm{T}f$ and $\bar{\mathrm{T}}f$, regarded as functions of a point z of the domain G, exist almost everywhere and belong to an arbitrary class $L_p(\bar{G}_*)$ where p is an arbitrary number satisfying the condition $1 \leqslant p < 2$ and G_* is an arbitrary bounded domain of the plane.*

PROOF. According to Lemma 1, the function

$$g_1(z) = \iint_G \frac{|g(\zeta)|\,d\xi\,d\eta}{|\zeta - z|} \tag{5.4}$$

is continuous if $g \in L_p(\bar{G})$, $p > 2$. Hence $|f|g_1 \in L_1(\bar{G})$ if $f \in L_1(\bar{G})$, and in view of Fubini's theorem

$$\iint_G |f|g_1\,dx\,dy = \iint_G |g|f_1\,dx\,dy\;,$$

$$f_1(z) = \iint_G \frac{|f(\zeta)|\,d\xi\,d\eta}{|\zeta - z|}\,.$$

This relation holds for an arbitrary function g of the class $L_p(\bar{G})$, $p > 2$. Therefore, in accordance with a known property of summable functions, $f_1 \in L_q(\bar{G})$, $q = \dfrac{p}{p-1} < 2$. Hence $\mathrm{T}f \in L_q(\bar{G})$, for $|\mathrm{T}f| \leqslant f_1$. Here q is an arbitrary number satisfying the condition $1 \leqslant q < 2$. Since the function $\mathrm{T}f$ is holomorphic outside $\bar{G}$ we have, obviously, $\mathrm{T}f \in L_q(G_*)$, G_* being an arbitrary bounded domain of the plane. Evidently, an analogous statement is true for $\bar{\mathrm{T}}f$.

THEOREM 1.14. *If $f \in L_1(\bar{G})$, then*

$$\iint_G \mathrm{T}f\frac{\partial\varphi}{\partial\bar{z}}\,dx\,dy + \iint_G f\varphi\,dx\,dy = 0\;, \tag{5.5}$$

$$\iint_G \bar{\mathrm{T}}f\frac{\partial\varphi}{\partial z}\,dx\,dy + \iint_G f\varphi\,dx\,dy = 0\;, \tag{5.6}$$

where φ is an arbitrary function of the class $D_1^0(G)$.

PROOF. If $\varphi \in D_1^0(G)$, then in view of the formulae (4.9) and (4.10) we have

$$\varphi(z) = -\frac{1}{\pi}\iint_G \frac{\partial\varphi(\zeta)}{\partial\bar{\zeta}}\,\frac{d\xi\,d\eta}{\zeta - z} \equiv \mathrm{T}\left(\frac{\partial\varphi}{\partial\bar{z}}\right),$$

$$\varphi(z) = -\frac{1}{\pi}\iint_G \frac{\partial\varphi}{\partial\zeta}\,\frac{d\xi\,d\eta}{\bar{\zeta} - \bar{z}} \equiv \bar{\mathrm{T}}\left(\frac{\partial\varphi}{\partial z}\right).$$

Hence

$$\iint_G \mathrm{T}f\frac{\partial\varphi}{\partial\bar z}\,dx\,dy = \frac{1}{\pi}\iint_G f(\zeta)\,d\xi\,d\eta \iint_G \frac{\partial\varphi}{\partial\bar z}\frac{dx\,dy}{z-\zeta}$$
$$= -\iint_G f\varphi\,dx\,dy\,,$$

$$\iint_G \overline{\mathrm{T}}f\frac{\partial\varphi}{\partial z}\,dx\,dy = \frac{1}{\pi}\iint_G f(\zeta)\,d\xi\,d\eta \iint_G \frac{\partial\varphi}{\partial z}\frac{dx\,dy}{\bar z-\bar\zeta}$$
$$= -\iint_G f\varphi\,dx\,dy\,.$$

This was to be proved.

5.2 Following Sobolev, [79a], we shall now introduce the concept of so-called generalized derivatives, [14d].

Definition. Let $f,\, g \in L_1(G)$. If f and g satisfy the relation

$$\iint_G g\frac{\partial\varphi}{\partial\bar z}\,dx\,dy + \iint_G f\varphi\,dx\,dy = 0$$
$$\left(\iint_G g\frac{\partial\varphi}{\partial z}\,dx\,dy + \iint_G f\varphi\,dx\,dy = 0\right), \tag{5.7}$$

where φ is an arbitrary function of the class $D_1^0(G)$, f is said to be the generalized derivative of g with respect to $\bar z$ (with respect to z). *

If $g \in C^1(G)$ and $f = \partial_{\bar z} g\,(f = \partial_z g)$ the relation (5.7) is obviously satisfied. Therefore, in what follows, we shall use the same notation for the generalized derivatives

* Usually, generalized derivatives with respect to real arguments x and y are considered. For our purpose, however, it is more convenient and more expedient to regard as original the differential operators $\partial_{\bar z}$ and ∂_z [14d]. It is easily seen that both definitions are entirely equivalent. Moreover, it should be observed that they are equivalent to the concept of generalized derivative introduced by means of the ordinary partial derivatives with respect to x and y [61]. Let us also notice that in (5.7) we may confine ourselves to functions φ belonging to the linear manifold $D_\infty^0(G)$.

with respect to $\bar{z}$ and z as for ordinary derivatives, i.e. $\partial_{\bar{z}}$ and ∂_z, respectively. In general, we settle on the notation

$$\partial_{\bar{z}} f \equiv \frac{\partial f}{\partial \bar{z}} \equiv f_{\bar{z}}, \quad \partial_z f \equiv \frac{\partial f}{\partial z} \equiv f_z .$$

The definition implies directly that functions having generalized derivatives with respect to $\bar{z}$ or z constitute linear manifolds which from now on will be denoted by $D_{\bar{z}}(G)$ and $D_z(G)$, respectively.

We shall discover below that the classes $D_{\bar{z}}(G)$ and $D_z(G)$ retain a number of important properties of functions differentiable in the ordinary sense.

It follows immediately from the relation (5.7) that if $g \in D_{\bar{z}}(G)$, then $\bar{g} \in D_z(G)$, and conversely. Thus, it is sufficient to investigate properties of one of these classes, $D_{\bar{z}}(G)$ for instance.

We also notice that according to Theorem 1.14 $\mathrm{T}f \in D_{\bar{z}}(G)$, $\overline{\mathrm{T}}f \in D_z(G)$ if $f \in L_1(\bar{G})$, and

$$\frac{\partial \mathrm{T}f}{\partial \bar{z}} = f, \quad \frac{\partial \overline{\mathrm{T}}f}{\partial z} = f. \tag{5.8}$$

We shall also consider classes of functions possessing generalized derivatives of an order higher than one, with respect to z and $\bar{z}$.

We shall say that a function $f(z)$ belongs to the class $D_{m,p}(G)$ if there exist inside G all generalized derivatives

$$\frac{\partial^{i+k} f}{\partial z^i \partial \bar{z}^k} \quad (i+k \leqslant m, \ i, k = 0, 1, \ldots)$$

and they belong to the class $L_p(\bar{G})$, $p \geqslant 1$, including the case of $p = \infty$. The symbol D_m will stand for $D_{m,1}$. Below we shall investigate a number of properties of functions of the class $D_{m,p}$ (§5.6, §6.1, 6.4). In particular, $D_{m,p}$ turns out to be a Banach type space if the norm in it is defined by the formula

$$D_{m,p}(f, G) = \sum_{i,k=0}^{i+k \leqslant m} L_p\left(\frac{\partial^{i+k} f}{\partial z^i \partial \bar{z}^k}\right). \tag{5.8a}$$

Spaces of the type $D_{m,p}$ were first introduced by Sobolev in the class of functions of an arbitrary number of variables, [79a], and denoted by $W_p^{(m)}$. Important properties of these spaces (imbedding theorems) were established in the papers of Sobolev and Kondrashev, [79a], Nikolski [61b, c], and others. Later on, in §6, we shall prove some of these properties for functions of two variables, principal attention being devoted to the properties used in the subsequent chapters. Our considerations will be based on a general representation of the class of functions $D_{\bar{z}}$, which will be derived in §5.4.

5.3. THEOREM 1.15. *If* $\partial_{\bar{z}} g = 0$, *then* $g(z)$ *is holomorphic inside* G, *i.e.* $g(z) \in \mathfrak{A}_0(G)$.

PROOF. It is sufficient to prove that g is holomorphic inside the vicinity of any fixed point z_0 of the domain G. With no loss of generality we can assume that $z_0 = 0$. Let us take a sufficiently small circle G_R with centre at the point $z = 0$, and radius R, $\bar{G}_R \subset G$, and consider the biharmonic Green function of this circle ([14b], §44)

$$Z(z, \zeta) = 2|z-\zeta|^2 \lg \frac{|R^2 - z\bar{\zeta}|}{R|z-\zeta|} - (R^2 - |z|^2)\left(1 - \frac{|\zeta|^2}{R^2}\right),$$

where z and ζ are arbitrary points of the circle G_R. Taking the point ζ inside G_R it is readily observed that for $z \neq \zeta$ $Z(z, \zeta)$ satisfies the biharmonic equation $\Delta\Delta Z = 0$ and the following boundary conditions

$$Z = \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial y} = 0 \quad \text{for} \quad |z| = R.$$

Furthermore, Z, Z_x and Z_y are continuous in the closed circle $|z| \leqslant R$. Consider the function

$$\varphi(z) = \begin{cases} Z(z, \zeta), & \text{for} \quad |z| \leqslant R, \\ 0, & \text{for} \quad |z| > R, \end{cases}$$

where ζ is a fixed point inside G_R. Evidently, $\varphi(z) \in D_1^0(G)$.

If $f = \partial_{\bar{z}} g = 0$, then according to the relation (5.7) we have

$$\iint_G g \frac{\partial \varphi}{\partial \bar{z}} \, dx\, dy = \iint_{G_R} g(z) \frac{\partial Z(z, \zeta)}{\partial \bar{z}} \, dx\, dy = 0 \, . \qquad (5.9)$$

This relation holds for an arbitrary point ζ inside G. If we now apply to both sides of the relation (5.9) the operations $\partial^2_{\zeta\bar{\zeta}}$ we have

$$\frac{\partial^2}{\partial \zeta \partial \bar{\zeta}} \iint_{G_R} g(z) \frac{\partial Z(z, \zeta)}{\partial \bar{z}} \, dx\, dy$$

$$= \iint_{G_R} g(z) \frac{\partial^3 Z(z, \zeta)}{\partial \bar{z} \partial \zeta \partial \bar{\zeta}} \, dx\, dy = 0 \, . \qquad (5.10)$$

The validity of the above change of the order of differentiation and integration can easily be proved.

Simple computation shows that

$$\frac{\partial^3 Z(z, \zeta)}{\partial \bar{z} \partial \zeta \partial \bar{\zeta}} = \frac{1}{\bar{\zeta} - \bar{z}} + \frac{R^2 z - 2R^2 \zeta + \bar{z}\zeta^2}{(R^2 - \bar{z}\zeta)^2} + \frac{z\bar{\zeta}}{R^2 - z\bar{\zeta}} \, .$$

Hence, we obtain from the relation (5.10)

$$\bar{\mathrm{T}} g \equiv -\frac{1}{\pi} \iint_{G_R} \frac{g(z)\, dx\, dy}{\bar{z} - \bar{\zeta}} = \Phi(\zeta) + \overline{\Phi_1(\zeta)} \, , \qquad (5.11)$$

where

$$\Phi(\zeta) = -\frac{1}{\pi} \iint_{G_R} g(z) \frac{R^2 z - 2R^2 \zeta + \bar{z}\zeta^2}{(R^2 - \bar{z}\zeta)^2} \, dx\, dy \, ,$$

$$\Phi_1(\zeta) = -\frac{1}{\pi R^2} \iint_{G_R} \frac{\bar{z}\zeta \, dx\, dy}{R^2 - \bar{z}\zeta} \, .$$

Since $\Phi(\zeta)$ and $\Phi_1(\zeta)$ are holomorphic inside G_R we have, in view of the formulae (4.5) and (5.8), from (5.11)

$$g(z) \equiv \frac{\partial \bar{\mathrm{T}} g}{\partial \zeta} = \Phi'(\zeta) \, ,$$

i.e. g is holomorphic inside G_R — the required result.

This theorem can also be proved by making use of properties of mean values of functions, [79a].

5.4. THEOREM 1.16. *If* $f = \partial_{\bar{z}} g \in L_1(\bar{G})$, *then*

$$g(z) = \Phi(z) - \frac{1}{\pi} \iint_G \frac{f(\zeta)\,d\xi\,d\eta}{\zeta - z} \equiv \Phi(z) + \mathrm{T}_G f\,, \quad (5.12)$$

where Φ *is a function holomorphic inside* G. *Conversely, if* $\Phi \in \mathfrak{A}_0(G)$ *and* $f \in L_1(\bar{G})$, *then the function* $g = \Phi + \mathrm{T}_G f \in D_{\bar{z}}(G)$, *and*

$$\frac{\partial g}{\partial \bar{z}} = f\,. \quad (5.13)$$

PROOF. The first part of the theorem follows from the preceding one, for $\partial_{\bar{z}}(g - \mathrm{T}f) = \partial_{\bar{z}} g - \partial_{\bar{z}} \mathrm{T}f = f - f = 0$. The second part is obvious. *

The formulae (5.12) and (5.13) imply directly the uniqueness of the generalized derivatives.

Let $D_{\bar{z}}(\bar{G})$ be a set of functions $g(z)$ for which $\partial_{\bar{z}} g \in L_1(\bar{G})$. Obviously, $D_{\bar{z}}(\bar{G}) \subset D_{\bar{z}}(G)$.

The formula (5.12) provides a general representation of functions of the class $D_{\bar{z}}(\bar{G})$. Denoting by $\mathrm{T}L_p(\bar{G})$ the set of functions of the form $\mathrm{T}f$ where $f \in L_p(\bar{G})$, $p \geqslant 1$, we obtain in view of the formula (5.12)

$$D_{\bar{z}}(\bar{G}) = \mathfrak{A}_0(G) + \mathrm{T}L_1(\bar{G})\,,$$

i.e. $D_{\bar{z}}(\bar{G})$ is a direct sum of the sets $\mathfrak{A}_0(G)$ and $\mathrm{T}L_1(\bar{G})$. In other words, every element of the set $D_{\bar{z}}(\bar{G})$ is uniquely representable in the form of a sum $\Phi + g$ where $\Phi \in \mathfrak{A}_0(G)$, $g \in \mathrm{T}L_1(\bar{G})$. Evidently, the sets $\mathfrak{A}_0(G)$ and $\mathrm{T}L_1(G)$ have no elements in common, except the zero element.

THEOREM 1.17. *If* $g \in D_{\bar{z}}(\bar{G})$, *then* $g \in D_{\bar{z}}(G_1)$, *where* G_1 *is an arbitrary subdomain of the domain* G.

* This theorem was proved in author's paper [14d].

PROOF. According to Theorem 1.16

$$g(z) = \Phi(z) - \frac{1}{\pi}\iint_G \frac{\partial g}{\partial \bar{\zeta}} \frac{d\xi\, d\eta}{\zeta - z}$$

$$= \Phi_1(z) - \frac{1}{\pi}\iint_{G_1} \frac{\partial g}{\partial \bar{\zeta}} \frac{d\xi\, d\eta}{\zeta - z}, \qquad (5.14)$$

where

$$\Phi_1(z) = \Phi(z) - \frac{1}{\pi}\iint_{G-G_1} \frac{\partial g}{\partial \bar{\zeta}} \frac{d\xi\, d\eta}{\zeta - z}.$$

Since $\Phi_1 \in \mathfrak{A}_0(G_1)$, according to Theorem 1.16 the right-hand side of the relation (5.14) belongs to $D_{\bar{z}}(G_1)$. This completes the proof.

This theorem also implies that *the property of differentiability of a function with respect to* $\bar{z}$ *(or* z*) in the generalized sense is a local property.*

5.5. Assume that $f(z)$ has a generalized derivative with respect to $\bar{z}$ at all points of a domain G. In other words, to every point $z_0 \in G$ there corresponds a neighbourhood G_0 such that

$$f(z) = \Phi_0(z) - \frac{1}{\pi}\iint_{G_0} \frac{g_0(\zeta)\, d\xi\, d\eta}{\zeta - z} \equiv \Phi_0 + \mathrm{T}_{G_0} g_0,$$

$$\Phi_0 \in \mathfrak{A}_0(G_0), \qquad g_0 \in L_1(\bar{G}_0).$$

inside G_0. *In this case* f *possesses a generalized derivative with respect to* $\bar{z}$ *in the whole domain, i.e.* $\partial_{\bar{z}} f = g \in L_1(G)$.

Let G_0 and G_1 be neighbourhoods of the points z_0 and z_1 of the domain G, their intersection $G_0 G_1$ being non-empty. Since in $G_0 G_1$

$$f = \Phi_0 + \mathrm{T}_{G_0} g_0 = \Phi_1 + \mathrm{T}_{G_1} g_1, \qquad \Phi_0, \Phi_1 \in \mathfrak{A}_0(G_0 G_1),$$

by differentiating both sides of these relations with respect to $\bar{z}$ and making use of the formulae (4.5) and (5.13) we obtain $g_0 = g_1$ in $G_0 G_1$.

Let G' be a closed subdomain of the domain G. Let us cover G' by a finite number of neighbourhoods $G_0, G_1, \ldots, G_m$ in the interior of which

$$f = \Phi_j + \mathrm{T}_{G_j} g_j, \qquad \Phi_j \in \mathfrak{A}_0(G_j), \qquad g_j \in L_1(\bar{G}_j),$$

and, because of the result proved above, $g_j = g_k$ in $G_j G_k \neq 0$. Assume that g is a function equal to g_j in G_j. Obviously, $g \in L_1(G')$. If $z \in G_j$, then

$$\partial_{\bar{z}}(f - \mathrm{T}_{G'} g) = \partial_{\bar{z}}(f - \mathrm{T}_{G_j} g - \mathrm{T}_{G'-G_j} g) = 0 \ ,$$

since $f - \mathrm{T}_{G_j} g_j = \Phi_j$ and $\mathrm{T}_{G'-G_j} g$ are holomorphic inside G_j. Consequently, $f = \Phi + \mathrm{T}_{G'} g$, $\Phi \in \mathfrak{A}_0(G')$, thus completing the proof. It is readily seen that g is independent of the subdomain G'.

5.6. Let $f_z \in D_{\bar{z}}(G)$. Then, in view of Theorem 1.16

$$f_z = \Phi_0(z) - \frac{1}{\pi} \iint\limits_{G_0} \frac{\partial}{\partial \bar{\zeta}} \left(\frac{\partial f}{\partial \zeta}\right) \frac{d\xi\, d\eta}{\zeta - z} \, , \qquad \Phi_0 \in \mathfrak{A}_0(G_0) \, ,$$

where G_0 is a subdomain of the domain G and $\bar{G}_0 \subset G$. This relation implies that

$$f(z) = u_0(x, y) + \frac{2}{\pi} \iint\limits_{G_0} \frac{\partial}{\partial \bar{\zeta}} \left(\frac{\partial f}{\partial \zeta}\right) \lg|\zeta - z|\, d\xi\, d\eta \, ,$$

where u_0 is a harmonic function in G_0. Differentiating both sides of the above relation with respect to $\bar{z}$ we obtain

$$f_{\bar{z}} = u_{0\bar{z}} - \frac{1}{\pi} \iint\limits_{G_0} \frac{\partial}{\partial \bar{\zeta}} \left(\frac{\partial f}{\partial \zeta}\right) \frac{d\xi\, d\eta}{\bar{\zeta} - \bar{z}} \, .$$

Applying to both sides the operations ∂_z we have according to Theorem 1.16.

$$\frac{\partial}{\partial z} \left(\frac{\partial f}{\partial \bar{z}}\right) = \frac{\partial}{\partial \bar{z}} \left(\frac{\partial f}{\partial z}\right) .$$

Thus, we have proved the following:

THEOREM 1.18. *If $f_z \in D_{\bar{z}}(G)$, i.e. if $f_{z\bar{z}}$ exists, then $f_{\bar{z}z}$ also exists and $f_{\bar{z}z} = f_{z\bar{z}}$.*

In other words, mixed generalized derivatives with respect to z and $\bar{z}$ are independent of the order of differentiation.

If $f \in C^2(G)$, then $f_{z\bar{z}} = \frac{1}{4}(f_{xx}+f_{yy}) \equiv \frac{1}{4}\Delta f$. Hence, employing also Theorem 1.18, we may now introduce the definition of the generalized Laplace operator Δ as follows:

$$\Delta f \equiv \frac{1}{4}\frac{\partial^2 f}{\partial z \partial \bar{z}}. \qquad (5.15)$$

5.7. We can now introduce wider classes of functions $D_{\bar{z}}^*$ and D_z^*. We say that $f \in D_{\bar{z}}^*(G)$ if $\partial_{\bar{z}} f \in \mathfrak{A}_0^* \times L(G)$. In an analogous way the definition of the class D_z^* is constructed.

It can easily be proved that the general representation of functions of the class $D_{\bar{z}}^*(\bar{G})$ is given by the formula

$$f(z) = \Phi(z) - \frac{\Phi_0(z)}{\pi}\iint_G \frac{g(\zeta)\,d\xi\,d\eta}{\zeta - z}, \qquad (5.16)$$

where Φ and Φ_0 are arbitrary analytic functions of the class $\mathfrak{A}_0^*(G)$ and g is an arbitrary function of the class $L(\bar{G})$. Also

$$g = \frac{1}{\Phi_0(z)}\frac{\partial f}{\partial \bar{z}}. \qquad (5.17)$$

Consider now the non-homogeneous Cauchy–Riemann equation

$$\frac{\partial w}{\partial \bar{z}} = f, \qquad (5.18)$$

and let us assume that $f \in \mathfrak{A}_0^* \times L(\bar{G})$, i.e.

$$f = \frac{f_0(z)}{\Phi_0(z)}, \quad f_0 \in L_1(\bar{G}), \quad \Phi_0 \in \mathfrak{A}_0^*(G).$$

By the term *generalized solution* of the equation (5.18) we shall understand any function $w(z)$ of the class $D_{\bar{z}}^*(G)$ satisfying the equation (5.18) almost everywhere. All such solutions are obviously given by the formula

$$w(z) = \Phi(z) - \frac{1}{\pi\Phi_0(z)}\iint_G \frac{\Phi_0(\zeta)f(\zeta)\,d\xi\,d\eta}{\zeta - z}, \qquad (5.19)$$

where Φ is an arbitrary function of the class $\mathfrak{A}_0^*(G)$.

§6. Properties of the operator $T_G f$

In this paragraph we shall investigate properties of the operator T_G with respect to various classes of functions.

6.1. THEOREM 1.19. *Let G be a bounded domain. If $f \in L_p(\bar{G})$, $p > 2$, then the function $g = T_G f$ satisfies the conditions*

$$|g(z)| \leqslant M_1 L_p(f, \bar{G}), \quad z \in E, \tag{6.1}$$

$$|g(z_1) - g(z_2)| \leqslant M_2 L_p(f, \bar{G}) |z_1 - z_2|^\alpha, \quad \alpha = \frac{p-2}{p}, \tag{6.2}$$

where z_1 and z_2 are arbitrary points of the plane, and M_1, M_2 are constants, M_1 depending on p and G, while M_2 depends on p only.

PROOF. Making use of the Hölder inequality (1.11) we obtain

$$\leqslant \frac{1}{\pi} \left(\iint_G |f(\zeta)|^p d\xi d\eta \right)^{1/p} \left(\iint_G |\zeta - z|^{-q} d\xi d\eta \right)^{1/q}, \tag{6.3}$$

$$\left(\frac{1}{p} + \frac{1}{q} = 1 \right).$$

Since $q < 2$ we have

$$\frac{1}{\pi} \left(\iint_G |\zeta - z|^{-q} d\xi d\eta \right)^{1/q} \leqslant \frac{1}{\pi} \left(\frac{2\pi}{\alpha q} \right)^{1/q} d^\alpha = M_1 \equiv M_1(p, G),$$

where d is the diameter of the domain G and $\alpha = \dfrac{p-2}{2}$. Therefore, (6.3) implies immediately the inequality (6.1).

Since

$$g(z_1) - g(z_2) = \frac{z_1 - z_2}{\pi} \iint \frac{f(\zeta) d\xi d\eta}{(\zeta - z_1)(\zeta - z_2)}, \quad z_1 \neq z_2, \tag{6.3a}$$

we have according to the Hölder inequality

$$|g(z_1) - g(z_2)|$$
$$\leqslant L_p(f, \bar{G}) \frac{|z_1 - z_2|}{\pi} \left(\iint_G (|\zeta - z_1| |\zeta - z_2|)^{-q} d\xi d\eta \right)^{1/q}. \tag{6.4}$$

We now estimate an integral of the form

$$J(\alpha,\beta)=\iint_G |\zeta-z_1|^{-\alpha}|\zeta-z_2|^{-\beta}d\xi\,d\eta\,,\quad \alpha<2\,,\ \beta<2\,. \tag{6.5}$$

About the point z_1 we draw a circle G_1 of radius $\varrho = 2|z_1-z_2|$ and a concentric circle G_0 of radius $2\varrho_0$, such that $\bar{G}\subset G_0$. If ζ lies outside G_1, then $2|\zeta-z_2|\geqslant|\zeta-z_1|$. Therefore

$$J_0=\iint_{G_0-G_1}|\zeta-z_1|^{-\alpha}|\zeta-z_2|^{-\beta}d\xi\,d\eta\leqslant\pi 2^{1+\beta}\int_{\varrho}^{2\varrho_0} r^{1-\alpha-\beta}dr$$

$$<\begin{cases}\dfrac{8\pi|z_1-z_2|^{2-\alpha-\beta}}{\alpha+\beta-2} & \text{when}\quad \alpha+\beta>2\,,\\[2ex] 8\pi\lg\dfrac{\varrho_0}{|z_1-z_2|} & \text{when}\quad \alpha+\beta=2\,,\\[2ex] \dfrac{32\pi}{2-\alpha-\beta}\varrho_0^{2-\alpha-\beta} & \text{when}\quad \alpha+\beta<2\,.\end{cases} \tag{6.6}$$

Furthermore,

$$J_1=\iint_{G_1}\frac{d\xi\,d\eta}{|\zeta-z_1|^{\alpha}|\zeta-z_2|^{\beta}}$$
$$=\frac{1}{|z_1-z_2|^{\alpha+\beta-2}}\iint_{|\zeta|\leqslant 2}\frac{d\xi\,d\eta}{|\zeta|^{\alpha}|\zeta-e^{i\theta}|^{\beta}}\leqslant\frac{M_{\alpha,\beta}}{|z_1-z_2|^{\alpha+\beta-2}}\,.$$

Since

$$J(\alpha,\beta)\leqslant J_0+J_1\,,$$

we have the estimates *

$$J(\alpha,\beta)\leqslant\begin{cases}M'_{\alpha,\beta}|z_1-z_2|^{2-\alpha-\beta} & \text{for}\quad \alpha+\beta>2\,,\\ M''_{\alpha,\beta}(G)+8\pi|\lg|z_1-z_2|| & \text{for}\quad \alpha+\beta=2\,,\\ M''_{\alpha,\beta}(G) & \text{for}\quad \alpha+\beta<2\,.\end{cases} \tag{6.7}$$

Returning now to the inequality (6.4) and remembering that $1<q<2$ we have in view of the first inequality (6.7)

$$\frac{|z_1-z_2|}{\pi}\left(\iint_G(|\zeta-z_1|\,|\zeta-z_2|)^{-q}d\xi\,d\eta\right)^{1/q}\leqslant M_p|z_1-z_2|^{(p-2)/p}\,.$$

* The method of deriving the inequalities (6.6) used here was given by J. Hadamard (see e.g. [30], §563).

Therefore, the inequality (6.2) follows from (6.4). This completes the proof.

The inequalities (6.1) and (6.2) indicate that Tf *is a linear completely continuous operator in the space* $L_p(\bar{G})$ *mapping this space onto* $C_\alpha(\bar{G})$, $\alpha = \frac{p-2}{p}$, $p > 2$, * *and*

$$C_\alpha(Tf, \bar{G}) \leqslant ML_p(f, \bar{G}), \qquad \alpha = \frac{p-2}{p}, \qquad p > 2\,. \quad (6.8)$$

Let $f \in C(\bar{G})$. Then the following inequalities follow from (6.3. (a)):

$$\left.\begin{aligned} &|g(z)| \leqslant MC(f, \bar{G})\,, \\ &|g(z_1)-g(z_2)| \leqslant MC(f, \bar{G})|z_1-z_2|\lg\frac{2d}{|z_1-z_2|}\,, \end{aligned}\right\} \quad (6.9)$$

where d is the diameter of the domain G. If, on the other hand, $f \in L_\infty(\bar{G})$, then we have

$$\left.\begin{aligned} &|g(z)| \leqslant ML_\infty(f, \bar{G})\,, \\ &|g(z_1)-g(z_2)| \leqslant ML_\infty(f, \bar{G})|z_1-z_2|\lg\frac{2d}{|z_1-z_2|}\,. \end{aligned}\right\} \quad (6.9a)$$

These inequalities imply that the operator $T_G f$ is completely continuous in the spaces $C(\bar{G})$ and $L_\infty(\bar{G})$, and it maps these spaces onto a class of functions satisfying Dini's condition.

The following result follows from Theorems 1.17 and 1.19:

if $f \in D_{1,p}$, $p > 2$, *then* $f(z)$ *belongs to the class* $C_{\frac{p-2}{p}}$ *inside* G.

We shall see later that if $p \leqslant 2$ the function $f(z)$ may turn out to have discontinuities.

Theorem 1.19 implies a more general:—

THEOREM 1.20. *If* $f \in D_{m,p}(G)$, $p > 2$, $m \geqslant 1$, *then* Tf *belongs inside* G *to the class* C_α^{m-1} *where* $\alpha = \frac{p-2}{p}$.

* Such operators are sometimes termed stronger completely continuous operators [79a].

To prove the theorem it is sufficient to express derivatives of f of the order $(m-1)$ by derivatives of the order m in accordance with the formula (5.12), and then to make use of Theorem 1.19.

We also observe that *the formula* (4.9) *is still valid under the following conditions*: (1) $G \in C$, (2) $w \in C(\bar{G})$ *and* $\partial_{\bar{z}} w \in L_p(\bar{G})$, $p > 2$.

In fact, in view of the formula (5.12), $w(z) = \Phi(z) + g(z)$ where $\Phi \in \mathfrak{A}_0(G)$ and $g(z) = \mathrm{T}_G^1(\partial_{\bar{z}} w)$. According to Theorem 1.19 $g \in C_a(E)$, $a = \frac{p-2}{p}$, is holomorphic outside $\bar{G}$ and vanishes at infinity. Since, by assumption, w is continuous in $\bar{G}$, $\Phi = w - g$ is also continuous in $\bar{G}$, and in view of the Cauchy formula and Cauchy theorem we have

$$\Phi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{w(\zeta) - g(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\Gamma} \frac{w(\zeta)}{\zeta - z} d\zeta .$$

Thus, *under conditions stated above, the following formula is valid*:

$$w(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{w(\zeta)\, d\zeta}{\zeta - z} - \frac{1}{\pi} \iint_{G} \frac{\partial_{\bar{\zeta}} w}{\zeta - z} d\xi\, d\eta . \qquad (6.10)$$

6.2. The inequalities (6.1) and (6.2) were derived under the assumption that G is a bounded domain. In the case of an unbounded domain the inequality (6.1) has no meaning since in general the constant M_1 depends on the dimensions of the domain G and tends to infinity when the diameter of G tends to infinity. The inequality (6.2), however, is still true because the constant M_2 is independent of G.

Consequently, *if* $f \in L_p(E)$, $p > 2$ *and* $\mathrm{T}f$ *exists at a fixed point* $z = z_0$ *then* $\mathrm{T}f \in H_a(E)$, $a = \frac{p-2}{p}$ *and*

$$\mathrm{T}f = O\left(|z|^{\frac{p-2}{p}}\right) \quad (\text{near } z = \infty) . \qquad (6.10\text{a})$$

It should be observed that the integral over the infinite domain is to be understood as the principal value of the integral.

We shall now prove a theorem for the infinite domain which also implies Theorem 1.19.

THEOREM 1.21. *Let* $f \in L_p L_{p'}(\bar{G})$ *where* $L_p L_{p'}(\bar{G})$ *is the intersection of the sets* $L_p(\bar{G})$ *and* $L_{p'}(\bar{G})$, *and* $p > 2, 1 < p' < 2$. *In this case the function* $g = \mathrm{T}f$ *satisfies the inequalities*

$$|g(z)| \leqslant M_{p,p'} L_p L_{p'}(f, \bar{G}), \quad z \in E, \tag{6.11}$$

$$|g(z_1) - g(z_2)| \leqslant M_{p,p'} L_p L_{p'}(f, \bar{G}) |z_1 - z_2|^{\frac{p-2}{p}}, \quad z_1, z_2 \in E, \tag{6.12}$$

i.e. $\mathrm{T}f \in C_\alpha(E)$, $\alpha = \dfrac{p-2}{p}$ *where*

$$L_p L_{p'}(f, \bar{G}) \equiv L_p(f, \bar{G}) + L_{p'}(f, \bar{G}). \tag{6.13}$$

PROOF. Bearing in mind that the constant M_2 is independent of the domain G, we see that the inequality (6.12) follows from (6.2) if we take formula (6.13) into account. It remains to prove the inequality (6.11). Assuming $f = 0$ outside G we have

$$\mathrm{T}f = -\frac{1}{\pi} \iint_{|\zeta| \leqslant 1} \frac{f(\zeta + z)}{\zeta} d\xi\, d\eta - \frac{1}{\pi} \iint_{|\zeta| \geqslant 1} \frac{f(\zeta + z)}{\zeta} d\xi\, d\eta.$$

Therefore, because of the Hölder inequality

$$\begin{aligned}
|\mathrm{T}f| &\leqslant \frac{1}{\pi} \Big(\iint_{|\zeta| \leqslant 1} |f(\zeta + z)|^p d\xi\, d\eta \Big)^{\frac{1}{p}} \Big(\iint_{|\zeta| \leqslant 1} |\zeta|^{-q} d\xi\, d\eta \Big)^{\frac{1}{q}} \\
&\quad + \frac{1}{\pi} \Big(\iint_{|\zeta| \geqslant 1} |f(\zeta + z)|^{p'} d\xi\, d\eta \Big)^{\frac{1}{p'}} \Big(\iint_{|\zeta| \geqslant 1} |\zeta|^{-q'} d\xi\, d\eta \Big)^{\frac{1}{q'}} \\
&\leqslant \frac{1}{\pi} \Big(\frac{2\pi}{2 - q} \Big)^{\frac{1}{q}} L_p(f, \bar{G}) + \frac{1}{\pi} \Big(\frac{2\pi}{q' - 2} \Big)^{\frac{1}{q'}} L_p(f, \bar{G}) \\
&\leqslant M_{p,p'} \big((L_p(f, \bar{G}) + L_{p'}(f, \bar{G}) \big) \equiv M_{p,p'} L_p L_{p'}(f, \bar{G}),
\end{aligned}$$

where

$$M_{p,p'} = \frac{1}{\pi}\left(\frac{2\pi}{2-q}\right)^{\frac{1}{q}} + \frac{1}{\pi}\left(\frac{2\pi}{q'-2}\right)^{\frac{1}{q'}},$$

$$q = \frac{p}{p-1} < 2\,, \qquad q' = \frac{p'}{p'-1} > 2\,.$$

This completes the proof.

Since in the case of a bounded domain $L_p(\bar{G}) \subset L_{p'}(\bar{G})$ and $L_{p'}(f, \bar{G}) \leqslant ML_p(f, \bar{G})$, the inequalities (6.1) and (6.2) follow from the inequalities (6.11) and (6.12).

Similarly to Theorem 1.21 we can prove

THEOREM 1.22. *If $f \in L_\infty L_{p'}(\bar{G})$, $1 < p' < 2$, then the function $g(z) = \mathrm{T}_G f$ satisfies the conditions*

$$|g(z)| \leqslant M_{p'} L_\infty L_{p'}(f, \bar{G})\,,$$

$$|g(z_1) - g(z_2)| \leqslant M_{p'} L_\infty L_{p'}(f, \bar{G})\,|z_1 - z_2|\,\big|\ln|z_1 - z_2|\big|\,.$$

Here $L_\infty L_{p'}(\bar{G})$ is the intersection of the sets $L_\infty(\bar{G})$ and $L_{p'}(\bar{G})$, $L_\infty L_{p'}(\bar{G})$ being a Banach space normed in the following way:

$$L_\infty L_{p'}(f, \bar{G}) \equiv \text{real}\max|f(z)| + L_{p'}(f, \bar{G})\,,$$

where G is an arbitrary (bounded or unbounded) domain of the plane. If G is a bounded set, $L_{\infty,p'}(f, \bar{G}) \leqslant ML_\infty(f, \bar{G})$.

We shall prove the following theorem for the case of the infinite plane:

THEOREM 1.23. *Let $f \in L_{p,2}(E)$, $p > 2$. Then the function $g(z) = \mathrm{T}_E f$ satisfies the conditions*

$$|g(z)| \leqslant M_p L_{p,2}(f)\,, \tag{6.14}$$

$$|g(z_1) - g(z_2)| \leqslant M_p L_{p,2}(f)\,|z_1 - z_2|^{\frac{p-2}{p}} \quad (z_1, z_2 \in E)\,. \tag{6.15}$$

Moreover, for a given $R > 1$ a number $M_{p,R}$ can be found, such that

$$|g(z)| \leqslant M_{p,R} L_{p,2}(f)|z|^{\frac{2-p}{p}} \quad \text{for} \quad |z| \geqslant R\,. \tag{6.16}$$

PROOF. Writing $T_E f$ in the form $T_{E_1}f + T_{E_2}f$ where $E_1 = \mathcal{E}(|z| \leqslant 1)$, $E_2 = \mathcal{E}(|z| \geqslant 1)$ * and replacing the variable ζ by $1/\zeta$ in the integral $T_{E_2}f$ we have

$$g(z) = -\frac{1}{\pi}\iint_{E_1} \frac{f(\zeta)\,d\xi\,d\eta}{\zeta - z} - \frac{1}{\pi}\iint_{E_2} \frac{f(\zeta)\,d\xi\,d\eta}{\zeta - z}$$

$$= -\frac{1}{\pi}\iint_{E_1} \frac{f(\zeta)\,d\xi\,d\eta}{\zeta - z} - \frac{1}{\pi}\iint_{E_2} \frac{f\left(\dfrac{1}{\zeta}\right)d\xi\,d\eta}{\bar{\zeta}^2\zeta(1-\zeta z)} \equiv g_1(z) + g_2(z) .$$

Also, it is readily observed that

$$g_2(z) = g_0(0) - g_0\left(\frac{1}{z}\right),$$

$$g_0(z) \equiv -\frac{1}{\pi}\iint_{E_1} \frac{f_0(\zeta)\,d\xi\,d\eta}{\zeta - z},$$

$$f_0(\zeta) \equiv \frac{f\left(\dfrac{1}{\zeta}\right)}{\zeta^2}.$$

According to Theorem 1.19 g_1 and g_0 satisfy conditions of the form (6.1), for f and $f_0 \in L_p(E_1)$, $p > 2$. Therefore

$$|g(z)| \leqslant |g_1(z)| + |g_0(0)| + \left|g_0\left(\frac{1}{z}\right)\right|$$
$$\leqslant M_p[L_p(f, E_1) + L_p(f_0, E_1)] \equiv M_p L_{p,2}(f) .$$

Further, $g_1(z)$ satisfies the inequality (6.2) and for $g_2(z)$ we have

$$|g_2(z_1) - g_2(z_2)| \leqslant \frac{|z_2 - z_1|}{\pi} \iint_{E_1} \frac{|f_0(\zeta)|\,d\xi\,d\eta}{|1-\zeta z_1||1-\zeta z_2|}. \quad (6.17)$$

If $|z_1|, |z_2| \leqslant \frac{1}{2}$ then $|1-\zeta z_1| \geqslant \frac{1}{2}$, $|1-\zeta z_2| \geqslant \frac{1}{2}$ when $|\zeta| \leqslant 1$. Hence, from (6.17), we obtain

$$|g_2(z_1) - g_2(z_2) \leqslant |M_p L_p(f_0, E_1)\,|z_1 - z_2|$$
$$\leqslant M_p L_{p,2}(f)\,|z_1 - z_2|^{\frac{p-2}{p}} \qquad (|z_1|, |z_2| \leqslant \tfrac{1}{2}) .$$

* $\mathcal{E}(....)$ denotes the set of elements satisfying the conditions in parenthesis.

If $|z_1| < \frac{1}{2}$, $|z_2| \geqslant \frac{1}{2}$, then

$$|g_2(z_1) - g_2(z_2)| \leqslant \frac{2|z_2 - z_1|}{\pi |z_2|} \iint_{E_1} \frac{|f_0(\zeta)|}{\left|\frac{1}{z_2} - \zeta\right|}$$

$$\leqslant M'_p L_p(f_0, E_1) |z_2 - z_1|^{\frac{p-2}{p}} \left|\frac{z_1}{z_2} - 1\right|^{\frac{2}{p}}$$

$$\leqslant M_p L_{p,2}(f, E) |z_2 - z_1|^{\frac{p-2}{p}},$$

because

$$|z_2| \geqslant \tfrac{1}{2}, \ |z_1| \leqslant \tfrac{1}{2}, \ \left|\frac{z_1}{z_2} - 1\right|^{\frac{2}{p}} \leqslant 2^{\frac{2}{p}}.$$

Finally, if $|z_1|, |z_2| \geqslant \frac{1}{2}$, then

$$|g_2(z_2) - g_2(z_1)| \leqslant \left|g_0\left(\frac{1}{z_2}\right) - g_0\left(\frac{1}{z_1}\right)\right|$$

$$\leqslant M'_p L_p(f_0, E_1) \left|\frac{1}{z_2} - \frac{1}{z_1}\right|^{\frac{p-2}{p}} \leqslant M_p L_{p,2}(f) |z_1 - z_2|^{\frac{p-2}{p}}.$$

Thus,

$$|g(z_1) - g(z_2)| \leqslant |g_1(z_1) - g_1(z_2)| + |g_2(z_1) - g_2(z_2)|$$

$$\leqslant M_p L_{p,2}(f) |z_1 - z_2|^{\frac{p-2}{p}}.$$

Further, when $|z| > 1$ we have

$$|g(z)| \leqslant |g_1(z)| + \left|g_0(0) - g_0\left(\frac{1}{z}\right)\right|$$

$$\leqslant \frac{M'_p L_p(f, E_1)}{|z| - 1} + M''_p L_p(f_0, E_1) |z|^{\frac{2-p}{p}}$$

$$\leqslant M_p L_{p,2}(f) \left[\frac{1}{|z| - 1} + |z|^{\frac{2-p}{p}}\right].$$

The above relation implies immediately the inequality (6.16). This completes the proof of Theorem 1.22.

Thus, *if* $f \in L_{p,2}(E)$, *then*

$$\mathrm{T}_E f \in C_{\frac{p-2}{p}}(E), \quad p > 2,$$

and near infinity $\mathrm{T}_E f$ *decreases as* $|z|^{\frac{2-p}{p}}$.

6.3. Theorem 1.23 implies the following

THEOREM 1.24. *Let $A(z) \in L_{p,2}(E)$, $p > 2$. Then an operator of the form*

$$\mathrm{P}f = \iint_E \frac{A(\zeta)f(\zeta)}{\zeta - z}\,d\xi\,d\eta \equiv -\pi \mathrm{T}_E(Af) \qquad (6.18)$$

is completely continuous in the space $C(E)$ and maps this space onto the space $C_\alpha(E)$, $\alpha = \frac{p-2}{p}$, and

$$C_\alpha(\mathrm{P}f, E) \leqslant M_p L_{p,2}(A)\, C(f, E)\,. \qquad (6.19)$$

Moreover, near infinity

$$|\mathrm{P}f| \leqslant M_p L_{p,2}(A)\, C(f, E) |z|^{\frac{2-p}{p}}, \qquad p > 2\,. \qquad (6.20)$$

PROOF. The inequalities (6.19) and (6.20) follow immediately from (6.15) and (6.16) if we take into account that $Af \in L_{p,2}(E)$, $p > 2$ when $f \in C(E)$, and $L_{p,2}(Af) \leqslant$ $\leqslant L_{p,2}(A)\, C(f, E)$.

If there exists a bounded set of functions $\{f\}$ (i.e. $|f| <$ $< M$), then by virtue of (6.19) the set $\{\mathrm{P}f\}$ is uniformly equi-continuous and uniformly bounded. Hence, by virtue of Arzela's theorem, it follows that the operator $\mathrm{P}f$ is completely continuous.

Also, the following theorem is true.

THEOREM 1.25. *Let $A(z) \in L^{\bullet}_{p,2}(E)$, $p > 2$. Then $\mathrm{P}f$ is completely continuous in the space $L_{q,0}(E)$ when $q \geqslant \frac{2p}{p-2}$; besides $\mathrm{P}f \in C_\alpha(E)$, $0 < \alpha = 1 - 2\left(\frac{1}{p} + \frac{1}{q}\right) \leqslant \frac{p-2}{p}$, and*

$$C_\alpha(\mathrm{P}f, E) \leqslant M_{p,q} L_{p,2}(A) L_{q,0}(f)\,. \qquad (6.21)$$

Moreover, near infinity,

$$|\mathrm{P}f| \leqslant M_{p,q} L_{p,2}(A) L_{q,0}(f) |z|^{-\alpha}, \quad \alpha = 1 - 2\left(\frac{1}{p} + \frac{q}{1}\right). \qquad (6.22)$$

Proof. If the condition $\frac{1}{q}+\frac{1}{p}<\frac{1}{2}$ is satisfied the function $Af \in L_{r,2}(E)$, $r=\frac{pq}{p+q}>2$, and, according to the inequality (1.11),

$$L_{r,2}(Af) \leqslant L_{p,2}(A)L_{q,0}(f) .$$

Therefore (6.21) and (6.22) follow at once from (6.15) and (6.16). The complete continuity of the operator Pf in the space $L_{q,0}(E)$ follows from the inequality (6.21).

6.4. Theorem 1.26. *If* $f \in L_p(\bar{G})$, $1 \leqslant p \leqslant 2$, *then* $g=\mathrm{T}_G f$ *belongs to* $L_\gamma^\alpha(\bar{G})$ *where* G *is a bounded domain*; γ *is an arbitrary number satisfying the inequality*

$$1<\gamma<\frac{2p}{2-p}. \tag{6.23}$$

In this case the following inequalities are satisfied:

$$L_\gamma(\mathrm{T}_G f, \bar{G}) \leqslant M_{p,\gamma}(G)L_p(f, \bar{G}) , \tag{6.24}$$

$$\Big(\iint_E |g(z+\Delta z)-g(z)|^\gamma dx\,dy\Big)^{\frac{1}{\gamma}} \leqslant M'_{p,\gamma}L_p(f, \bar{G})\,|\Delta z|^\alpha , \tag{6.25}$$

$$\alpha=\frac{1}{\gamma}-\frac{2-p}{2p}>0 .$$

Proof. First assume that $p<\gamma<\frac{2p}{2-p}$. Then we have

$$|\mathrm{T}_G f| \leqslant \frac{1}{\pi}\iint_G |f(\zeta)|^{\frac{p}{\gamma}}|\zeta-z|^{-\frac{2}{\gamma}+2}|f(\zeta)|^{p\left(\frac{1}{p}-\frac{1}{\gamma}\right)}|\zeta-z|^{-\frac{2}{q}+\alpha}d\xi\,d\eta ,$$

$$q=\frac{p}{p-1},$$

where $\alpha=\frac{1}{\gamma}-\frac{1}{p}+\frac{1}{2}>0$. Since $\frac{1}{\gamma}+\frac{\gamma-p}{p\gamma}+\frac{1}{q}=1$, with the help of the Hölder inequality we have

$$|\mathrm{T}_G f| \leqslant \frac{1}{\pi}\Big(\iint_G |f(\zeta)|^p|\zeta-z|^{-2+\gamma\alpha}d\xi\,d\eta\Big)^{\frac{1}{\gamma}}\times$$

$$\times\Big(\iint_G |f(\zeta)|^p d\xi\,d\eta\Big)^{\frac{1}{p}-\frac{1}{\gamma}}\Big(\iint_G |\zeta-z|^{-2+\alpha q}d\xi\,d\eta\Big)^{\frac{1}{q}}. \tag{6.26}$$

Since for $\lambda > 0$ the constant $M(\lambda, G)$ is given by

$$M(\lambda, G) = \sup_{z\in E} \iint_G |\zeta - z|^{-2+\lambda} d\xi\, d\eta < \infty ,$$

we easily obtain from (6.26).

$$\iint_G |\mathrm{T}_G f|^\gamma dx\, dy \leqslant \frac{1}{\pi^\gamma} \left(M(qa, \bar{G})\right)^{\frac{\gamma}{q}} \left(L_p(f, \bar{G})\right)^{\gamma - p} \times$$

$$\times \iint_G |f(\zeta)|^p d\xi\, d\eta \iint_G |\zeta - z|^{-2+\gamma a} dx\, dy$$

$$\leqslant \frac{1}{\pi^\gamma} \left(M(qa, G)\right)^{\frac{\gamma}{q}} M(\gamma a, G) \left(L_p(f, \bar{G})\right)^\gamma .$$

The last result at once implies the inequality (6.24). * Evidently, the restriction $\gamma > p$ may now be abandoned.

Let $g(z) = \mathrm{T}_G f$. Then

$$|g(z + \Delta z) - g(z)| \leqslant \frac{|\Delta z|}{\pi} \iint_G \frac{|f(\zeta)|\, d\xi\, d\eta}{|\zeta - z|\, |\zeta - z - \Delta z|}$$

$$\leqslant \frac{|\Delta z|}{\pi} \left(\iint_G |f(\zeta)|^p \left(|\zeta - z|\, |\zeta - z - \Delta z|\right)^{-2+\gamma a} d\xi\, d\eta \right)^{\frac{1}{\gamma}} \times$$

$$\times \left(L_p(f, G)\right)^{1 - \frac{p}{\gamma}} \left(\iint_G |\zeta - z|\, |\zeta - z - \Delta z|^{-2+qa} d\xi\, d\eta \right)^{\frac{1}{q}}$$

$$\leqslant M'_{p,\gamma} |\Delta z|^{\frac{2}{\gamma}} \left(L_p(f, G)\right)^{1 - \frac{p}{\gamma}} \times$$

$$\times \left(\iint_G |f(\zeta)|^p \left(|\zeta - z|\, |\zeta - z - \Delta z|\right)^{-2+\gamma a} d\xi\, d\eta \right)^{\frac{1}{\gamma}} .$$

* Making use of an important inequality of Sobolev ([79b], p. 481) we may prove that (6.24) remains valid also for $\gamma = \dfrac{2p}{2-p}$.

From the last inequality we obtain

$$\Big(\iint_E |g(z+\Delta z)-g(z)|^\gamma dx\,dy\Big)^{\frac{1}{\gamma}} \leqslant M_{p,\gamma}|\Delta z|^{\frac{2}{\gamma}}\big(L_p(f, G)\big)^{1-\frac{p}{\gamma}} \times$$

$$\Big(\iint_G |f(\zeta)|^p d\xi\,d\eta \iint_G \big(|\zeta-z|\,|\zeta-z-\Delta z|\big)^{-2+\gamma a} dx\,dy\Big)^{\frac{1}{\gamma}}$$

$$\leqslant M_{p,\gamma} L_p(f, \bar{G})|\Delta z|^{\frac{1}{\gamma}-\frac{1}{p}+\frac{1}{2}}.$$

It is important to observe that the constant $M_{p,\gamma}$ in this inequality is entirely independent of the domain G if $\gamma > 2$. This can always be achieved for $p > 1$. Thus, Theorem 1.26 has been proved completely.

If follows from the inequalities (6.24) and (6.25) that $T_G f$ *is a completely continuous linear operator in the space* $L_p(\bar{G})$, $1 \leqslant p \leqslant 2$, *and maps this space onto the space* $L_\gamma^a(E)$ *where* $a = \dfrac{1}{\gamma} - \dfrac{1}{p} + \dfrac{1}{2}$, γ *being an arbitrary number satisfying the condition* $p \leqslant \gamma < \dfrac{2p}{2-p}$.

We notice that if G is a bounded domain and γ satisfies the condition $2 < \gamma < \dfrac{2p}{2-p}$, then the inequality (6.24) is replaced by the stronger one

$$L_\gamma(T_G f, E) \leqslant M'_{p,\gamma}(G) L_p(f, \bar{G}), \quad 1 < p \leqslant 2. \qquad (6.27)$$

From Theorem 1.26 in conjunction with Theorems 1.19 and 1.20 the following theorems follow:

THEOREM 1.27. *If* $f \in D_{m,p}(G)$, $m \geqslant 2$, $1 < p \leqslant 2$, *then, inside* G, f *belongs to the classes* $D_{m-1,\gamma}$ *and* $C^{m-2}_{\frac{\gamma-2}{\gamma}}$, *where* γ *is an arbitrary number satisfying the inequality* $2 < \gamma < \dfrac{2p}{2-p}$.

THEOREM 1.28. *If* $f \in D_{m,1}(G)$, $m \geqslant 3$, *then, inside* G, f *belongs to the classes* $D_{m-1,\gamma}$, $D_{m-2,\frac{2\gamma}{2-\gamma}}$ *and* $C^{m-3}_{\frac{2(\gamma-1)}{\gamma}}$, *where* γ *is an arbitrary number satisfying the inequality* $1 < \gamma < 2$.

6.5. THEOREM 1.29. *Let G be a bounded open set and $A(z) \in L_p(\bar{G})$, $p > 2$. Then the operator*

$$\mathrm{P}f = \iint_G \frac{A(\zeta)f(\zeta)\,d\xi\,d\eta}{\zeta - z} \equiv -\pi \mathrm{T}_G(Af) \tag{6.18}$$

is completely continuous in $L_q(\bar{G})$ if $\frac{1}{2} \leqslant \frac{1}{p} + \frac{1}{q} \leqslant 1$. *Moreover, if the integer n satisfies the condition*

$$n > \frac{2p}{p-2}\left(\frac{1}{p} + \frac{1}{q} - \frac{1}{2}\right) \geqslant n - 1\,, \tag{6.28}$$

then

$$L^{a}_{\gamma_k}(\mathrm{P}^k f, E) \leqslant M_{p,q,a}(G) L_p(A, \bar{G}) L_p(f, \bar{G}) \tag{6.29}$$
$$(k = 1, \ldots, n)\,,$$

$$C_\beta(\mathrm{P}^{n+1} f, E) \leqslant M_{p,q,a}(G) L_p(A, \bar{G}) L_q(f, \bar{G})\,, \tag{6.30}$$

where

$$\gamma_k = \frac{1}{\frac{1}{q} + \frac{k}{p} - \frac{k}{2} + ka} \quad (k = 1, \ldots, n)\,, \tag{6.31}$$

$$\beta = 1 - 2\left(\frac{1}{q} + \frac{n+1}{p} - \frac{n}{2} + na\right), \tag{6.32}$$

a being an arbitrary positive number satisfying the inequality

$$0 < a < \frac{p-2}{2p} - \frac{1}{n}\left(\frac{1}{p} + \frac{1}{q} - \frac{1}{2}\right). \tag{6.33}$$

PROOF. Since $A \in L_p(\bar{G})$, $f \in L_q(\bar{G})$, $\frac{1}{2} \leqslant \frac{1}{p} + \frac{1}{q} \leqslant 1$, then $Af \in L_{r_1}(\bar{G})$ where $r_1 = \dfrac{pq}{p+q}$ and $1 \leqslant r_1 \leqslant 2$. By virtue of Theorem 1.26 the function $\mathrm{P}f \in L^{a}_{\gamma_1}(G)$ where $\dfrac{1}{\gamma_1} = \dfrac{1}{q} + \dfrac{1}{p} - \dfrac{1}{2} + a$, a being an arbitrarily small positive number. Hence $A\mathrm{P}f \in L_{r_2}(\bar{G})$ where $r_2 = \dfrac{p\gamma_1}{p + \gamma_1}$ and $\mathrm{P}^2 f = \pi^2 \mathrm{T}(A\mathrm{P}f) \in L^{a}_{\gamma_2}(\bar{G})$ where $\dfrac{1}{\gamma_2} = \dfrac{1}{q} + \dfrac{2}{p} - 1 + 2a$. Re-

peating this reasoning we obtain $\mathrm{P}\, f \in L^{\alpha}_{\gamma_k}(\bar{G})$ $(k = 1, \ldots, n)$ where γ_k is determined by the relation (6.31). Therefore, $A\mathrm{P}^n f \in L_{r_{n+1}}(\bar{G})$ where $r_{n+1} = \dfrac{p\gamma_n}{p+\gamma_n}$. Taking into account the inequalities (6.28) we find that $r_{n+1} > 2$ if α satisfies the inequality (6.33). Consequently, in view of Theorem 1.19. $\mathrm{P}^{n+1}f = -\pi\mathrm{T}(A\mathrm{P}^n f) \in C_\beta(E)$ where β is given by the relation (6.32). The inequalities (6.29) and (6.30) follow from the inequalities (6.1), (6.2), (6.23) and (6.25).

6.6. THEOREM 1.30. *Let the boundary Γ of a domain G be the union of a finite number of piecewise smooth contours. If $f \in L_p(\bar{G})$, $1 < p \leqslant 2$, then $\mathrm{T}_G f \in L_\gamma(\Gamma)$ where γ is an arbitrary number satisfying the condition*

$$1 < \gamma < \frac{p}{2-p}, \qquad 1 < p \leqslant 2. \tag{6.34}$$

Further,

$$\Big(\int_\Gamma |\mathrm{T}_G f|^\gamma ds\Big)^{\frac{1}{\gamma}} \leqslant M_{p,\gamma}(G)\Big(\iint_G |f(\zeta)|^p d\xi\, d\eta\Big)^{\frac{1}{p}},$$

i.e.

$$L_\gamma(\mathrm{T}_G f, \Gamma) \leqslant M_{p,\gamma}(G) L_p(f, G). \tag{6.35}$$

PROOF. First let us assume that $p < \gamma < \dfrac{p}{2-p}$. Then

$$|\mathrm{T}_G f| \leqslant \frac{1}{\pi}\iint_G |f(\zeta)|^{\frac{p}{\gamma}} |\zeta - z|^{-\frac{1}{\gamma}+\alpha} |f(\zeta)|^{p\left(\frac{1}{p}-\frac{1}{\gamma}\right)} |\zeta - z|^{-\frac{2}{q}+\alpha} d\xi\, d\eta,$$

where $2\alpha = \dfrac{1}{\gamma} - \dfrac{2}{p} + 1$. Since $\dfrac{1}{\gamma} + \dfrac{\gamma - p}{\gamma p} + \dfrac{1}{q} = 1$, applying the Hölder inequality we obtain

$$|\mathrm{T}_G f| \leqslant \frac{1}{\pi}\Big(\iint_G |f(\zeta)|^p |\zeta - z|^{-1+\gamma\alpha} d\xi\, d\eta\Big)^{\frac{1}{\gamma}} \times$$

$$\times \Big(\iint_G |f(\zeta)|^p d\xi\, d\eta\Big)^{\frac{1}{p}-\frac{1}{\gamma}} \Big(\iint_G |\zeta - z|^{-2+q\alpha} d\xi\, d\eta\Big)^{\frac{1}{q}},$$

whence

$$\int_{\Gamma} |T_G f|^{\gamma} ds \leqslant \pi^{-\gamma}\big(M(q\alpha, G)\big)^{\frac{\gamma}{q}}\big(L_p(f, \bar{G})\big)^{\gamma}\int_{\Gamma} |\zeta - z|^{-1+\gamma\alpha} ds . \qquad (6.36)$$

Since for $\lambda < 1$ we have for the constant $M(\lambda, \Gamma)$

$$M(\lambda, \Gamma) = \sup_{z \in E} \int_{\Gamma} |z-\zeta|^{-\lambda} ds < \infty , \, *$$

the relation (6.36) implies immediately the inequality (6.35). Obviously, we may now omit the restriction $p < \gamma < \frac{p}{2-p}$. †

It is seen from (6.23) and (6.34) that if $p = 2$, in the inequalities (6.24), (6.25), (6.35) γ may stand for an arbitrarily large positive number. It does not mean, however, that $T_G f \in L_\infty(\bar{G})$ and $L_\infty(\Gamma)$ when $f \in L_2(\bar{G})$. For example, if G is the circle $|z| \leqslant d < 1$ the function

$$f(z) = \partial_{\bar{z}} \ln\ln\frac{1}{r} = \frac{e^{i\varphi}}{2r\ln r}, \qquad z = re^{i\varphi} ,$$

belongs to the class $L_2(\bar{G})$ but $T_G f = \ln\ln\frac{1}{r}$ is unbounded.

§7. Green's formula for the class of functions $D_{1,p}$. Areal derivative

7.1. Let us now consider a domain G the boundary Γ of which consists of a finite number of simple piecewise smooth Jordan curves. Let $f(z) \in L_p(\bar{G})$, $p > 1$. Then the following formula holds

$$-\frac{1}{2\pi i}\int_{\Gamma} \frac{T_G f}{\zeta - z} d\zeta = \begin{cases} -T_G f(z) , & \text{if} \quad z \,\bar{\in}\, G+\Gamma , \\ 0 , & \text{if} \quad z \in G . \end{cases} \qquad (7.1)$$

* This can easily be proved by making use of some simple properties of piecewise smooth contours ([60a], Ch. II, §2; appendix 1).

† The inequalities (6.24), (6.25), and (6.35) are particular cases of inequalities of Sobolev and Kondrashev ([79a], Ch. I, §6).

We shall prove this assertion. Let G_n be a sequence of domains satisfying the conditions: (1) $\bar{G}_n \subset G_{n+1} \subset \bar{G}_{n+1} \subset \bar{G}$ (2) $G_n \to G$ when $n \to \infty$.

Consider now the functions

$$T_n f = -\frac{1}{\pi} \iint\limits_{G_n} \frac{f(\zeta)}{\zeta - z} d\xi f\eta ,$$

which are obviously holomorphic outside $\bar{G}_n$ and vanish at infinity. Therefore, according to the Cauchy formula, we have the relations

$$\frac{1}{2\pi i} \int\limits_{\Gamma} \frac{\mathrm{T}_n f d\zeta}{\zeta - z} = \begin{cases} -\mathrm{T}_n f(z) , & \text{if} \quad z \in G + \Gamma , \\ 0 , & \text{if} \quad z \in G . \end{cases} \tag{7.2}$$

It is obvious that if the value of z is fixed, $\lim\limits_{n\to\infty} \mathrm{T}_n f = = \mathrm{T}_G f$. Besides, according to the inequality (6.35), the sequence $\mathrm{T}_n f$ converges on Γ to $\mathrm{T}_G f$ in the mean of the order γ, i.e. $L_\gamma(\mathrm{T}_G f - \mathrm{T}_n f, \Gamma) \to 0$, $\gamma > 1$. Therefore, passing to the limit in (7.2) we arrive at the relation (7.1).

Setting in (7.1) $z \to \infty$ we obtain

$$\frac{1}{2i} \int\limits_{\Gamma} \mathrm{T}_G f dz = \iint\limits_{G} f(z) dx dy ,$$

i.e.

$$\frac{1}{2i} \int\limits_{\Gamma} \mathrm{T}_G f dz = \iint\limits_{G} \frac{\partial \mathrm{T}_G f}{\partial \bar{z}} dx dy . \tag{7.3}$$

In an analogous way we can prove that

$$\frac{1}{2i} \int\limits_{\Gamma} \overline{\mathrm{T}}_G f d\bar{z} = - \iint\limits_{G} \frac{\partial \overline{\mathrm{T}}_G f}{\partial z} dx dy = - \iint\limits_{G} f dx dy , \tag{7.4}$$

if $f \in L_p(\bar{G})$, $p > 1$.

If $f \in L_p(\bar{G})$, $p > 2$, the formulae (7.3) and (7.4) are valid for the more general case of Γ consisting of a finite number of rectifiable, simple Jordan curves. The proof of this presents no difficulties, since in this case, according to Theorem 1.19, the function $\mathrm{T}_G f \in C_\alpha(E)$, $\alpha = \dfrac{p-2}{p}$.

Suppose now that $\partial_{\bar z} w \in L_p(G_0)$, $p > 1$, where G_0 is a domain containing the domain G, i.e. $\bar G \subset G_0$. In this case we have the formula

$$\frac{1}{2i}\int_\Gamma w(z)\,dz = \iint_G \frac{\partial w}{\partial \bar z}\,dx\,dy\,. \tag{7.5}$$

If, on the other hand, $\partial_z w \in L_p(G_0)$, $p > 1$, then we obtain

$$\frac{1}{2l}\int_\Gamma w(z)\,d\bar z = -\iint_G \frac{\partial w}{\partial z}\,dx\,dy\,. \tag{7.6}$$

We shall prove these formulae. They were employed before (§4) for $w \in C^1(G)$ and $C(\bar G)$. It is sufficient to prove (7.5).

Let G' be a subdomain of the domain G_0 satisfying the condition $\bar G \subset G' \subset \bar G' \subset G_0$. Then, inside G'

$$\begin{aligned} w(z) &= \Phi(z) - \frac{1}{\pi}\iint_{G'} \frac{\partial w}{\partial \bar\zeta}\frac{d\xi\,d\eta}{\zeta - z} \\ &= \Phi(z) - \frac{1}{\pi}\iint_{G} \frac{\partial w}{\partial \bar\zeta}\frac{d\xi\,d\eta}{\zeta - z} - \frac{1}{\pi}\iint_{G'-G} \frac{\partial w}{\partial \bar\zeta}\frac{d\xi\,d\eta}{\zeta - z}\,, \end{aligned}$$

Φ being a function holomorphic inside G'. Consequently,

$$\begin{aligned} \frac{1}{2i}\int_\Gamma w(z)\,dz &= \frac{1}{2i}\int_\Gamma \Phi(z)\,dz + \frac{1}{2i}\int_\Gamma \mathrm{T}_G\left(\frac{\partial w}{\partial \bar z}\right)dz \\ &\quad - \frac{1}{2\pi i}\int_\Gamma \left(\iint_{G'-G} \frac{\partial w}{\partial \bar\zeta}\frac{d\xi\,d\eta}{\zeta - z}\right)dz\,. \end{aligned} \tag{7.7}$$

But, by virtue of the Cauchy theorem and the formula (7.1)

$$\int_\Gamma \Phi(z)\,dz = 0\,, \qquad \int_\Gamma \left(\iint_{G'-G} \frac{\partial w}{\partial \bar\zeta}\frac{d\xi\,d\eta}{\zeta - z}\right)dz = 0\,,$$

Therefore, according to the formula (7.1), (7.7) implies the relation (7.5).

Let us observe that the formulae (7.5) and (7.6) are still valid when Γ consists of a finite number of rectifiable

Jordan curves if $w \in C(\bar{G})$, $\partial_{\bar{z}} w \in L_p(\bar{G})$ or $\partial_z w \in L_p(\bar{G})$, $p > 2$, respectively.

7.2. Let us consider a so-called regular sequence of domains G_n which contracts to the point $\zeta \in G$, ζ belonging to all G_n ([78], Ch. IV, §2). Then, according to Lebesgue's theorem ([78], Ch. IV, §5), we have for an arbitrary sequence of domains G_n of this kind, and for an arbitrary function $f \in L_p(G)$

$$\lim_{n\to\infty} \frac{1}{\operatorname{mes} G_n} \iint_{G_n} f(z)\,dx\,dy = f(z) \tag{7.8}$$

(almost everywhere in G).

Hence, we obtain from (7.5): *if* $\partial_{\bar{z}} w \in L_p(\bar{G})$, $p > 1$, *then*

$$\frac{\partial w}{\partial \bar{z}} = \lim_{n\to\infty} \frac{1}{\operatorname{mes} G_n} \frac{1}{2i} \int_{\Gamma_n} w(z)\,dz \tag{7.9}$$

(*almost everywhere in* G).

The right-hand side of the last relation is termed *the derivative of w in the sense of Pompeiu*, or *areal derivative*, subject to the obvious condition of existence and independence of the regular sequence of domains G_n contracting to the point ζ. *

Thus, we are led to

THEOREM 1.31. *If the generalized derivative $\partial_{\bar{z}} w \in L_p(\bar{G})$, $p > 1$, then the function $w(z)$ has almost everywhere in G a derivative in the sense of Pompeiu, the latter being equal to the generalized derivative in the sense of Sobolev $\partial_{\bar{z}} w$.*

In particular, if $\partial_{\bar{z}} w \in C(\bar{G})$, then $w(z)$ belongs to the class $C_{\bar{z}}(G)$ (and conversely). The properties of this class of functions are investigated in author's paper [14a] (see also [90], [94]).

* This generalization of the notion of the derivative was established by the Rumanian mathematician D. Pompeiu, [71], (1912) who named it "derivative with respect to area" (la dérivée aréolaire). We shall use the term areal derivative [82a, b].

§8. On differential properties of functions of the form $T_G f$. Operator Πf

8.1. In this paragraph conditions will be derived under which functions of the form $T_G f$ possess derivatives in the classical sense. We have already learnt that the function $T_G f$ has the generalized derivative with respect to $\bar{z}$ equal to $f(z)$ if $f \in L_1(\bar{G})$. It is interesting to derive conditions for the existence of the generalized derivative of $T_G f$ with respect to z. The latter derivative, if it exists, will be denoted by

$$\Pi_G f \equiv \frac{\partial T_G f}{\partial z} \quad \text{or} \quad \Pi f \equiv \frac{\partial T f}{\partial z}. \tag{8.1}$$

THEOREM 1.32. *Let $G \in C_\alpha^{m+1}$, $f(z) \in C_\alpha^m(\bar{G})$, $0 < \alpha < 1$, $m \geqslant 0$. Then the function $h(z) = T_G f$ belongs to the class $C_\alpha^{m+1}(\bar{G})$ and $T_G f$ is a completely continuous operator in $C_\alpha^m(\bar{G})$. Moreover,*

$$\frac{\partial h}{\partial x} = f + \Pi f, \quad \frac{\partial h}{\partial y} = -if + i\Pi f, \tag{8.2}$$

where

$$\Pi f \equiv \Pi_G f = -\frac{1}{\pi}\iint_G \frac{f(\zeta)}{(\zeta - z)^2} d\xi\, d\eta. \tag{8.3}$$

This singular integral exists in the sense of the Cauchy principal value and belongs to the class $C_\alpha^m(\bar{G})$. Besides, Πf represents a linear bounded operator in $C_\alpha^m(\bar{G})$ mapping this space onto itself.

PROOF. Let z be a fixed point of the domain G. Then we have

$$\begin{aligned}\Pi_G f &\equiv -\frac{1}{\pi}\iint_G \frac{f(\zeta)}{(\zeta - z)^2} d\xi\, d\eta = -\lim_{\varepsilon\to 0}\iint_{G_\varepsilon} \frac{f(\zeta)\, d\xi\, d\eta}{(\zeta - z)^2} \\ &= \lim_{\varepsilon\to 0}\frac{1}{\pi}\iint_{G_\varepsilon} \frac{f(z) - f(\zeta)}{(\zeta - z)^2} d\xi\, d\eta - \\ &\qquad - f(z)\lim_{\varepsilon\to 0}\frac{1}{\pi}\iint_{G_\varepsilon} \frac{d\xi\, d\eta}{(\zeta - z)^2}. \end{aligned}\tag{8.4}$$

On the other hand, applying the formula (7.6) we may write

$$\frac{1}{\pi}\iint\limits_{G_\varepsilon}\frac{d\xi\,d\eta}{(\zeta-z)^2}=-\frac{1}{\pi}\iint\limits_{G_\varepsilon}\frac{\partial}{\partial\zeta}\left(\frac{1}{\zeta-z}\right)d\xi\,d\eta$$

$$=\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{d\bar\zeta}{\zeta-z}-\frac{1}{2\pi i}\int\limits_{\Gamma_\varepsilon}\frac{d\bar\zeta}{\zeta-z}=\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{d\bar\zeta}{\zeta-z}$$

$$=\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{\bar\zeta\,d\zeta}{(\zeta-z)^2}=\Phi'_\Gamma(z)\,, \qquad (8.5)$$

where

$$\Phi_\Gamma(z)=\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{\bar\zeta\,d\zeta}{\zeta-z}\,. \qquad (8.6)$$

Let us observe that if Γ is the circle $|\zeta-z_0|=R$ and z lies inside it, $|z-z_0|<R$, then $\Phi_\Gamma(z)\equiv 0$.

Thus, in view of (8.4) and (8.5),

$$\Pi_G f=-\frac{1}{\pi}\iint\limits_{G}\frac{f(\zeta)-f(z)}{(\zeta-z)^2}\,d\xi\,d\eta-f(z)\Phi'_\Gamma(z)\,. \qquad (8.7)$$

Since $f\in C_\alpha(\bar G)$, the double integral in the right-hand side of the last relation is to be regarded as an ordinary improper integral.

Consequently, for an arbitrary function f of the class $C_\alpha(\bar G)$ the singular integral (8.3) exists in the sense of the Cauchy principal value at every point z inside the domain G, and it is represented by the formula (8.7).

Let us now prove that $g(z)\equiv\Pi_G f\in C_\alpha(\bar G)$, $0<\alpha<1$, when $f\in C_\alpha(\bar G)$.

If z and z_1 belong to G, and $z\neq z_1$, then

$$g(z_1)-g(z)=\frac{z_1-z}{\pi}\iint\limits_{G}\frac{f(\zeta)-f(z)}{(\zeta-z)^2(\zeta-z_1)}\,d\xi\,d\eta+$$

$$+\frac{z_1-z}{\pi}\iint\limits_{G}\frac{f(\zeta)-f(z_1)}{(\zeta-z)(\zeta-z_1)^2}\,d\xi\,d\eta+$$

$$+\frac{f(z)(z_1-z)}{\pi}\iint\limits_{G}\frac{d\xi\,d\eta}{(\zeta-z)^2(\zeta-z_1)}+$$

$$+\frac{f(z_1)(z_1-z)}{\pi}\iint\limits_{G}\frac{d\xi\,d\eta}{(\zeta-z)(\zeta-z_1)^2}\,. \qquad (8.8)$$

Since

$$\frac{1}{\pi}\iint_G \frac{d\xi\,d\eta}{(\zeta-z)^2(\zeta-z_1)} = \frac{1}{\pi(z-z_1)}\iint_G \frac{d\xi\,d\eta}{(\zeta-z)^2} - \frac{1}{\pi(z-z_1)^2}\left(\iint_G \frac{d\xi\,d\eta}{\zeta-z} - \iint_G \frac{d\xi\,d\eta}{\zeta-z_1}\right),$$

$$\frac{1}{\pi}\iint_G \frac{d\xi\,d\eta}{\zeta-z} = -\bar{z} + \frac{1}{2\pi i}\int_\Gamma \frac{\bar{\zeta}\,d\zeta}{\zeta-z} \equiv -\bar{z} + \Phi_\Gamma(z),$$

we have, taking into account the relation (8.5),

$$\frac{1}{\pi}\iint_G \frac{d\xi\,d\eta}{(\zeta-z)^2(\zeta-z_1)} = \frac{\Phi_\Gamma'(z)}{z-z_1} + \frac{\bar{z}-\bar{z}_1}{(z-z_1)^2} + \frac{\Phi_\Gamma(z_1)-\Phi_\Gamma(z)}{(z-z_1)^2}, \quad z \neq z_1. \tag{8.8a}$$

On the basis of this formula the relation (8.8) may now be written in the form

$$\begin{aligned} g(z_1)-g(z) &= \frac{z_1-z}{\pi}\iint_G \frac{f(\zeta)-f(z)}{(\zeta-z)^2(\zeta-z_1)}\,d\xi\,d\eta + \\ &+ \frac{z_1-z}{\pi}\iint_G \frac{f(\zeta)-f(z_1)}{(\zeta-z)(\zeta-z_1)^2}\,d\xi\,d\eta + \\ &+ [f(z)-f(z_1)]\left(\frac{\bar{z}-\bar{z}_1}{z_1-z} + \frac{\Phi_\Gamma(z_1)-\Phi_\Gamma(z)}{z_1-z} - \Phi_\Gamma'(z)\right) + \\ &+ f(z_1)[\Phi_\Gamma'(z_1)-\Phi_\Gamma'(z)]. \end{aligned} \tag{8.9}$$

Since $\Gamma \in C_\alpha^{m+1}$, $0 < \alpha < 1$, the integral of the Cauchy type $\Phi_\Gamma(z)$ given by the relation (8.6) belongs to $C_\alpha^{m+1}(\bar{G})$. Hence, $\Phi_\Gamma'(z) \in C_\alpha^m(\bar{G})$ (§3).

Let $f \in C_\alpha(\bar{G})$, i.e.

$$|f(z)-f(z_1)| \leqslant H(f, \alpha, \bar{G})|z-z_1|^\alpha, \quad 0 < \alpha < 1.$$

Then, taking into account the inequality (6.7),

$$\begin{aligned} \left|\frac{1}{\pi}\iint_G \frac{f(\zeta)-f(z)}{(\zeta-z)^2(\zeta-z_1)}\,d\xi\,d\eta\right| &\leqslant \frac{H(f, \alpha, \bar{G})}{\pi}\iint_G \frac{d\xi\,d\eta}{|\zeta-z|^{2-\alpha}|\zeta-z_1|} \\ &\leqslant M_\alpha' H(f, \alpha, \bar{G})|z-z_1|^{\alpha-1}, \end{aligned} \tag{8.9a}$$

where M'_α is independent of the domain G. Making use of this inequality we obtain from (8.9)

$$|g(z)-g(z_1)| \leqslant M_\alpha(G)\, C_\alpha(f, \bar{G})|z-z_1|^\alpha , \tag{8.10}$$

where

$$M_\alpha(G) = 1+2M'_\alpha + C_\alpha(\Phi'_\Gamma, \bar{G})+H(\Phi_\Gamma, \bar{G}) .$$

The relation (8.7) implies also the inequality

$$\begin{aligned} |\Pi_G f| &\leqslant M'_\alpha(G) H(f, \alpha, \bar{G}) + C(f, \bar{G})\, C(\Phi_\Gamma, \bar{G}) \\ &\leqslant M''_\alpha(G)\, C_\alpha(f, \bar{G}) . \end{aligned} \tag{8.11}$$

From (8.10) and (8.11) it follows that

$$C_\alpha(\Pi_G f, \bar{G}) \leqslant M_\alpha(G)\, C_\alpha(f, \bar{G}) . \tag{8.12}$$

Thus, *we have established that $\Pi_G f \in C_\alpha(\bar{G})$ when $f(z) \in C_\alpha(\bar{G})$.* * *Moreover, $\Pi_G f$ is a linear operator in $C_\alpha(\bar{G})$ mapping this space onto itself.*

We turn now to the derivation of the formulae (8.2). Denoting $T_G f$ by $h(z)$, for $z \neq z_1$, $z, z_1 \in G$ we have employing the formula (8.8a)

$$\begin{aligned} \frac{h(z_1)-h(z)}{z_1-z} - \Pi_G f &= \frac{z-z_1}{\pi} \iint_G \frac{f(\zeta)-f(z)}{(\zeta-z)^2(\zeta-z_1)}\, d\xi\, d\eta + \\ &\quad + \frac{f(z)(z-z_1)}{\pi} \iint_G \frac{d\xi\, d\eta}{(\zeta-z)^2(\zeta-z_1)} \\ &= \frac{z-z_1}{\pi} \iint_G \frac{f(\zeta)-f(z)}{(\zeta-z)^2(\zeta-z_1)}\, d\xi\, d\eta + \\ &\quad + \left(\frac{\bar{z}-\bar{z}_1}{z-z_1} + \Phi'_\Gamma(z) - \frac{\Phi_\Gamma(z)-\Phi_\Gamma(z_1)}{z-z_1}\right) f(z) . \end{aligned} \tag{8.13}$$

If $f \in C_\alpha(\bar{G})$ it follows immediately from (8.13) that

$$\begin{aligned} \left|\frac{h(z_1)-h(z)}{z_1-z} - \Pi_G f - \frac{\bar{z}-\bar{z}_1}{z-z_1} f(z)\right| &\leqslant M_\alpha H(f, \alpha, \bar{G})|z-z_1|^\alpha + \\ &\quad + \left|\Phi'_\Gamma(z) - \frac{\Phi_\Gamma(z)-\Phi_\Gamma(z_1)}{z-z_1}\right| C(f, \bar{G}) . \end{aligned} \tag{8.14}$$

* This proposition was first proved by Giraud [34] (cf. also [54], [56a]).

If now z_1 tends to z along a radius making an angle ϑ with the real axis, $z_1-z=|z_1-z|e^{i\vartheta}$ we have

$$\lim_{z_1\to z}\frac{h(z_1)-h(z)}{z_1-z}=\Pi_G f+e^{-2i\vartheta}f(z)\,. \tag{8.15}$$

Setting here $\vartheta=0$ and $\vartheta=\frac{\pi}{2}$ we obtain the formulae (8.2) which can be written in the form

$$\frac{\partial \mathrm{T}_F f}{\partial \bar z}=f(z)\,,\qquad \frac{\partial \mathrm{T}_G f}{\partial z}=\Pi_G f\,. \tag{8.16}$$

Thus, if $G\in C^1_\alpha$, $f\in C_\alpha(\bar G)$, then

$$\mathrm{T}_G f\in C^1_\alpha(\bar G)\,,\qquad \Pi_G f=\frac{\partial \mathrm{T}_G f}{\partial z}\in C_\alpha(\bar G)\,. \tag{8.17}$$

We have also the following inequalities

$$C_\alpha(\Pi_G f)\leqslant C^1_\alpha(\mathrm{T}_G f)\leqslant M_\alpha C_\alpha(f)\,. \tag{8.18}$$

This indicates that $\mathrm{T}_G f$ *is a completely continuous operator in* $C_\alpha(\bar G)$ *and maps this space onto* $C^1_\alpha(\bar G)$.

Assume now that $f\in C^1(\bar G)$. Then by means of the formula (7.6) we find

$$\begin{aligned}\Pi_G f&=-\lim_{\varepsilon\to 0}\frac{1}{\pi}\iint_{G_\varepsilon}\frac{f(\zeta)}{(\zeta-z)^2}\,d\xi\,d\eta\\&=\lim_{\varepsilon\to 0}\frac{1}{\pi}\iint_{G_\varepsilon}\frac{\partial}{\partial\zeta}\left(\frac{1}{\zeta-z}\right)f(\zeta)\,d\xi\,d\eta\\&=-\lim_{\varepsilon\to 0}\frac{1}{\pi}\iint_{G_\varepsilon}\frac{\partial f}{\partial\zeta}\,\frac{d\xi\,d\eta}{\zeta-z}-\frac{1}{2\pi i}\int_\Gamma\frac{f(\zeta)\,d\bar\zeta}{\zeta-z}+\\&\qquad\qquad+\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{f(\zeta)\,d\bar\zeta}{\zeta-z}\,.\end{aligned} \tag{8.19}$$

Since

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{f(\zeta)\,d\bar\zeta}{\zeta-z}=0\,,$$

$$\lim_{\varepsilon\to 0}\frac{1}{\pi}\iint_{G_\varepsilon}\frac{\partial f}{\partial\zeta}\,\frac{d\xi\,d\eta}{\zeta-z}=\frac{1}{\pi}\iint_G\frac{\partial f}{\partial\zeta}\,\frac{d\xi\,d\eta}{\zeta-z}\,,$$

we obtain in view of (8.19)

$$\Pi_G f = \mathrm{T}_G\left(\frac{\partial f}{\partial z}\right) - \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)\,d\bar{\zeta}}{\zeta - z}. \tag{8.20}$$

Hence, according to the first formula (8.16),

$$\frac{\partial \Pi_G f}{\partial \bar{z}} = \frac{\partial f(z)}{\partial z}, \quad f \in C^1(\bar{G}). \tag{8.21}$$

Besides, if $f \in C^1_\alpha(\bar{G})$, $0 < \alpha < 1$, from (8.20), in consequence of (8.16), we have

$$\frac{\partial \Pi_G f}{\partial z} = \Pi_G\left(\frac{\partial f}{\partial z}\right) - \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)\,d\bar{\zeta}}{(\zeta - z)^2}. \tag{8.22}$$

The formulae (8.21) and (8.22) indicate that if $G \in C^2_\alpha$, $f \in C^1_\alpha(\bar{G})$, then $\Pi_G f \in C^1_\alpha(\bar{G})$. By continuing a similar reasoning we conclude that if $G \in C^{m+1}_\alpha$, $f \in C^m_\alpha(\bar{G})$, $0 < \alpha < 1$, then $\Pi_G f \in C^m_\alpha(\bar{G})$, $\mathrm{T}_G f \in C^{m+1}_\alpha(\bar{G})$, and

$$C^m_\alpha(\Pi_G f) \leqslant C^{m+1}_\alpha(\mathrm{T}_G f) \leqslant M_{m,\alpha} C^m_\alpha(f, \bar{G}). \tag{8.23}$$

This inequality implies that $\mathrm{T}_G f$ is a completely continuous operator in any $C^m_\alpha(\bar{G})$ when $0 \leqslant \alpha < 1$, $m = 0, 1, \ldots$. Thus, Theorem 1.32 has been completely proved. It is to be borne in mind that by $C^m_0(\bar{G})$ we understand $C^m(\bar{G})$.

COROLLARY OF THEOREM 1.32. *Let $A(z) \in C^m_\alpha(\bar{G})$. Then $\mathrm{P}_G f \equiv \mathrm{T}_G(Af)$ will also be a completely continuous operator in $C^m_\alpha(\bar{G})$, and if $f \in C^m_\alpha(\bar{G})$, then $\mathrm{P}_G f \in C^{m+1}_\alpha(\bar{G})$.*

Indeed, in view of (8.23),

$$C^{m+1}_\alpha(\mathrm{P}_G f) \leqslant M_{m,\alpha} C^m_\alpha(Af, \bar{G}).$$

But, making use of the inequality (1.6(a)) we have

$$C^{m+1}_\alpha(\mathrm{P}_G f) \leqslant M'_{m,\alpha} C^m_\alpha(A)\, C^m_\alpha(f) \leqslant M''_{m,\alpha} C^m_\alpha(f).$$

It follows readily that $\mathrm{P}_G f$ is a completely continuous operator mapping $C^m_\alpha(\bar{G})$ onto $C^{m+1}_\alpha(\bar{G})$.

The relation (8.15) implies also the following

THEOREM 1.33. *If $f \in C_\alpha(\bar{G})$ and $f(z_0) = 0$, $z_0 \in G$, then the function $\mathrm{T}_G f$ is differentiable at the point z_0, i.e. at this point there exists a derivative of the function $\mathrm{T}_G f$ with respect to the complex argument; moreover, we have*

$$\frac{d\mathrm{T}_F f}{dz_0} \equiv \Pi_G f = -\frac{1}{\pi}\iint\limits_G \frac{f(\zeta)\,d\xi\,d\eta}{(\zeta-z_0)^2}. \tag{8.24}$$

8.2. We have seen above that the function $\mathrm{T}_G f$ is continuous on the entire plane if $f \in L_p(\bar{G})$, $p > 2$. The function $\Pi_G f$ in general has a discontinuity on the contour Γ.

Let $\Gamma \in C^2$ and $f \in C_\alpha(\Gamma)$. Then we may regard the function f as continued outside $\bar{G}$, the class being conserved. Therefore the formula (8.20) holds for both $z \in G$ and $z \bar{\in} G+\Gamma$.

Let a function $\psi(z)$ be given both inside G and outside $\bar{G}$. Then, if there exist limits of $\psi(z)$ when z tends to a point t of the contour from inside G or from outside $\bar{G}$, they will be denoted by $\psi^+(t)$ and $\psi^-(t)$, respectively. It is known that for an integral of the Cauchy type (3.1) the following formulae take place ([60a], Ch. I, §17):

$$\begin{aligned} \Phi^+(t) &= \frac{1}{2}f(t) + \frac{1}{2\pi i}\int\limits_\Gamma \frac{f(\zeta)\,d\zeta}{\zeta - t}, \\ \Phi^-(t) &= -\frac{1}{2}f(t) + \frac{1}{2\pi i}\int\limits_\Gamma \frac{f(\zeta)\,d\zeta}{\zeta - t}. \end{aligned} \tag{8.25}$$

Making use of these formulae, we obtain from (8.20) the relation, [11a], [96],

$$(\Pi_G f)^+ - (\Pi_G f)^- = -f(t)\left(\frac{d\bar{t}}{ds}\right)^2, \quad t \in \Gamma, \tag{8.26}$$

representing the jump of the function $\Pi_G f$ on the contour Γ.

8.3. Let us now consider the case in which G covers the entire plane E. This case has certain peculiarities; it is important, therefore, to examine it in more detail.

First, let us observe that Theorem 1.32 still holds if $f(z) \subset L_p C_\alpha^m(E)$, $1 \leqslant p < 2$. We also have

THEOREM 1.34. *If* $f \in L_p C_\alpha^m(E)$, *then* $\mathrm{T}_E f \in L_p C_\alpha^{m+1}(E)$ *and* $\Pi_E f \in C_\alpha^m(E)$, $0 < \alpha < 1$.

PROOF. Let G be the circle $|z| < R$. Then

$$\Phi_\Gamma(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\bar{\zeta}\, d\zeta}{\zeta - z} = 0\,, \qquad |z| < R$$

and, in view of the inequality (8.9(a)), we obtain from (8.9)

$$|g(z_1) - g(z)| \leqslant M_\alpha H(f, \alpha, E)\,|z_1 - z|^\alpha\,, \qquad M_\alpha = 1 + 2M'_\alpha\,. \tag{8.27}$$

Since the right-hand side of this inequality is independent of R it is also true even when $R \to \infty$. Consequently, $\Pi_E f \in H_\alpha(E)$ when $f \in H_\alpha(E)$. Further, since

$$\begin{aligned}\Pi_E f &= -\frac{1}{\pi} \iint_E \frac{f(\zeta)\, d\xi\, d\eta}{(\zeta - z)^2} \\ &= -\frac{1}{\pi} \iint_{|\zeta| \leqslant 1} \frac{f(\zeta + z) - f(z)}{\zeta^2}\, d\xi\, d\eta - \frac{1}{\pi} \iint_{|\zeta| \geqslant 1} \frac{f(\zeta + z)}{\zeta^2}\, d\xi\, d\eta\,,\end{aligned}$$

we have

$$\begin{aligned}|\Pi_E f| &\leqslant \frac{2}{\alpha} H(f, \alpha, E) + \frac{1}{\pi}\left(\frac{2\pi}{q-1}\right)^{1/q} L_p(f, E) \\ &\leqslant M''_{p,\alpha}\big(L_p(f, E) + H(f, \alpha, E)\big)\,.\end{aligned} \tag{8.28}$$

It follows from (8.27) and (8.28) that

$$C_\alpha(\Pi_E f, E) \leqslant M'''_{p,\alpha}[L_p(f, E) + H(f, E)]\,.$$

Thus, $\Pi_E f \in C_\alpha(E)$ when $f \in L_p C_\alpha(E)$, $p \geqslant 1$.

According to (8.14) the following inequality holds inside the circle $|z| \leqslant R$:

$$\left|\frac{h(z_1) - h(z)}{z_1 - z} - \Pi_E f - \frac{\bar{z} - \bar{z}_1}{z - z_1} f(z)\right| \leqslant M_\alpha H(f, \alpha, E)|z - z_1|^\alpha.$$

This inequality is clearly true also for $R \to \infty$. Consequently, the relations (8.16) still hold when $f \in L_p C_\alpha(E)$, $p \geqslant 1$ i.e. $\mathrm{T}_E f \in C_\alpha^1(E)$ if $f \in L_p C_\alpha(E)$.

This completes the proof of Theorem 1.34 in the case $m = 0$. In an analogous way the theorem may be proved for an arbitrary $m > 0$.

§9. Extension of the operator Πf

In this section we shall show that the operator Πf can be extended to a linear (bounded) operator in the spaces L_p, $p > 1$.

9.1. We first prove three lemmas.

LEMMA 1. *If f and $g \in D_\infty^0(E)$, then*

$$(\Pi f, g) = (f, \overline{\Pi} g)\,, \qquad (f, g) = \iint\limits_E f(z)\overline{g(z)}\,dx\,dy\,, \quad (9.1)$$

where

$$\Pi f \equiv -\frac{1}{\pi}\iint\limits_E \frac{f(\zeta)\,d\xi\,d\eta}{(\zeta - z)^2}\,, \qquad \overline{\Pi} f \equiv \frac{1}{\pi}\iint\limits_E \frac{f(\zeta)\,d\xi\,d\eta}{(\bar{\zeta} - \bar{z})^2}\,. \quad (9.2)$$

PROOF. If $f \in D_\infty^0(E)$, then, according to the formulae (8.20) and (8.21),

$$\Pi f = \frac{\partial \mathrm{T} f}{\partial z} = \mathrm{T}\left(\frac{\partial f}{\partial z}\right), \qquad \frac{\partial \Pi f}{\partial \bar{z}} = \frac{\partial f}{\partial z}\,. \quad (9.3)$$

Integrating by parts and making use of the formulae (7.5) and (7.6) we obtain

$$\begin{aligned}(\Pi f, g) &= \lim_{R\to\infty}\iint\limits_{|z|\leqslant R} \Pi f \bar{g}\,dx\,dy = \lim_{R\to\infty}\iint\limits_{|z|\leqslant R} \frac{\partial \mathrm{T} f}{\partial z}\bar{g}\,dx\,dy \\ &= -\lim_{R\to\infty}\iint\limits_{|z|\leqslant R} \mathrm{T} f\frac{\partial \bar{g}}{\partial z}\,dx\,dy = \lim_{R\to\infty}\iint\limits_{|z|\leqslant R} f\mathrm{T}\left(\frac{\partial \bar{g}}{\partial z}\right)dx\,dy \\ &= \lim_{R\to\infty}\iint\limits_{|z|\leqslant R} f\Pi\bar{g}\,dx\,dy = (f, \overline{\Pi} g)\,.\end{aligned}$$

In deriving the above formula we have omitted the curvilinear integral over the circumference $|z| = R$ since it vanishes as a result of the vanishing of g for sufficiently large R.

The formula (9.1) indicates that Π and $\overline{\Pi}$ are adjont operators in the linear manifold $D^0_\infty(E)$.

LEMMA 2. *If* $f \in D^0_\infty(E)$, *then*

$$\overline{\Pi}\Pi f = f . \tag{9.4}$$

PROOF. In view of the property (9.3) of the operator Πf we have

$$\overline{\Pi}\Pi f = \partial_{\bar{z}}\overline{\mathrm{T}}(\partial_z \mathrm{T} f) . \tag{9.5}$$

By virtue of the formula (4.10), inside the circle $|z| < R$

$$\mathrm{T}f = -\frac{1}{2\pi i}\iint_{|\zeta|=R}\frac{\mathrm{T}f\,d\bar{\zeta}}{\bar{\zeta}-\bar{z}} - \frac{1}{\pi}\iint_{|\zeta|\leqslant R}\frac{\partial \mathrm{T}f}{\partial\zeta}\frac{d\xi\,d\eta}{\bar{\zeta}-\bar{z}} .$$

Since $\mathrm{T}f = 0(|z|^{-1})$, then

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{|\zeta|=R}\frac{\mathrm{T}f\,d\bar{\zeta}}{\bar{\zeta}-\bar{z}} = 0 .$$

Consequently,

$$\mathrm{T}f = -\frac{1}{\pi}\iint_E \frac{\partial \mathrm{T}f}{\partial\zeta}\frac{d\xi\,d\eta}{\bar{\zeta}-\bar{z}} = \overline{\mathrm{T}}(\partial_z \mathrm{T}f) .$$

Differentiating both sides of this relation with respect to $\bar{z}$, according to the relation (9.5) we obtain

$$f = \partial_{\bar{z}}\overline{\mathrm{T}}(\partial_z \mathrm{T}f) \equiv \overline{\Pi}\Pi f .$$

The relation (9.4) proves that in the linear manifold $D^0_\infty(E)$ the operator Π has an inverse Π^{-1} which coincides with its adjont $\overline{\Pi}$.

LEMMA 3. *If* f *and* $h \in D^0_\infty(E)$, *then*

$$(\Pi f, \Pi h) = (f, h) . \tag{9.6}$$

PROOF. If we set $g = \Pi h$ in the formula (9.1) and take into account (9.4) we obtain (9.6). *

If $f = h$ the formula (9.6) assumes the form

$$(\Pi f, \Pi f) = (f, f), \quad \text{i.e.} \quad L_2(\Pi f, E) = L_2(f, E). \quad (9.7)$$

As a consequence of (9.6) the operator Π does not affect the invariancy of the scalar product of elements of the linear manifold $D^0_\infty(E)$, and according to (9.1) and (9.4) the adjont operator $\overline{\Pi}$ coincides with the inverse operator Π^{-1}.

Thus, in the linear manifold $D^0_\infty(E)$ the operator Π possesses properties of unitary operator. Since the linear manifold $D^0_\infty(E)$ is dense in $L_2(E)$, according to a known theorem of functional analysis, *operators Π and $\overline{\Pi}$ can be uniquely continued to mutually adjoint linear unitary operators of the space $L_2(E)$.*

9.2. Let us consider the operator

$$\Pi_* g \equiv \partial_{\bar{z}} \mathrm{T}_* g, \quad \text{where} \quad \mathrm{T}_* g = \frac{1}{\pi i} \iint_E \frac{g(\zeta)\, dE_\zeta}{|\zeta - z|}. \quad (9.8)$$

Applying the operator Π_* to finite functions g belonging to the linear manofold $D^0_\infty(E)$ we obtain infinitely many times differentiable functions behaving at infinity like $|z|^{-2}$. Therefore we may consider combinations of the form $\Pi_* \Pi^* g, \Pi^* \Pi_* g$ where

$$\Pi^* g = \partial_z \mathrm{T}_* g, \quad (9.9)$$

If $g \in D^0_m(E), m \geqslant 1$, we can show that

$$\partial_{\bar{z}} \mathrm{T}_* g = \mathrm{T}_*(\partial_{\bar{z}} g), \quad \partial_z \mathrm{T}_* g = \mathrm{T}_*(\partial_z g). \quad (9.10)$$

Bearing in mind that

$$\mathrm{T}_* g = \frac{1}{\pi i} \iint_E \frac{g(\zeta + z)}{|\zeta|} d\xi\, d\eta,$$

* We should not be disturbed by the fact that generally $\Pi f \bar{\in} D^0_\infty(E)$. It is readily observed that the formula (9.1) is also valid in the case of $g = \Pi h, h \in D^0_\infty(E)$, since near infinity $g = 0(|z|^{-2})$, $\mathrm{T} f = 0(|z|^{-1})$ and the above proof is still valid.

the relations (9.10) are established by a differentiation of the integrand.

By virtue of the formulae (9.10) we have

$$-\pi^2\Pi_*\Pi^*g = \partial_{\bar z}\iint_E \frac{dE_\zeta}{|\zeta-z|}\iint_E \frac{\partial_t g(t)}{|t-\zeta|}dE_t$$

$$= \partial_{\bar z}\iint_E \frac{dE_\zeta}{|\zeta|}\iint_E \frac{\partial_t g(t)}{|t-\zeta-z|}dE_t$$

$$= \partial_{\bar z}\left\{\lim_{R\to\infty}\iint_E \partial_t g(t)\,dE_t \iint_{|\zeta|\leqslant R}\frac{dE\zeta}{|\zeta|\,|t-\zeta-z|}\right\}.$$

By means of integration by parts we obtain

$$\iint_E \partial_t g(t)\,dE_t \iint_{|\zeta|\leqslant R}\frac{dE_\zeta}{|\zeta|\,|t-\zeta-z|}$$

$$= -\iint_E g(t)\,dE_t\frac{\partial}{\partial t}\iint_{|\zeta|\leqslant R}\frac{dE_\zeta}{|\zeta|\,|t-\zeta-z|}.$$

Consequently,

$$\Pi_*\Pi^*g = \partial_{\bar z}^{\frac{1}{z}}\iint_E K(z,\zeta)g(\zeta)\,dE_\zeta,$$

where

$$K(z,\zeta) = \lim_{R\to\infty}\frac{1}{\pi^2}\frac{\partial}{\partial\zeta}\iint_{|t|\leqslant R}\frac{dE_t}{|t|\,|t+(z-\zeta)|}.$$

In computing the last integral we may assume that the points z and ζ lie on the real axis. Performing a change of the variables of integration according to the formula $t = |z-\zeta|\varrho e^{i\vartheta}$, $(z\neq\zeta)$, we obtain

$$K(z,\zeta) = \frac{1}{\pi^2}\lim_{R\to\infty}\frac{\partial}{\partial\zeta}\int_0^{\frac{R}{|z-\zeta|}} d\varrho\int_0^{2\pi}\frac{d\varphi}{|1-\varrho e^{i\varphi}|}$$

$$= -\frac{1}{\pi^2}\frac{\partial\ln|z-\zeta|}{\partial\zeta}\lim_{R\to\infty}\int_0^{2\pi}\frac{d\varphi}{1-\frac{|z-\zeta|}{R}e^{-i\varphi}} = -\frac{1}{\pi(\zeta-z)}.$$

Consequently,

$$\Pi_* \Pi^* g = \partial_{\bar{z}}\left(-\frac{1}{\pi}\iint\limits_E \frac{g(\zeta)}{\zeta - z}\,dE_\zeta\right) = g(z)\,. \qquad (9.11)$$

Replacing in this relation g by $\bar{g}$ and passing then to the complex conjugate relation we obtain $\Pi^*\Pi_* g = g$. It is also easy to establish the formulae

$$(\Pi_* f, g) = (f, \Pi^* g)\,, \qquad (\Pi_* f, \Pi_* g) = (f, g)\,. \quad (9.12)$$

By means of the relations (9.11) and (9.12), which are true for arbitrary finite functions, we infer that the operators Π_* and Π^* may be uniquely continued to mutually adjont linear unitary operators of the Hilbert space $L_2(E)$.

We shall now prove the following relation in the linear manifold $D^0_\infty(E)$:

$$L_p(\Pi_* g, E) \leqslant \Lambda^*_p L_p(g, E)\,, \qquad p > 1\,, \qquad (9.13)$$

where Λ^*_p is a positive constant depending on p but independent of g.

We have

$$\begin{aligned}\Pi_* g &= \frac{1}{\pi i}\iint\limits_E \frac{\partial_{\bar{\zeta}} g(\zeta + z)}{|\zeta|}\,dE_\zeta \\ &= \lim_{\varepsilon\to 0}\left(\frac{1}{\pi i}\iint\limits_{|\zeta|\geqslant\varepsilon}\left[\frac{\partial}{\partial\bar{\zeta}}\frac{g(\zeta+z)}{|\zeta|} + \frac{\zeta g(\zeta+z)}{2|\zeta|^3}\right]dE_\zeta\right) \\ &= -\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int\limits_{|\zeta|=\varepsilon}\frac{g(\zeta+z)}{|\zeta|}\,d\zeta + \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\iint\limits_{|\zeta|\geqslant\varepsilon}\frac{\zeta g(\zeta+z)}{|\zeta|^3}\,dE\zeta\,.\end{aligned}$$

But

$$\int\limits_{|\zeta|=\varepsilon}\frac{g(\zeta+z)}{|\zeta|}\,d\zeta \underset{\varepsilon\to 0}{\to} i\int\limits_0^{2\pi} g(z)e^{i\varphi}d\varphi = 0\,.$$

Changing now the variables of integration in the second integral according to the formula $\zeta = \varrho e^{i\varphi}$ and passing to the limit, $\varepsilon\to 0$, we obtain

$$\Pi_* g = \frac{1}{2\pi i}\int\limits_0^\infty \frac{d\varrho}{\varrho}\int\limits_0^{2\pi} e^{i\varphi} g(\varrho e^{i\varphi} + z)\,d\varphi\,. \qquad (9.14)$$

Taking into account that

$$\int_0^{2\pi} e^{i\varphi} g(\varrho e^{i\varphi}+z)\,d\varphi = \int_0^{\pi} e^{i\varphi}[g(\varrho e^{i\varphi}+z)-g(-\varrho e^{i\varphi}+z)]\,d\varphi\,,$$

the relation (9.14) may be written in the form

$$\Pi_* g = \frac{1}{2}\int_0^{\pi} \tilde{g}(z, e^{i\varphi})\,d\varphi\,, \tag{9.15}$$

where

$$\tilde{g}(z, e^{i\varphi}) = \frac{e^{i\varphi}}{\pi i}\int_{-\infty}^{+\infty} \frac{g(\varrho e^{i\varphi}+z)}{\varrho}\,d\varrho\,.$$

Replacing z by $(\tau+i\sigma)e^{i\varphi}$ we obtain

$$\tilde{g}[(\tau+i\sigma)e^{i\varphi}, e^{i\varphi}] = \frac{e^{i\varphi}}{\pi i}\int_{-\infty}^{+\infty} \frac{g[(\varrho+i\sigma)e^{i\varphi}]}{\varrho-\tau}\,d\varrho\,. \tag{9.16}$$

We recall that for the transformations of the form

$$\psi(x) = \frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{\chi(t)}{t-x}\,dt$$

the following inequality, due to Riesz [77], holds

$$\int_{-\infty}^{+\infty} |\psi(x)|^p\,dx \leqslant \tilde{\Lambda}_p^p \int_{-\infty}^{+\infty} |\chi(t)|^p\,dt\,, \quad p>1\,,$$

where $\tilde{\Lambda}_p$ is a positive constant depending only on p. Making use of this inequality we obtain from (9.16)

$$\int_{-\infty}^{+\infty} |\tilde{g}[(\tau+i\sigma)e^{i\varphi}, e^{i\varphi}]|^p\,d\tau \leqslant \tilde{\Lambda}_p^p \int_{-\infty}^{+\infty} |g[(\tau+i\sigma)e^{i\varphi}]|^p\,d\tau\,.$$

Integrating this inequality again with respect to σ from $-\infty$ to $+\infty$ and performing a change of the variables of integration according to the formula $\zeta=(\tau+i\sigma)e^{i\varphi}$ we obtain

$$\iint_E |\tilde{g}(\zeta, e^{i\varphi})|^p\,dE_\zeta \leqslant \tilde{\Lambda}_p^p \iint_E |g(\zeta)|^p\,dE_\zeta\,. \tag{9.17}$$

From the formula (9.15) with the help of the Hölder inequality we have

$$|\Pi_* g|^p \leqslant \frac{\pi^{p/q}}{2^p} \int_0^\pi |\tilde{g}(z, e^{i\varphi})|^p d\varphi .$$

Hence, in view of the inequality (9.17), we arrive at the inequality

$$\iint_E |\Pi_* g|^p E_z \leqslant \frac{\pi^p}{2^p} \tilde{\Lambda}_p^p \iint_E |g(z)|^p dE_z ,$$

implying immediately (9.13).

By means of the inequality (9.13) we can assure an extension of the operator Π_* to a linear operator acting from $L_p(E)$ into $L_p(E)$ for an arbitrary $p > 1$.

Let g be an element of $L_p(E)$, $p > 1$. Let g_n be a sequence of elements of $D^0_\infty(E)$ which converges in the mean to g, i.e. $L_p(g-g_n) \to 0$. Then, in view of the inequality (9.13), the sequence $\Pi_* g_n$ converges in the mean to an element of $L_p(E)$ which will be denoted by $\Pi_* g$, i.e. $L_p(\Pi_* g - -\Pi_* g_n) \to 0$ if $n \to \infty$. If two sequences g_n and g'_n of the linear manifold $D^0_\infty(E)$ converge to g, then Πg_n and $\Pi g'_n$ also converge to the same limit. It is evident that the operator $\Pi_* g$ is additive and homogeneous. Moreover, in the case of this operator the inequality (9.13) holds. Consequently, $\Pi_* g$ is a linear bounded operator in an arbitrary $L_p(E)$, $p > 1$.

Let us now consider the problem of extension of the operator Π. We have in the linear manifold $D^0_\infty(E)$ in view of (9.11), (9.8) and (9.9),

$$\begin{aligned} \Pi g &= \Pi(\Pi_* \Pi^* g) = \partial_z \mathrm{T}(\partial_{\bar{z}} \mathrm{T}_* \Pi^* g) \\ &= \partial_z \mathrm{T}_* \Pi^* g = \Pi^{*2} g . \end{aligned} \tag{9.18}$$

The formula

$$\mathrm{T}(\partial_{\bar{z}} \mathrm{T}_* \Pi^* g) = \mathrm{T}_* \Pi^* g ,$$

has been employed above; its validity follows easily from Theorem 1.16 if we take into account that $\mathrm{T}_* \Pi^* g$ vanishes for $z = \infty$.

Evidently, the formula (9.18) allows us to consider Π as a linear operator in an arbitrary $L_p(E)$, $p > 1$.

Let us now denote by Λ_p the norm of the operator Π in $L_p(E)$; $\Lambda_p = L_p(\Pi)$, $p > 1$. We have seen before that Π is a unitary operator in $L_2(E)$. Consequently, $\Lambda_2 = 1$. But, according to an important theorem of Riesz [77], (see also [35], Ch. IX), Λ_p^p is a logarithmically convex function of p. Therefore a number $\delta(\varepsilon) > 0$ may be found for an arbitrary $\varepsilon > 0$, such that

$$\Lambda_p - 1 < \varepsilon, \quad \text{if} \quad |p-2| < \delta(\varepsilon). \tag{9.19}$$

This property of the norm Λ_p of the operator Π will frequently be used below.

REMARK. The reasoning carried out above is basically adopted from the papers of Zygmund and Calderon [36a, b]; they examined properties of more general many-dimensional singular integrals (see also [56b]).

In the foregoing, the following formula was laid down as the basis of the definition of the operator Π:

$$\Pi f = \partial_z \mathrm{T} f. \tag{9.20}$$

It is so far rigorously justified only for functions continuous in the Hölder sense. Naturally, the question arises: is it preserved in more general spaces $L_p(E)$, $p > 1$? This problem is solved by

THEOREM 1.35. *If $f \in L_p(E)$, $p > 1$, then $\mathrm{T}f$ has a generalized derivative with respect to z equal to Πf, i.e. in this case the formula* (9.20) *holds which may be written thus:*

$$\frac{\partial \mathrm{T} f}{\partial x} = f + \Pi f, \quad \frac{\partial \mathrm{T} f}{\partial y} = -if + i\Pi f. \tag{9.21}$$

PROOF. It is to be proved that

$$I = \iint_G (\mathrm{T} f \partial_z \varphi + \Pi f \cdot \varphi)\, dx\, dy = 0, \tag{9.22}$$

where φ is an arbitrary element of $D_1^0(G)$.

Let f_n be a sequence of elements of the linear manifold $D^0_\infty(G)$, which converges in the mean to f, i.e. $L_p(f-f_n)\to 0$ when $n\to\infty$. Since

$$I_n = \iint_G (\mathrm{T}f_n \partial_z \varphi + \varphi \Pi f_n)\,dx\,dy = 0 \qquad (n = 1, 2, \ldots)\,, \quad (9.23)$$

taking into account that the sequences $\mathrm{T}f_n$ and Πf_n converge in the mean to $\mathrm{T}f$ and Πf, respectively, we obtain from (9.23) by the limiting process (9.22).

Theorems 1.16 and 1.35 imply the following:

THEOREM 1.36. *If $\partial_{\bar z} f \in L_p(G)$, $p > 1$, then $\partial_z f$ exists and also belongs to $L_p(G)$.*

PROOF. Since $f = \Phi + \mathrm{T}f_{\bar z}$, $\Phi \in \mathfrak{A}_0(G_1)$, then $f_z = \Phi' + \Pi f_{\bar z} \in L_p(G_1)$ where G_1 is an arbitrary subdomain of G.

We have seen in the preceding paragraph that the operator Πf in the space C_α may be represented by an integral of the form

$$\Pi f = -\frac{1}{\pi}\iint_E \frac{f(\zeta)}{(\zeta - z)^2}\,d\xi\,d\eta\,, \qquad (9.24)$$

the integral being understood as the Cauchy principal value. Naturally, the question arises: is this formula preserved in the case of the space $L_p(E)$, $p > 1$. Omitting details of the proof we note that on the basis of the general results of Zygmund and Calderon, [36a, b], the formula (9.24) can be justified, namely: *if $f \in L_p(E)$, $p > 1$, then the right-hand side of the relation* (9.24) *exists in the sense of the principal value almost everywhere and it is equal to Πf.*

§10. Some other properties of functions of the classes $D_z(G)$ and $D_{\bar z}(G)$

In this section, on the basis of the previous results, we shall prove a number of properties of functions belonging to the classes $D_z(G)$ and $D_{\bar z}(G)$. Evidently, it is sufficient to consider the class $D_{\bar z}$ only.

THEOREM 1.37. *Let* $f \in D_{1,p}(G)$, $1 < p < 2$, $g \in D_{1,p'}(G)$, $p' = \dfrac{2p}{3p-2}$. *Then the product* $fg \in D_{\bar{z}}(G)$ *and*

$$\partial_{\bar{z}}(fg) = f\partial_{\bar{z}}g + g\partial_{\bar{z}}f \,. \tag{10.1}$$

PROOF. Let G_1 be a subdomain of the domain G, $G_1 \subset G$. Then, according to the formula (5.12)

$$f(z) = \Phi(z) + \mathrm{T}f_1 \,, \qquad g(z) = \Psi(z) + \mathrm{T}g_1$$

$$\big(f_1 = \partial_{\bar{z}}f \,, \quad g_1 = \partial_{\bar{z}}g \,, \quad \mathrm{T}f \equiv \mathrm{T}_{G_1}f \,, \quad \Phi \,,\ \Psi \in \mathfrak{A}_0(G_1)\big) \,.$$

Consequently,

$$fg = h + \mathrm{T}f_1\mathrm{T}g_1 \,, \qquad h = \Phi\Psi + \Phi\mathrm{T}g_1 + \Psi\mathrm{T}f_1 \,.$$

We note that the formula (10.1) holds if at least one of the functions f and g belongs to $C^1(G)$. Now we have

$$\partial_{\bar{z}}h = \Phi\partial_{\bar{z}}\mathrm{T}g_1 + \Psi\partial_{\bar{z}}\mathrm{T}g_1 \equiv \Phi g_1 + \Psi f_1 \,.$$

It remains, therefore, to prove the formula (10.1) only for the product $\mathrm{T}f_1\mathrm{T}g_1$, i.e. we have to show that if $\varphi \in D_1^0(G_1)$, then

$$\iint\limits_G [(\mathrm{T}f_1\mathrm{T}g_1)\partial_{\bar{z}}\varphi + (f_1\mathrm{T}g_1 + g_1\mathrm{T}f_1)\varphi]\,dx\,dy = 0 \,. \tag{10.2}$$

Let f_n be a sequence of functions of the linear manifold $D_\infty(G_1)$, which is convergent in the mean of the order p to f_1. Evidently, we have

$$\iint\limits_G [(\mathrm{T}f_n\mathrm{T}g_1)\partial_{\bar{z}}\varphi + (f_n\mathrm{T}g_1 + g_1\mathrm{T}f_n)\varphi]\,dx\,dy = 0 \,. \tag{10.3}$$

Since $f_1 \in L_p(G)$, $\mathrm{T}g_1 \in L_{\frac{p}{p-1}}(G)$ and $g_1 \in L_{p'}(G)$, $\mathrm{T}f_1 \in L_{\frac{p'}{p'-1}}(G)$, then after passing to the limit (10.2) is obtained from (10.3). It should be taken into account that f_n and $\mathrm{T}f_n$ are convergent in the mean (and, consequently, also weakly) to f_1 and $\mathrm{T}f_1$, respectively.

10.2. THEOREM 1.38. *Let* $\partial_{\bar{z}}f \in L_2(G)$. *Let the function* $z = w(\zeta)$ *establish a one to one continuous mapping of a domain* G' *of the* ζ*-plane onto a domain* G *of the* z*-plane.*

If $w(\zeta) \in C^1(G')$ *and the Jacobian of the transformation does not vanish inside the domain, then the implicit function* $f(w(\zeta)) \in D_{\bar\zeta}(G'), D_\zeta(G')$, *and*

$$\partial_{\bar\zeta} f(w(\zeta)) = \partial_z f(z)\partial_{\bar\zeta} w + \partial_{\bar z} f(z)\partial_{\bar\zeta}\overline{w}\,, \tag{10.4}$$

$$\partial_\zeta f(w(\zeta)) = \partial_z f(z)\partial_\zeta w + \partial_{\bar z} f(z)\partial_\zeta\overline{w}\,. \tag{10.5}$$

PROOF. It is sufficient to prove the first of the above formulae; the second will then follow in a similar way. We first observe that, according to Theorem 1.36, the condition $\partial_{\bar z} f \in L_2(G)$ implies that $\partial_z f \in L_2(G)$. Since inside any subdomain G_1 of the domain G the function f can be represented in the form

$$f = \Phi + \mathrm{T}g\,, \qquad \text{where} \qquad \Phi \in \mathfrak{A}_0(G_1)\,, \qquad g = \partial_{\bar z} f\,,$$

it is sufficient to prove the formula (10.4) only for a function of the form $\mathrm{T}g$, where $g \in L_2(G)$. In doing so it is to be borne in mind that the formula (10.4) is true if $f \in C^1(G)$. Let g_n be a sequence of elements of $D^0_\infty(G)$, which converges in the mean to g. Then the sequence $f_n(w(\zeta))$, where $f_n(z) = \mathrm{T}g_n$, will converge in the mean to $f(w(\zeta))$ over the domain G' of the plane ζ.

Therefore, if $\varphi \in D^0_1(G')$, then

$$\begin{aligned}
&\iint_{G'} f(w(\zeta))\partial_{\bar\zeta}\varphi(\zeta)\,d\xi\,d\eta \\
&\qquad = \lim_{n\to\infty} \iint_{G'} f_n(w(\zeta))\partial_{\bar\zeta}\varphi(\zeta)\,d\xi\,d\eta \\
&\qquad = -\lim_{n\to\infty} \iint_{G'} \varphi(\zeta)\partial_{\bar\zeta} f_n(w(\zeta))\,d\xi\,d\eta \\
&\qquad = -\lim_{n\to\infty} \iint_{G'} \varphi(\zeta)\left(\partial_z f_n(z)\partial_{\bar\zeta} w + \partial_{\bar z} f_n(z)\partial_{\bar\zeta}\overline{w}\right)d\xi\,d\eta \\
&\qquad = -\lim_{n\to\infty} \iint_{G'} \varphi(\zeta)[\Pi g_n\partial_{\bar\zeta} w + g_n\partial_{\bar\zeta}\overline{w}]\,d\xi\,d\eta \\
&\qquad = -\iint_{G'} \varphi(\zeta)[\Pi_{G_1} g\partial_{\bar\zeta} w + g\partial_{\bar\zeta}\overline{w}]\,d\xi\,d\eta \\
&\qquad = -\iint_{G'} \varphi(\zeta)[\partial_z \mathrm{T}g\partial_{\bar\zeta} w + \partial_{\bar z}\mathrm{T}g\partial_{\bar\zeta}\overline{w}]\,d\xi\,d\eta\,,
\end{aligned}$$

which completes the proof of the formula (10.4).

THEOREM 1.39. *Let the values of the function* $z_* = f(z)$ *of the class* $D_{\bar z}(G)$ *belong to a bounded domain* G_*, *and* $g = \partial_{\bar z} f \in L_p(G)$, $p > 2$.

Let the function $\Phi(z_*)$ *be holomorphic inside* G_*^0, *and* $\bar G_* \subset G_*^0$. *Then the implicit function* $\Phi[f(z)] = f_*(z)$ *belongs to* $D_{\bar z}(G)$ *and*

$$\partial_{\bar z} f_*(z) = \Phi'(f(z))\partial_{\bar z} f(z)\,, \tag{10.6}$$

$$\partial_z f_*(z) = \Phi'(f(z))\partial_z f(z)\,. \tag{10.7}$$

PROOF. Inside the subdomain G_1 of the domain G we have

$$f(z) = \Phi(z) + \mathrm{T}_{G_1} g\,, \qquad \Phi \in \mathfrak{A}_0(G_1)\,, \qquad g = \partial_{\bar z} f\,.$$

Let g_n be a sequence of elements of $D_\infty(G_1)$ which converges in the mean of the order p to g. Then the sequence $f_n = \Phi + \mathrm{T}_{G_1} g_n$ converges uniformly to f in $\bar G_1$. Therefore, the sequences $\Phi(f_n(z))$ and $\Phi'(f_n(z))$ will converge uniformly inside G_1 to $\Phi(f(z))$ and $\Phi'(f(z))$, respectively. Therefore, if $\varphi \in D_1^0(G_1)$, then

$$\begin{aligned}
&\iint_{G_1} \Phi(f(z))\partial_{\bar z}\varphi\,dx\,dy \\
&\qquad = \lim_{n\to\infty}\iint_{G_1} \Phi(f_n(z))\partial_{\bar z}\varphi\,dx\,dy \\
&\qquad = -\lim_{n\to\infty}\iint_{G_1} \varphi(z)\Phi'(f_n(z))\partial_{\bar z} f_n(z)\,dx\,dy \\
&\qquad = -\lim_{n\to\infty}\iint_{G_1} \varphi(z)\Phi'(f_n(z))\,g_n(z)\,dx\,dy \\
&\qquad = -\iint_{G_1} \varphi\Phi'(f(z))\,g(z)\,dx\,dy\,,
\end{aligned}$$

which completes the proof of the formula (10.6). The formula (10.7) can be proved in a similar way.

CHAPTER II

REDUCTION OF A POSITIVE DIFFERENTIAL QUADRATIC FORM TO THE CANONICAL FORM. BELTRAMI'S EQUATION. GEOMETRIC APPLICATIONS

§1. Introductory remarks. Homeomorphisms of a quadratic form

In this chapter our attention is focussed mainly on the investigation of the problem of the reduction of the quadratic form

$$F \equiv a(x, y)\,dx^2 + 2b(x, y)\,dx\,dy + c(x, y)\,dy^2\,, \qquad \Delta \equiv ac - b^2 > 0 \tag{1.1}$$

to the canonical form

$$F \equiv \Lambda(du^2 + dv^2)\,, \qquad \Lambda \neq 0\,. \tag{1.2}$$

This problem consists in proving the existence of univalent solutions (homeomorphisms) of Beltrami's system of equations

$$\left.\begin{aligned} \sqrt{\Delta}\frac{\partial u}{\partial x} - a\frac{\partial v}{\partial y} + b\frac{\partial v}{\partial x} &= 0\,, \\ \sqrt{\Delta}\frac{\partial u}{\partial y} - b\frac{\partial v}{\partial y} + c\frac{\partial v}{\partial x} &= 0\,. \end{aligned}\right\} \tag{1.3}$$

Many problems of analysis and geometry may be reduced to the above problem (for instance, the problem of the conformal mapping of a surface on the plane, reduction of elliptic differential equations to canonical form in a two-dimensional domain, etc.).

At present this problem has been solved under very general assumptions in respect of the coefficients of the

quadratic form. The complete solution was probably first obtained by Lichtenstein [48], who basically employed for this purpose Koebe's results on the theory of uniformization; Lichtenstein assumed that the coefficients were continuous in the Hölder sense. In a more general case, when the coefficients are simply continuous, the problem was solved in a different way by Lavrentyev, [45a, b]. Further generalizations may be found in the papers [46a], [59], [18], [93].

In this chapter a method indicated previously by the author, [14c], is presented, which enables us to obtain a full solution of the problem under very wide assumptions about the coefficients of the quadratic form (1.1). A similar method of proof was proposed independently somewhat later by Ahlfors, [3a] (see also [92], [5d]). Further development and applications of the method to the problems of quasi-conformal mappings can be found in papers of Bojarski [11b, c, d, e].

We shall assume henceforth that the following conditions are always satisfied:

(1) $a(x, y)$, $b(x, y)$, $c(x, y)$ are measurable bounded functions on the entire plane E;

(2) $\Delta = ac - b^2 \geqslant \Delta_0 > 0$, $a > 0$ almost everywhere on E ($\Delta_0 = \text{const}$).

It follows from the condition (2) that

$$\frac{(a-\sqrt{\Delta})^2+b^2}{(a+\sqrt{\Delta})^2+b^2} \leqslant q_0 < 1\,, \qquad q_0 = \text{const}\,. \tag{1.4}$$

We note that the condition (1.4) is always satisfied if in the vicinity of the point at infinity $a = c \neq 0$, $b = 0$. Then the quadratic form (1.1) has in the vicinity of the point $z = \infty$ the canonical form (1.2). If a, b, c are given in a bounded closed domain $\bar{G}$ in which the conditions (1) and (2) are satisfied, they can always be continued to the entire plane preserving the above conditions. To this end it is for instance sufficient to set outside G $a = c \neq 0$, $b = 0$.

The main object of the present chapter is to prove the existence of a transformation of the independent variables

$$u = u(x, y), \quad v = v(x, y), \tag{1.5}$$

which would imply, first, the setting up of a one-to-one continuous (homeomorphic) mapping of a given domain G_z of the plane $z = x+iy$ onto a domain G_w of the plane $w = u+iv$, and, secondly, the reduction of the quadratic form (1.1) to the canonical form. If such a transformation exists the complex function $w = u+iv$ will be called the *global homeomorphism* of the quadratic form (1.1). If the domains G_z and G_w under consideration cover the entire z- and w-planes, respectively, then the corresponding global homeomorphism will be termed the *complete homeomorphism* of the quadratic form.

It is also useful to introduce the concept of the *local homeomorphism*. The function $w(z)$ is said to be the local homeomorphism of the quadratic form (1.1) if it establishes homeomorphic mappings of a neighbourhood of the point z_0 onto a neighbourhood of the point $w_0 = w(z_0)$, and the form (1.1) assumes the form (1.2).

Below we shall, under certain specified conditions, prove theorems on the existence of the homeomorphisms of the quadratic form (1.1). We shall also examine differential properties of the homeomorphism depending on the differential properties of the coefficients of the form (1.1).

§2. Beltrami's system of equations

It is evident that

$$aF \equiv \left(a\,dx + \left(b + i\sqrt{\Delta}\right)dy\right)\left(a\,dx + \left(b - i\sqrt{\Delta}\right)dy\right).$$

If the functions $\mu = \mu(z)$ and $w = u+iv$ are found which satisfy the relation

$$\mu\,dw = a\,dx + \left(b + i\sqrt{\Delta}\right)dy, \quad \Delta = ac - b^2 > 0, \tag{2,1}$$

then we have

$$F \equiv \frac{\mu\bar{\mu}}{a} dw\,d\bar{w} \equiv \Lambda(du^2+dv^2)\,, \qquad \Lambda = \frac{\mu\bar{\mu}}{a}\,. \tag{2.2}$$

It is seen from (2.1) that w satisfies the following (complex) equation:

$$a\frac{\partial w}{\partial y} - \left(b + i\sqrt{\Delta}\right)\frac{\partial w}{\partial x} = 0\,, \tag{2.3}$$

or

$$\partial_{\bar{z}} w - q(z)\partial_z w = 0\,, \tag{2.4}$$

where

$$q(z) = \frac{a - \sqrt{\Delta} + ib}{a + \sqrt{\Delta} - ib} \equiv \frac{a - c + 2ib}{a + c + 2\sqrt{\Delta}}\,. \tag{2.4a}$$

The equation (2.4) is equivalent to the following system of two real equations

$$\sqrt{\Delta}\frac{\partial u}{\partial x} - a\frac{\partial v}{\partial y} + b\frac{\partial v}{\partial x} = 0\,, \qquad \sqrt{\Delta}\frac{\partial u}{\partial y} - b\frac{\partial v}{\partial y} + c\frac{\partial v}{\partial x} = 0\,. \tag{2.5}$$

This system is named *Beltrami's system of differential equations*. Evidently, it constitutes a generalization of the Cauchy–Riemann system. As we shall see later (§4.4) a close connection exists between these two systems.

Thus, the problem of determining the homeomorphisms of the quadratic form (1.1) is equivalent to proving the existence of univalent solutions of Beltrami's system of equations (2.2). These solutions will also be named *the homeomorphisms of this system.*

§3. Construction of the basic homeomorphism of Beltrami's equation

The function $w(z)$ is said to be a generalized solution of the equation (2.4) in a domain G if there exist generalized derivatives $\partial_z w$ and $\partial_{\bar{z}} w$ of the class $L_p(G)$, $p \geqslant 1$, satisfying the equation (2.4) almost everywhere in G. First, we shall

prove the existence of generalized solutions of the class $D_{1,p}$, $p > 2$. *

We decided above to always assume that $q(z)$ is a measurable and bounded function on the entire plane and satisfies the condition

$$|q(z)| \leqslant q_0 < 1 . \tag{3.1}$$

Moreover, let us assume that $q(z)$ belongs to a $L_{p'}(E)$, $p' < 2$. Then the inequality $|q(z)|^p \leqslant |q(z)|^{p'}$, $p \geqslant p'$, implies that q belongs to any class $L_p(E)$, $p \geqslant p'$. Later on we shall abandon this assumption imposing certain restrictions on the behaviour of the function $q(z)$ near infinity. We shall now prove that under the indicated conditions Beltrami's equation (2.4) has a solution of the form

$$W(z) = z - \frac{1}{\pi} \iint_E \frac{f(\zeta)}{\zeta - z} dE_\zeta \equiv z + \mathrm{T}f , \tag{3.2}$$

where f is the function to be determined, belonging to a class $L_p(E)$, $p \geqslant p'$.

According to the formulae (8.16), Ch. I, we have

$$\partial_{\bar{z}} W = f , \quad \partial_z W = 1 + \partial_z \mathrm{T}f = 1 + \Pi f . \tag{3.3}$$

Inserting these expressions into the equation (2.4) we obtain

$$f - q\Pi f = q . \tag{3.4}$$

This equation belongs to a class of two-dimensional singular integral equations investigated by Tricomi, [85b]. We shall consider it now as a linear equation in the space $L_p(E)$ and we shall prove its solubility.

In fact, since $L_p(\Pi f) \leqslant \Lambda_p L_p(f)$, $\Lambda_p = L_p(\Pi)$, then $L_p(q\Pi f) \leqslant q_0 L_p(\Pi f) \leqslant q_0 \Lambda_p L_p(f)$. But the constant Λ_p is

* Below we shall derive a formula enabling us to construct even wider classes of generalized solutions possessing isolated singularities, branch points, etc. (§4.4).

a continuous function of p, [77]; therefore, taking into account that $q_0 < 1$ and $\Lambda_2 = 1$ (see formula (9.7), Ch. I), a number $\varepsilon > 0$ may be found such that when $2-\varepsilon \leqslant \leqslant p \leqslant 2+\varepsilon$ the inequality $q_0 \Lambda_p < 1$ is satisfied. Hence, according to the principle of contraction mappings, there exists a solution of the equation (3.4) and it belongs to any class $L_p(E)$, $|p-2| \leqslant \varepsilon$. In view of Theorem 1.21 the function $W - z = \mathrm{T}f$ belongs to $C_\alpha(E)$. Moreover, it belongs evidently to $D_{1,2+\varepsilon}$.

It is important to note that the equation (3.4) can be solved by the method of successive approximations, according to the scheme

$$f_0 = g\,, \; f_{n+1} = q + q\Pi f_n \quad (n = 0, 1, \ldots)\,. \tag{3.5}$$

This procedure yields the series

$$f = q + q\Pi f + q\Pi(q\Pi q) + \ldots \tag{3.6}$$

This method makes possible a fairly easy determination of an approximation of the function f with an arbitrarily large (assumed beforehand) degree of accuracy.

Below, (in §5), we shall prove that the function (3.2) constitutes a complete homeomorphism and satisfies at infinity the following conditions:

$$W(\infty) = \infty\,, \quad z^{-1}W(z) \to 1 \quad \text{with} \quad z \to \infty\,. \tag{3.7}$$

These conditions follow immediately from the formula (3.2) if we take into account the formula (6.10a) of Ch. I. We shall find out later on that these conditions determine the complete homeomorphism in a unique way (§5.3) Therefore, the homeomorphism represented by the formula (3.2) will be referred to as *the basic homeomorphism* of Beltrami's equation (2.4). In order to prove the above fundamental assertion we have to establish the existence of local homeomorphisms and to examine some of their properties; the following section, therefore, will be devoted to this problem.

§4. Proof of existence of a local homeomorphism

4.1. We have the following

THEOREM 2.1. *Let G_0 be the vicinity of a fixed point z_0. If $q(z) \in C_\alpha(G_0)$, $0 < \alpha < 1$, then there exists in a small neighbourhood G_0' of the point $z_0(G_0' \subset G_0)$ a local homeomorphism $W_0(z)$ of the equation* (2.4), *belonging to the class $C_\alpha^1(\bar{G}_0')$, $0 < \alpha < 1$.*

PROOF. By an application of the non-singular affine transformation

$$\zeta = z - z_0 + q(z_0)(\bar{z} - \bar{z}_0) \tag{4.1}$$

we see that the equation (2.4) assumes the form

$$\partial_{\bar{\zeta}} W - \varrho(\zeta)\partial_\zeta W = 0\,, \qquad \varrho(\zeta) = \frac{q(z) - q(z_0)}{1 - q(z)\overline{q(z_0)}}\,. \tag{4.2}$$

Since $|q(z)| \leqslant q_0 < 1$, it is easy to obtain

$$|\varrho(\zeta)| \leqslant q_0' < 1\,, \qquad |\varrho(\zeta_1) - \varrho(\zeta_2)| \leqslant \frac{q(z_1) - q(z_2)|}{1 - q_0}\,. \tag{4.3}$$

Since $\varrho(0) = 0$ there exists a closed circle G_δ, $|\zeta| \leqslant \delta$, in which the following inequalities are satisfied:

$$|\varrho(\zeta)| \leqslant M|\zeta|^\alpha\,, \qquad |\varrho(\zeta_1) - \varrho(\zeta_2)| \leqslant M|\zeta_1 - \zeta_2|^\alpha\,, \tag{4.4}$$

$$M = \frac{H(q,\, \alpha,\, \bar{G}_0)}{(1 - q_0)^2}\,.$$

Let us now consider the function

$$\varrho_\delta(\zeta) = \begin{cases} \varrho(\zeta)\,, & \text{for} \quad |\zeta| < \tfrac{1}{2}\delta\,, \\ 2\varrho(\zeta)\left(1 - \dfrac{|\zeta|}{\delta}\right), & \text{for} \quad \dfrac{1}{2}\delta \leqslant |\zeta| \leqslant \delta\,, \\ 0\,, & \text{for} \quad |\zeta| > \delta\,. \end{cases} \tag{4.5}$$

Evidently, $\varrho_\delta(\zeta) \in C_\alpha(E)$ and it vanishes outside the circle G_δ. From (4.4) and (4.5) we obtain

$$|\varrho_\delta(\zeta)| \leqslant M|\zeta|^\alpha\,, \qquad |\varrho_\delta(\zeta_1) - \varrho_\delta(\zeta_2)| \leqslant 5M|\zeta_1 - \zeta_2|^\alpha\,.$$

Therefore, assuming that $\delta \leqslant 1$, we have

$$\begin{aligned} C(\varrho_\delta, G_\delta) &\leqslant M\delta^a, \\ C_a(\varrho_\delta, G_\delta) &\leqslant M(5+\delta^a) \leqslant 6M. \end{aligned} \tag{4.6}$$

Let us now denote by $C_a^0(G_\delta)$ the set of elements belonging to $C_a(E)$ and vanishing outside G_δ. Since $C_a^0(G_\delta)$ is a closed linear manifold of elements of $C_a(E)$ it may be regarded as a space of the Banach type.

Consider now the operator $\Pi_\delta f \equiv \varrho_\delta(\zeta)\Pi f$ where $f \in C_a^0(G_\delta)$. Evidently, $\Pi_\delta f$ is a linear bounded operator mapping the space $C_a^0(G_\delta)$ onto itself. If $f \in C_a^0(G_\delta)$, then according to the formula (8.7), Ch. I,

$$g(z) \equiv \Pi f = -\frac{1}{\pi}\iint\limits_{E} \frac{f(\zeta)\,d\xi\,d\eta}{(\zeta-z)^2} = -\frac{1}{\pi}\iint\limits_{G_\delta} \frac{f(\zeta)-f(z)}{(\zeta-z)^2}\,d\xi\,d\eta.$$

Hence

$$|g(z)| \leqslant \frac{1}{\pi}H(f, a, G_\delta)\iint\limits_{G_\delta}\frac{d\xi\,d\eta}{|\zeta-z|^{2-a}} \leqslant \frac{4\delta^a}{a}H(f, a, G_\delta). \tag{4.7}$$

By virtue of the formula (8.9) of Ch. I, we have also

$$|g(z_1)-g(z_2)| \leqslant M\alpha|z_1-z_2|^a H(f, a, G_\delta),$$

i.e.

$$H_a(\Pi f, G) \leqslant M_a H(f, a, G_\delta). \tag{4.8}$$

It follows from the inequalities (4.8) and (4.7) that

$$C_a(\Pi f, G_\delta) \leqslant \left(M_a+\frac{4\delta^a}{a}\right)H(f, a, G_\delta) \leqslant \hat{M}_a C_a(f, G_\delta), \tag{4.9}$$

$$\hat{M}_a = M_a + \frac{4}{a}.$$

According to the formula (1.7) of Ch. I,

$$C_a(\varrho_\delta \Pi f, G_\delta) \leqslant C_a(\varrho_\delta, G_\delta)C(\Pi f, G_\delta) + C(\varrho_\delta, G_\delta)C_a(\Pi f, G_\delta), \tag{4.10}$$

Therefore, in view of (4.6), (4.8), and (4.10) we have

$$C_a(\varrho_\delta \Pi f, G_\delta) \leqslant \widetilde{M}_\delta C_a(f, G_\delta), \quad \widetilde{M}_\delta = M\left(M_a+\frac{28}{a}\right)\delta^a. \tag{4.11}$$

Let us choose δ so that the inequalities

$$\delta < 1\,, \quad \delta^{\alpha} < \frac{\alpha}{M(52+M_{\alpha})} \tag{4.12}$$

are satisfied. Then it is readily seen that

$$\widetilde{M}_{\delta} < \frac{M(M_{\alpha}+52)}{\alpha}\,\delta^{\alpha} < 1\,. \tag{4.13}$$

We now seek the solution of the equation (4.2) in the form

$$W(\zeta) = \zeta - \frac{1}{\pi}\iint\limits_{G_\delta} \frac{f(z)\,dx\,dy}{z-\zeta} \equiv \zeta + \mathrm{T}f\,, \quad f \in C_{\alpha}^{0}(G_{\delta})\,. \tag{4.14}$$

Since

$$\partial_{\bar\zeta} W = f(\zeta)\,, \quad \partial_{\zeta} W = 1 + \Pi f \equiv 1 - \frac{1}{\pi}\iint\limits_{G_\delta} \frac{f(z)\,dx\,dy}{(z-\zeta)^2}\,,$$

we obtain for f the equation

$$f(\zeta) - \varrho_{\delta}(\zeta)\Pi f = \varrho_{\delta}(\zeta)\,. \tag{4.15}$$

On the basis of the inequalities (4.11) and (4.13), applying the principle of contraction mappings we find that the equation (4.15) has a unique solution $f(\zeta)$ belonging to $C_{\alpha}^{0}(G_{\delta})$.

From (4.15), with the help of the inequalities (4.6) and (4.10), we obtain

$$C_{\alpha}(f, G_{\delta}) \leqslant C_{\alpha}(\varrho_{\delta}, G_{\delta}) + C_{\alpha}(\varrho_{\delta}\Pi f, G_{\delta}) \leqslant 6M + \widetilde{M}_{\delta} C_{\alpha}(f, G_{\delta})\,.$$

Hence, in view of (4.13) and (4.12),

$$C_{\alpha}(f, G_{\delta}) \leqslant \frac{6M}{1-\widetilde{M}_{\delta}} < \frac{\alpha}{4\delta^{\alpha}}\,, \quad \text{i.e.} \quad \frac{4\delta^{\alpha}}{\alpha} C_{\alpha}(f, G_{\delta}) < 1\,. \tag{4.16}$$

Taking into account that $f \in C_{\alpha}^{0}(G_{\delta})$, according to Theorem 2.1 the function $W(\zeta) = \zeta + \mathrm{T}f \in C_{\alpha}^{1}(G_{\delta})$. We shall now prove that $W(\zeta)$ represents a homeomorphic mapping of the vicinity of the point $\zeta = 0$ onto a vicinity of the point $W(0)$.

According to the equation (4.2) and the relation (4.15) the Jacobian of the transformation is given by the formula

$$J_0(\zeta) = \left|\frac{\partial W}{\partial \zeta}\right|^2 - \left|\frac{\partial W}{\partial \bar{\zeta}}\right|^2 = (1-|\varrho_\delta(\zeta)|^2)\,|1+\Pi f|^2 . \qquad (4.17)$$

Bearing in mind the inequalities (4.3), (4.7) and (4.16), we obtain

$$\begin{aligned} J_0(\zeta) &\geqslant (1-q_0^2)(1-|\Pi f|)^2 \\ &\geqslant (1-q_0^2)\left(1-\frac{4\delta^a}{a}\, C\alpha(f, G_\delta)\right)^2 > 0 , \end{aligned} \qquad (4.18)$$

Therefore, for $|\zeta| \leqslant \delta$ the Jacobian $J_0(\zeta) > 0$. Consequently, the function $W(\zeta) = \zeta + \mathrm{T}f$, where f is a solution of the equation (4.15), represents a one to one continuous mapping of a circle $|\zeta| \leqslant \delta_0 < \delta$ onto a neighbourhood of the point $W(0)$.

Returning to the variable $z = x+iy$, according to (4.1) we have the function

$$W_0(z) = W\big(z - z_0 + q(z_0)(\bar{z} - \bar{z}_0)\big) , \qquad (4.19)$$

which, obviously, is a solution of the equation (2.4) belonging to the class $C^1_a(G_0')$, where G_0' is an ellips with the centre at the point z_0 onto which the circle $|\zeta| \leqslant \delta_0$ is mapped by means of the affine transformation (4.1). Moreover, by means of the function $W_0(z)$ this ellipse is mapped homeomorphically onto a neighbourhood of the point $W_0(z_0)$. Therefore, $W_0(z)$ constitutes a local homeomorphism of the equation (2.4) in a neighbourhood of the point z_0. This completes the proof. *

REMARK. It can be proved that if $q(z) \in C_a(E)$, $0 < < a < 1$, the standard circle G_0 of the radius δ_0 independent

* The principle of the argument which will be presented below (p. 91) implies that the local homeomorphism of the equation (2.4) constructed by means of the formulae (4.14) and (4.16) in fact maps the entire plane z onto the plane W. However, it satisfies the equation (2.4) only in the vicinity of the point z_0.

of the position of the point z_0, may be taken for the vicinity in which the local homeomorphism exists.

4.2. THEOREM 2.2. *If $q(z) \in C_a^m(\overline{G}_0)$ where G_0 is a neighbourhood of the point z_0, then the local homeomorphism $W_0(z)$ which was constructed above belongs to the class* †

$$C_a^{m+1}(E)\,, \quad 0 < a < 1\,.$$

PROOF. Obviously, the theorem will be proved if we discover that the function $W(\zeta) = \zeta + \mathrm{T}f$, where f is a solution of the equation (4.15), belongs to the class $C_a^{m+1}(E)$.

It is easily seen that we may assume that the function $\varrho_\delta(\zeta)$ belongs to $C_a^m(E)$, $\varrho_\delta(\zeta) = \varrho(\zeta)$ for $|\zeta| \leqslant \frac{1}{2}\delta$, and it vanishes identically outside the circle $|\zeta| \leqslant \delta$. Such a continuation of the function $\varrho_\delta(\zeta)$ outside the circle $|\zeta| \leqslant \frac{1}{2}\delta$ is, evidently, always possible. We shall hereafter write simply $q(\zeta)$ instead of ϱ_δ. The proof will be carried out for the case $m = 1$. It is readily seen that the general case can be examined in a similar way.

Let us now assume that $q \in C_a^1(E)$, $0 < a < 1$, and consider the equation

$$g - q\Pi g - q_z \mathrm{T} g = q_z\,. \tag{4.20}$$

It is a linear equation in $C_a(E)$. Since $\mathrm{I} - q\Pi$ has the inverse operator the equation (4.20) reduces to the equation

$$g - (\mathrm{I} - q\Pi)^{-1} q_z \mathrm{T} g = (\mathrm{I} - q\Pi)^{-1} q_z\,. \tag{4.21}$$

Since T is a completely continuous operator and $(\mathrm{I} - q\Pi)^{-1}$ is linear, $(\mathrm{I} - q\Pi)^{-1} q_z \mathrm{T} g$ is completely continuous in $C_a(E)$. Let us prove that the equation (4.21) has always a solution in $C_a(E)$ (its right-hand side, obviously, also belongs to $C_a(E)$). Let g_0 be a solution of the corresponding homogeneous equation which is equivalent to the homogeneous singular equation

$$g_0 - q\Pi g_0 - q_z \mathrm{T} g_0 = 0\,, \quad g_0 \in C_a(E)\,. \tag{4.22}$$

† Strictly speaking the function $W_0(z) - (z - z_0 + q(z_0)(\bar{z} - \bar{z}_0))$ belongs to the class $C_a^{m+1}(E)$.

It follows from (4.22) that $g_0 = 0$ in the vicinity of the point $z = \infty$. Introducing the function

$$\overline{\mathrm{T}}g_0 = -\frac{1}{\pi}\iint\limits_E \frac{g_0(z)}{\zeta - \bar{z}}\, d\xi\, d\eta$$

and taking into account that (cf. the formulae (8.20), (8.22) of Ch. I)

$$\partial_z \overline{\mathrm{T}}g_0 = g_0\,, \qquad q\Pi g_0 + q_z \mathrm{T}g_0 = \partial_z(q\Pi\overline{\mathrm{T}}g_0)\,,$$

the relation (4.22) may be written in the form

$$\partial_z[\overline{\mathrm{T}}g_0 - q\Pi\overline{\mathrm{T}}g_0] = 0\,,$$

i.e. $\overline{\mathrm{T}}g_0 - q\Pi\overline{\mathrm{T}}g_0 = \overline{\Phi_0(z)}$ where $\Phi_0(z)$ is an entire function. But $\overline{\mathrm{T}}g_0 - q\Pi\overline{\mathrm{T}}g_0$ vanishes at infinity. Therefore, according to Liouville's theorem, $\Phi_0(z) \equiv 0$, i.e. $\overline{\mathrm{T}}g_0 - q\Pi\overline{\mathrm{T}}g_0 = 0$. Hence, $\overline{\mathrm{T}}g_0 \equiv 0$ and $g_0 \equiv 0$. This result implies that the equation (4.21) or, what is equivalent, the equation (4.20) has a solution g belonging to the class $C_\alpha(E)$, $0 < \alpha < 1$. On the other hand, the equation (4.20) can be written in the form

$$\partial_z[\mathrm{T}g - q\Pi\mathrm{T}g - q] = 0\,.$$

We therefore infer that $\overline{\mathrm{T}}g - q\Pi\overline{\mathrm{T}}g = q$. Thus, the solution of the equation (4.15) has the form $f = \overline{\mathrm{T}}g$. Since $g \in C_\alpha(E)$, in view of Theorem 1.32, $f \in C^1_\alpha(E)$. Consequently, $\mathrm{T}f \in C^2_\alpha(E)$ thus completing the proof.

Similar reasoning leads to the proof of the following

THEOREM 2.3. *If $q(z) \in D_{m,p}(G_0)$, $m \geqslant 1$, $p > 2$, where G_0 is a neighbourhood of the point z_0, then the local homeomorphism $W_0(z)$, which was constructed above, belongs to the class $D_{m+1,p}(E)$.*

4.3. THEOREM 2.4. *Let $q(z)$ belong to C^m_α, $0 < \alpha < 1$, $m \geqslant 0$ in a neighbourhood of the point z_0. Let $W_0(z)$ be a homeomorphism of the equation (4.20) corresponding to*

the vicinity G_0 of the point z_0; then every generalized solution of the equation

$$\frac{\partial w}{\partial \bar{z}} - q(z)\frac{\partial w}{\partial z} = 0 \tag{4.23}$$

in the domain G_0 is represented by the formula

$$w(z) = \Phi[W_0(z)] \tag{4.24}$$

and, consequently, belongs to the class $C_\alpha^{m+1}(G_0)$. Here $\Phi(w)$ is an arbitrary holomorphic function of the complex argument in the domain $W_0(G_0)$.

PROOF. First of all we can verify by a direct substitution that a function of the form (4.24) really represents a solution of the equation (4.23). It is also evident that $\Phi[W_0(z)] \in C_\alpha^{m+1}$ in a neighbourhood of the point z_0, since $W_0(z) \in C_\alpha^{m+1}$. Now, it remains to prove that every generalized solution of the equation (4.23) can be represented in the form (4.24) in the vicinity of the point z_0. In fact, regarding $W_0 = W_0(z)$ as an independent variable and z as its function we have an implicit function $w(z(W_0))$, which, according to Theorem 1.39, has generalized derivatives with respect to $\overline{W}_0$ and W_0. Taking into account that $w(z)$ satisfies the equation (4.23) we obtain

$$\begin{aligned}\frac{\partial w\big(z(W_0)\big)}{\partial \overline{W}_0} &= \frac{\partial w}{\partial z}\frac{\partial z}{\partial \overline{W}_0} + \frac{\partial w}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial \overline{W}_0} \\ &= \frac{\partial w}{\partial z}\left[\frac{\partial z}{\partial \overline{W}_0} + q(z)\frac{\partial \bar{z}}{\partial \overline{W}_0}\right].\end{aligned} \tag{4.25}$$

Since $W_0(z)$ is a solution of the equation (4.23),

$$1 = \frac{\partial W_0}{\partial W_0} = \frac{\partial W_0}{\partial z}\left(\frac{\partial z}{\partial W_0} + q(z)\frac{\partial \bar{z}}{\partial W_0}\right),$$

$$0 = \frac{\partial W_0}{\partial \overline{W}_0} = \frac{\partial W_0}{\partial z}\left(\frac{\partial z}{\partial \overline{W}_0} + q(z)\frac{\partial \bar{z}}{\partial \overline{W}_0}\right).$$

It follows from the first relation that $\frac{\partial W_0}{\partial z} \neq 0$. Consequently, in view of the second relation, $\frac{\partial z}{\partial \overline{W}_0} + q(z)\frac{\partial \bar{z}}{\partial \overline{W}_0} = 0$. Thus, by virtue of (4.25),

$$\frac{\partial w\big(z(W_0)\big)}{\partial \overline{W}_0} = 0 .$$

This implies, according to Theorem 1.15, that $w\big(z(W_0)\big)$ is a holomorphic function in W_0. This completes the proof.

As a consequence of Theorem 2.3, the formula (4.24) immediately implies the following:

THEOREM 2.5. *If* $q(z) \in D_{m,p}(G_0)$, $m \geqslant 1$, $p > 2$, *then every generalized solution of the equation* (4.23) *in the domain* G_0 *belongs to the class* $D_{m+1,p}(G_0)$.

4.4. THEOREM 2.6. *If* $q(z) \in C_\alpha(G)$, $0 < \alpha < 1$, *then the zeros of a non-constant solution* $w(z)$ *of the equation* (4.23) *are isolated inside* G, *i.e. every zero has a neighbourhood inside which there are no other zeros of the solution under consideration. If* $w(z_0) = 0$ *at a point* z_0, *then in the vicinity of this point*

$$w(z) = [(z-z_0) + q(z_0)(\bar{z}-\bar{z}_0)]^n \widetilde{w}(z) , \qquad (4.26)$$

where n *is a positive integer and* $w(z)$ *is a function continuous in the Hölder sense in a neighbourhood of the point* z_0 *where it does not vanish.*

PROOF. It follows from the formula (4.24) that in the vicinity of the point z_0

$$w(z) = [W_0(z) - W_0(z_0)]^n \Phi_0\big(W_0(z)\big) , \qquad (4.27)$$

where $\Phi_0(\zeta)$ is an analytic function in the vicinity of the point $\zeta_0 = W_0(z_0)$, and $\Phi_0(\zeta_0) \neq 0$. Now, according to the formulae (4.19) and (4.14), we have

$$W_0(z) - W_0(z_0) = W(\zeta) - W(0) = \zeta W_*(\zeta) , \qquad (4.28)$$

where $\zeta = z - z_0 + q(z_0)(\bar{z} - \bar{z}_0)$,

$$W_*(\zeta) = 1 - \frac{1}{\pi} \iint\limits_{|\zeta| \leqslant \delta} \frac{f(\zeta')\, dE_{\zeta'}}{\zeta'(\zeta' - \zeta)}.$$

Since $|f(\zeta)| \leqslant G_\alpha(f, G_\delta)|\zeta|^\alpha$, we have, obviously,

$$\frac{f(\zeta)}{\zeta} \in L_p(E), \quad 2 < p < \frac{1}{1-\alpha}.$$

Therefore, in view of Theorem 1.19 $W_* \in C_\beta(E)$, $\beta = \dfrac{p-2}{p}$. We shall now prove that $W_*(0) \neq 0$. In fact

$$|W_*(\zeta)| \geqslant 1 - \frac{1}{\pi} \iint\limits_{G_\delta} \frac{|f(\zeta')| dE_{\zeta'}}{|\zeta'|\,|\zeta' - \zeta|}$$

$$\geqslant 1 - \frac{C_\alpha(f, G_\delta)}{\pi} \iint\limits_{G_\delta} \frac{dE_{\zeta'}}{|\zeta'|^{1-\alpha} |\zeta' - \zeta|}.$$

Hence, taking into account the inequality (4.16),

$$|W_*(0)| \geqslant 1 - \frac{C_\alpha(f, G_\delta)}{\pi} \iint\limits_{G_\delta} \frac{dE_\zeta}{|\zeta|^{2-\alpha}} = 1 - \frac{2C_\alpha(f, G_\delta)\delta^\alpha}{\alpha} > \frac{1}{2}.$$

The formula (4.26) follows directly from the formulae (4.28) and (4.27). The number n will be called the multiplicity of the zero z_0.

4.5. The formula (4.26) implies at once the following

THEOREM 2.7. *Let $q(z) \in C_\alpha(0 < \alpha < 1)$ in a neighbourhood of a point z_0. If $w(z)$ is a solution of Beltrami's equation* (4.23) *in the vicinity of z_0, then when z tends to z_0 along the radius* $\varphi = \arg(z - z_0) = \text{const.}$,

$$\frac{w(z) - w(z_0)}{z - z_0} \to A_0\left(1 - q(z_0) e^{-2i\varphi}\right), \tag{4.29}$$

where A_0 is a constant independent of φ.

It follows from the relationship (4.29) that a solution of Beltrami's equation has at a point z_0 a derivative with respect to the complex argument z if and only if $q(z_0) = 0$.

4.6. THEOREM 2.8. (PRINCIPLE OF THE ARGUMENT). *Let* $q(z) \in C_a(\bar{G})$, $0 < a < 1$. *Let* $w(z)$ *be a solution of Beltrami's equation in the domain* G, *which satisfies the following conditions*: (1) $w(z)$ *is continuous in* $G+\Gamma$ *where* Γ *is the boundary of the domain* G, *and* (2) $w(z) \neq 0$ *everywhere on* Γ. *Such being the case*, $w(z)$ *may have inside* G *only a finite number of zeros, this number being given by the formula*

$$N = \frac{1}{2\pi} \Delta_\Gamma \arg w(z) . \tag{4.30}$$

Every zero is counted n *times, where* n *is its multiplicity.*

PROOF. The boundedness of the number of zeros of the function $w(z)$ follows immediately from Theorem 2.6. Let now $z_1, \dots, z_N$ be the zeros of this function, every zero being repeated n times (n is its multiplicity). Then, in view of the formula (4.26), we have

$$w(z) = w_0(z) \prod_{k=1}^{n} [z - z_k + q(z_k)(\bar{z} - \bar{z}_k)] ; \tag{4.31}$$

obviously, $w_0(z)$ is continuous in $G+\Gamma$ and vanishes nowhere. Now, taking into account that

$$\frac{1}{2\pi}\Delta_\Gamma \arg w_0(z) = 0 , \quad \frac{1}{2\pi}\Delta_\Gamma \arg \{(z - z_k) + q(z_k)(\bar{z} - \bar{z}_k)\} = 1 ,$$

we obtain the relation (4.30) directly from (4.31).

4.7. In conclusion, we consider one more important property of the homeomorphisms of Beltrami's equation.

THEOREM 2.9. *If* $q(z) \in C_a(G)$, $0 < a < 1$, *and the solution* $w(z)$ *of the equation* (4.23) *represents a one-to-one mapping of a neighbourhood of the point* $z_0 \in G$ *onto a neighbourhood of the point* $w_0 = w(z_0)$, *then the corresponding Jacobian of the transformation does not vanish at the point under consideration*, i.e.

$$J = \left|\frac{\partial w}{\partial z}\right|^2 - \left|\frac{\partial w}{\partial \bar{z}}\right|^2 > 0 . \tag{4.32}$$

This result follows immediately from Theorem 2.4 and the inequality (4.18).

§5. Proof of the existence of a complete homeomorphism

5.1. In §3 we constructed a solution of Beltrami's equation in the form

$$W(z) = z - \frac{1}{\pi} \iint_E \frac{f(\zeta)}{\zeta - z} dE_\zeta \equiv z + \mathrm{T}f \,, \tag{5.1}$$

where f satisfies the equation

$$f - q\Pi f = q \,, \qquad \Pi f \equiv \partial_z \mathrm{T}f \,. \tag{5.2}$$

We have assumed that

$$|q(z)| \leqslant q_0 < 1 \,, \qquad q(z) \in L_{p'}(E) \,, \qquad p' < 2 \,, \tag{5.3}$$

Therefore f belongs to any class $L_p(E)$ where $2 - \varepsilon \leqslant p \leqslant \leqslant 2 + \varepsilon$. Hence, in view of the inequality (6.11), Ch. 1, we have near the point at infinity

$$W(z) = z[1 + O(|z|^{-1})] \,. \tag{5.4}$$

We shall now prove that $W(z)$ represents a complete homeomorphism, under the additional assumption that $q(z) \in C_\alpha^m(E)$, $0 < \alpha < 1$, $m \geqslant 0$; further, in §5.4, this assumption will be abandoned.

If $q(z) \in C_\alpha^m(E)$, then, in accordance with Theorem 2.4, the function $W(z) \in C_\alpha^{m+1}$ in the vicinity of any fixed point of the plane. It follows immediately, therefore, that $z - W(z) \in C_\alpha^{m+1}(E)$.

Let us now prove that $W(z)$ assumes once and only once every fixed value W_0. In fact, the function $W_*(z) = W(z) - W_0$ which, evidently, satisfies Beltrami's equation

$$\partial_{\bar{z}} w - q(z) \partial_z w = 0 \,, \tag{5.5}$$

cannot be identically equal to a constant, since, by virtue of (5.4), near the point $z = \infty$ it has the form $W_* = = z[1 + O(|z|^{-1})]$. Therefore the increment of $\dfrac{1}{2\pi} \arg W_*(z)$

on a circle of a sufficiently large radius with the centre at the point $z = 0$ is equal to unity. Hence, according to the principle of the argument (Theorem 2.8) there exists only one point z_0 at which the function $W_*(z)$ has a zero of the first order, i.e. at the point z_0 and only at this point the function $W(z)$ assumes the given value W_0. We have, therefore, proved the following

THEOREM 2.10. *If* 1) $|q(z)| \leqslant q_0 < 1$, *and* 2) $q(z) \in L_{p'}C_a^m(E)$, $0 < a < 1$, $m \geqslant 0$, $p' < 2$, *then the function* (5.1) *is a solution of Beltrami's equation representing a complete homeomorphism of the plane onto the plane. This homeomorphism has the following property*:

$$W(z) - z \in C_a^{m+1}(E) .$$

5.2. The complete homeomorphism (5.1) may be regarded as a local homeomorphism in an arbitrary neighbourhood of any fixed point z_0. Therefore, Theorem 2.4 implies directly the following

THEOREM 2.11. *If the following conditions are satisfied*: (1) $|q(z)| \leqslant q_0 < 1$, *and* (2) $q(z) \in L_{p'}C_a(E)$, $0 < a < 1$, $p' < 2$, *then every function satisfying the equation* (5.5) *in a domain* G *is represented by the formula*

$$w(z) = \Phi\big(W(z)\big) , \tag{5.6}$$

where $\Phi(\zeta)$ *is an arbitrary analytic function in the domain* $G_\zeta = W(G)$.

Theorem 2.10 will now be completed by the following

THEOREM 2.12. *If* (1) $|q(z)| \leqslant q_0 < 1$, (2) $q(z) \in L_{p'}C_a(E)$, $0 < a < 1$, $p' < 2$ *and* (3) $q(z) \in C_a^m(G)$, $0 < a < 1$, $m \geqslant 1$ *where* G *is a domain of the plane* z, *then the complete homeomorphism* (5.1) *satisfies the conditions*: (1) $W(z) \in C_a^1(E)$ *and* (2) $W(z) \in C_a^{m+1}(G)$. *Moreover, all continuous solutions of the equation* (5.5) *in the domain* G *belong to the class* $C_a^{m+1}(G)$.

PROOF. Let G' be a closed subdomain of G. Let us take a polygonal domain G_0 such that $G' \subset G_0$, $\bar{G}_0 \subset G$

and let us continue the function $q(z)$ outside G_0 preserving the three conditions of the theorem—it is known that this continuation is possible, [44a]. Let us denote the function thus defined by $q_0(z)$ and the corresponding homeomorphism of the equation $w_{\bar{z}}-q_0 w_z = 0$ by $w_0(z)$. According to Theorem 2.10 $w_0(z) \in C_\alpha^{m+1}(E)$. Since $q_0 = q$ in G_0, $W(z)$ and $w_0(z)$, as solutions of Beltrami's equation (5.5), in view of the formula (5.6) satisfy the relation $W(z) = \Phi_0(w_0(z))$, where $\Phi_0(w_0)$ is a function holomorphic in the domain $G_{w_0} = w_0(G_0)$. Therefore $W \in C_\alpha^{m+1}(G')$. Now, G' being an arbitrary subdomain of G, we have $W \in C_\alpha^{m+1}(G)$. The remaining part of the theorem is obvious.

5.3. It was already mentioned in §3 that the complete homeomorphism (5.1) satisfies the conditions

$$W(\infty) = \infty\,, \quad z^{-1}W(z) \to 1 \quad \text{for} \quad z \to \infty\,. \tag{5.7}$$

In other words, the complete homeomorphism (5.1) preserves the point $z = \infty$ and the directions parallel to the real axis at this point. These requirements determine uniquely the complete homeomorphism. This result follows immediately from

THEOREM 2.13. *Every complete homeomorphism of the equation* (5.5) *has the form*

$$W_*(z) = \frac{\alpha W(z)+\beta}{\gamma W(z)+\delta}\,, \quad \alpha\delta-\gamma\beta \neq 0\,, \tag{5.8}$$

where $W(z)$ is the complete homeomorphism (5.1).

PROOF. According to Theorem 2.11 every complete homeomorphism of the equation (5.5) can be represented in the form

$$W_*(z) = \Phi\big(W(z)\big)\,, \tag{5.9}$$

where $W(z)$ is the basic homeomorphism and $\Phi(W)$ is an analytic function of the complex argument W which should establish a homeomorphic mapping of the plane W onto the plane W_*; it is known, however, that only linear fractional functions possess this property. This completes the proof of the formula (5.8).

It is now evident that the conditions (5.7) determine a complete homeomorphism uniquely; this was the reason for naming the homeomorphism (5.1) the basic homeomorphism.

5.4. We shall now abandon the requirement of continuity in the Hölder sense of the function $q(z)$. If $q(z)$ satisfies only the conditions (5.3), then the function $W(z) = z + \mathrm{T}f \in D_{1,p}(E)$, $|p-2| \leqslant \varepsilon$. According to Theorem 1.21 this shows that $\mathrm{T}f \in C_\varepsilon(E)$. It was proved by Bojarski, [11b], that also in this case the function $W = z + \mathrm{T}f$ represents a one-to-one continuous mapping of the plane z onto the plane W. We shall here reproduce Bojarski's proof, [11d].

THEOREM 2.14. *Let $q(z)$ be a measurable function satisfying the following conditions*: (1) $|q(z)| \leqslant q_0 < 1$ ($q_0 = \text{const.}$) *and* (2) $q(z) \equiv 0$ *outside a fixed circle K with the centre at the origin of coordinates. Then the function $W(z) = z + \mathrm{T}f$, where f is a solution of the equation* (5.2), *establishes a homeomorphic mapping of the plane z onto the plane W. The function $W = W(z)$ and the inverse function $z = z(W)$ belong to a space $C_\alpha(E)$, $0 < \alpha < 1$, α depending only on the circle K and the constant q_0, i.e. $\alpha = \alpha(K, q_0)$.*

PROOF. Let $q_n(z)$, $n = 1, 2, \ldots$ be a sequence of continuously differentiable functions on the entire plane, satisfying the conditions

$$\begin{aligned} &q_n(z) \to q(z) \quad \text{almost everywhere} \\ &|q_n(z)| \leqslant q_0\,, \\ &q_n(z) \equiv 0 \quad \text{outside } K\,. \end{aligned} \tag{5.10}$$

Such a sequence can be obtained, for instance, by taking the mean value of the function $q(z)$. By virtue of (5.10) we have

$$L_p(q_n - q) \to 0 \quad \text{for any} \quad p > 0\,. \tag{5.11}$$

Let us consider the sequence of the functions

$$W_n(z) = z + \mathrm{T}(f_n)\,, \tag{5.12}$$

where f_n is a solution of the integral equation

$$f_n - q_n \Pi f_n = q_n . \tag{5.13}$$

Evidently, $f_n \equiv 0$ outside K; $\dfrac{\partial W_n}{\partial \bar{z}} - q_n(z)\dfrac{\partial W_n}{\partial z} = 0$. From (5.13) we obtain the estimate $L_p(f_n) \leqslant q_0 \Lambda_p L_p(f_n) + L_p(q_n)$, or for p satisfying the condition $q_0 \Lambda_p < 1$, $|p-2| \leqslant \varepsilon$

$$L_p(f_n) \leqslant \frac{L_p(q_n)}{1-q_0\Lambda_p} < \frac{c}{1-q_0\Lambda_p}, \tag{5.14}$$

where c is a constant independent of both n and p. Further, we obtain from (5.13)

$$f_n - f_m = q_n \Pi(f_n - f_m) + (q_n - q_m)\Pi f_m + q_n - q_m ,$$

whence, for p satisfying the condition $q_0 \Lambda_p < 1$,

$$(1-q_0\Lambda_p)L_p(f_n - f_m) \leqslant L_p\big((q_n - q_m)\Pi f_m\big) + L_p(q_n - q_m) .$$

On the other hand

$$L_p[(q_n - q_m)\Pi f_m] \leqslant L_{pq'}(q_n - q_m) L_{pp'}(\Pi f_m) ,$$

where $\dfrac{1}{p'} + \dfrac{1}{q'} = 1$. Therefore, choosing p' so close to unity that $q_0 \Lambda_{pp'} < 1$, and taking into account (5.11) and (5.14), we find that

$$L_p(f_n - f_m) \leqslant \varepsilon_{n,m} c_1 , \qquad \varepsilon_{n,m} \to 0 \quad \text{when} \quad n, m \to \infty ,$$

where $\varepsilon_{n,m} = L_{pq'}(q_n - q_m)$ and c is a constant (depending on p but independent of n and m). Let

$$f = \lim_{n\to\infty} f_n \quad \text{in} \quad L_p(E) , \qquad f \in L_p(K) , \qquad f \equiv 0 \text{ outside } K$$

Evidently, $f - q\Pi f = q$. Assuming $W(z) = z + \mathrm{T}f$ we have $C_\alpha(W - W_n) \leqslant ML_p(f - f_n)$, $\alpha = \dfrac{p-2}{p}$, $p > 2$. Consequently, $W_n \to W(z)$ uniformly on the entire plane. It is evident that $W(z) \in C_\alpha(E)$. According to Theorem 2.10 $W_n(z)$ is a continuously differentiable homeomorphism of

the plane z onto the plane W. Let us prove that $W(z)$ is also a homeomorphism of the plane z onto the plane W. To this end let us consider the sequence of continuously differentiable functions $z = z_n(W)$—inverse to the functions of the sequence (5.12). We have

$$z_n\big(W_n(z)\big) \equiv z \quad \text{and} \quad W_n\big(z_n(W)\big) \equiv W \tag{5.14a}$$

for all W and z.

It is easy to verify the formulae

$$\overline{\frac{\partial z_n}{\partial W}} = \frac{1}{J_n}\frac{\partial W_n}{\partial z}, \qquad \frac{\partial z_n}{\partial \overline{W}} = -\frac{1}{J_n}\frac{\partial W_n}{\partial \bar{z}} \tag{5.15}$$

(J_n is the Jacobian of the transformation $W = W_n(z)$, $J_n = \left|\frac{\partial W_n}{\partial z}\right|^2 - \left|\frac{\partial W_n}{\partial \bar{z}}\right|^2$) which imply, by (5.13), that $z = z_n(W)$ satisfies the quasi-linear equation

$$\frac{\partial z_n}{\partial \overline{W}} + q_n\big(z_n(W)\big)\overline{\frac{\partial z_n}{\partial W}} = 0\,. \tag{5.16}$$

As a consequence of the inequality (5.14) we derive from (5.13)

$$|W_n(z) - z| < M_0 L_p(f_n) \leqslant M\,, \tag{5.16a}$$

where the constant M is independent of n. Taking (5.15) into account we see that $\frac{\partial z}{\partial \overline{W}} \equiv 0$ outside a fixed circle K_1 independent of n. Therefore, by Theorem 1.16, $z_n(W)$ can be represented thus:

$$\begin{aligned} z_n(W) &= W + \Phi_n(W) + \mathrm{T}(\tilde{f}_n) \\ &= W + \Phi_n(W) - \frac{1}{\pi}\iint_E \frac{\tilde{f}_n(\zeta)}{\zeta - W}\,d\xi\,d\eta\,, \end{aligned} \tag{5.17}$$

where $\tilde{f}_n(\zeta) \equiv 0$ outside K_1 and the function $\Phi_n(W)$ is holomorphic on the entire plane.

When n is fixed $\mathrm{T}(\tilde{f}_n)$ is bounded and $\lim \mathrm{T}(\tilde{f}_n) = 0$ when $W \to \infty$. By virtue of (5.16a) $\Phi_n(W)$ is also bounded and since $\lim |W_n(z) - z| = 0$, $z \to \infty$, i.e. $\lim |W - z_n(W)| = 0$,

$W \to \infty$, then $\Phi_n(W) \to 0$ if $W \to \infty$, i.e. $\Phi_n(W) \equiv 0$. Therefore, (5.17) assumes the form

$$z_n(W) = W + \mathrm{T}(\tilde{f}_n), \quad \tilde{f}_n \equiv 0 \quad \text{outside} \quad K_1. \tag{5.18}$$

Making use of (5.16) and the last formula we obtain for $\tilde{f}_n$ the equation

$$\tilde{f}_n + q_n\big(z_n(W)\big)\overline{\Pi \tilde{f}_n} = -q_n\big(z_n(W)\big),$$

Besides, $\left|q_n\big(z_n(W)\big)\right| \leqslant q_0 < 1$.

Thus, similarly to (5.14), we obtain the estimate

$$L_p(\tilde{f}_n) \leqslant c_2$$

(c_2 is independent of n) valid for $p > 2$ and satisfying the condition $q_0 \Lambda_p < 1$. According to the inequalities (6.1) and (6.2) of Ch. I, the operator $\mathrm{T}f$ transforms the space L_p into a space of functions satisfying the Hölder condition in a completely continuous way. Therefore, from the sequence (5.18), a sequence $z_{n_k}(W)$ can be extracted, such that it converges uniformly to a function $z(W)$ satisfying the Hölder condition. By a limiting process in (5.14a) with respect to the subsequence $n_k, k \to \infty$, we obtain $W\big(z(W)\big) \equiv W$ and $z\big(W(z)\big) \equiv z$, i.e. $w = W(z)$ is a homeomorphic mapping of the z-plane onto the w-plane possessing, as well as its inverse $z = z(w)$, all the required properties. This completes the proof of the theorem.

Remark. The uniqueness of the limit $z(W)$ of the subsequence $z_{nk}(W)$ and the compactness of the sequence $z_n(W)$ imply immediately that $z_n(W) \to z(W)$ uniformly.

5.5. We have so far been assuming that $q(z) = 0$ in a neighbourhood of the point $z = \infty$. Now we can abandon this restriction. We have the following

Theorem 2.15. *If $q(z)$ is a measurable function on the entire plane and the condition $|q(z)| \leqslant q_0 < 1$ is satisfied, then there exists a complete homeomorphism $W(z)$ of Beltrami's equation $\partial_{\bar{z}} w - q(z)\partial_z w = 0$ belonging to a class $C_\alpha(E)$, $0 < \alpha < 1$.*

PROOF. Let $W_R(z)$ be the basic homeomorphism of the equation

$$\partial_{\bar z} w - q_R(z)\partial_z w = 0 , \tag{5.19}$$

where

$$q_R(z) = \begin{cases} q(z) , & \text{for} \quad |z| \leqslant R , \\ 0 , & \text{for} \quad |z| > R . \end{cases} \tag{5.20}$$

According to Theorem 2.14 the function W_R exists and belongs to a class $C_\alpha(E)$, $0 < \alpha < 1$. Performing a change of variables in equation (5.19) according to the formula

$$\zeta = \zeta(z) \equiv \frac{1}{W_R(z) - W_R(0)} , \tag{5.21}$$

we obtain

$$\partial_{\bar\zeta} w - q_1(\zeta)\partial_\zeta w = 0 , \tag{5.22}$$

where

$$q_1(\zeta) = \frac{q(z) - q_R(z)}{1 - q(z)\overline{q_R(z)}} \frac{\overline{\partial_z \zeta}}{\partial_{\bar z}\bar\zeta} . \tag{5.23}$$

As is seen from (5.21), $\zeta(z)$ represents a univalent continuous mapping of the z-plane onto the ζ-plane, the vicinity of the point $z = 0$ being mapped onto the vicinity of the point $\zeta = \infty$, and conversely. By virtue of (5.20) and (5.23) $q_1(\zeta) \equiv 0$ in a fixed neighbourhood of the point $\zeta = \infty$. Moreover, it is evident that $|q_1(\zeta)| \leqslant q_0' < 1$. Therefore $q_1(\zeta)$ satisfies all the conditions of Theorem 2.13. Consequently, there exists a basic homeomorphism of the equation (5.22), which will be denoted by $W_1(\zeta)$. Considering now the function

$$W_*(z) = W_1[\zeta(z)] ,$$

we obtain a complete homeomorphism of the original equation (5.5).

If we take into account that $\zeta(z)$ is analytic in z outside the circle $|z| \leqslant R$ and $W_1(\zeta)$ is analytic in ζ in the domain in which $q_1(\zeta) = 0$ (this domain is the image of the circle $|z| < R$) we obtain the following differentiation formulae:

$$\partial_z W_* = \partial_\zeta W_1(\zeta)\partial_z\zeta , \quad \partial_{\bar z} W_* = \partial_{\bar\zeta} W_1(\zeta)\partial_{\bar z}\bar\zeta , \quad \text{for} \quad |z| > R ,$$

and

$$\partial_z W_* = \partial_\zeta W_1(\zeta)\partial_z \zeta\,, \quad \partial_{\bar z} W_* = \partial_\zeta W(\zeta)\partial_{\bar z}\zeta\,, \quad \text{for} \quad |z| < R\,.$$

It easily follows from these formulae that W_* satisfies Beltrami's equation (5.5). Taking into account that $\overset{*}{W}_*(\infty) = W_1(\zeta(\infty)) = W_1(0)$ we find that the fractional function

$$W(z) = \frac{1}{W_1[\zeta(z)] - W_1(0)}$$

represents a univalent and continuous mapping of the z-plane onto the W-plane, the point $z = \infty$ being preserved. Since $\zeta(z)$ and $W_1(\zeta)$ are functions continuous in the Hölder sense, evidently $W(z)$ also satisfies the Hölder condition in any finite part of the plane. Thus, Theorem 2.15 has been proved completely.

5.6. The above mapping of the z-plane onto the W-plane will not, in general, be continuously differentiable. We know only that it belongs to the class $D_{1,p}$ for a $p > 2$. Nevertheless, it was shown in [11b, 11d] that the mapping $W(z) = z + \mathrm{T}f$ has a number of properties analogous to those of continuously differentiable mappings. With respect to some fundamental operations of analysis (integration, generalized differentiation, etc.) it behaves in exactly the same way as continuously differentiable mappings. These statements allow us to investigate the properties of solutions of Beltrami's equation $w_{\bar z} - q(z)w_z = 0$ under the only assumption that $q(z)$ is a measurable function satisfying the condition $|q(z)| \leqslant q_0 < 1$. For example, we have the following

THEOREM 2.16. *Under the above conditions all solutions (of the class $D_{1,p}(G)$, $p > 2$) of Beltrami's equation*

$$w_{\bar z} - q(z)w_z = 0$$

are given by the formula

$$w(z) = \Phi\big(W(z)\big), \quad W(z) = z + \mathrm{T}f\,, \tag{5.24}$$

where f is a solution of the equation (5.2) and Φ is an arbitrary analytic in W function in the domain $W(G)$.

It is evident that the last formula makes possible the construction of solutions of Beltrami's equation possessing arbitrary singularities of the pole type, essential singularities, branch-points of various types, etc. In particular, by means of the formula (5.24) we can construct various univalent solutions mapping homeomorphically a domain G onto canonical domains of appropriate type. A number of important properties of this kind of univalent mappings were established in the paper [11d].

Finally, we state without proof the following theorem, [11d]:

THEOREM 2.17. *Let* $G_n, n = 1, 2, \ldots$ *be an increasing sequence of subdomains of the domain* G, $G_n \subset G_{n+1}$, $G = \lim G_n$ *when* $n \to \infty$. *Let there be given in the domain* G *an equation of the form* (5.5). *Assume that* $\sup_{z \in G_n} |q(z)| = q_n < 1$, *and it is possible that* $q_n \to 1$ *when* $n \to \infty$. *Then there exists in the domain* G *a global homeomorphism of the equation* (5.5). *In every bounded and closed subdomain of the domain* G *the properties of this homeomorphism are analogous to those of the homeomorphisms investigated above.*

Under the assumptions of Theorem 2.17 the global homeomorphism cannot, in general, be represented by means of the simple formulae (5.1) with $f \in L_p(G)$, $p > 2$. Nevertheless, it can be constructed from the homeomorphisms of the form (5.1) by a limiting process.

§6. Reduction of a positive quadratic differential form to the canonical form. Isometric and isometric–conjugate coordinate systems on a surface

6.1. Let us now again consider the quadratic form

$$F \equiv a\,dx^2 + 2b\,dx\,dy + c\,dy^2 . \tag{6.1}$$

Let us assume that: (1) a, b, c are bounded and belong to the class C_α^m in a domain G $(m \geqslant 0, 0 < \alpha < 1)$, (2) $\Delta = ac - b^2 \geqslant \Delta_0 > 0$ in $\bar{G}$, $\Delta_0 = \text{const}$. Considering the

global homeomorphism $W(z) = u(x, y) + iv(x, y)$ of the system of equations (1.3) we can reduce the quadratic form (6.1) in the whole domain G to the canonical form

$$F \equiv \Lambda(u, v)(du^2 + dv^2), \tag{6.2}$$

where

$$\Lambda = \frac{4\sqrt{\Delta}}{J}, \quad J = |W_z|^2(1 - |q(z)|^2) \tag{6.3}$$

or

$$\Lambda = ax_u^2 + 2bx_u x_v + cx_v^2 \equiv ay_u^2 + 2by_u y_v + cy_v^2. \tag{6.4}$$

In view of Theorem 2.10 every global homeomorphism of the quadratic form (6.1) belongs to the class $C_\alpha^{m+1}(G)$. Therefore, it is seen from (6.3) that $\Lambda \in C_\alpha^m$ in the appropriate domain.

If $a, b, c \in D_{m+1,p}(\bar{G})$, $m \geqslant 0$, $p \geqslant 2$, then, according to Theorem 2.5, the global homeomorphisms of the form (6.1) belong to the class $D_{m+2,p}$, and, consequently, $\Lambda \in D_{m+1,p}$.

By means of conformal transformations of the first or of the second kind

$$w_* = u_* + iv_* = \Phi_*(W(z)) \quad \text{or} \quad w_* = \overline{\Phi_*(W(z))},$$

where $\Phi_*(W)$ is an arbitrary univalent holomorphic function in $W(G)$, we can obtain any global homeomorphism of the quadratic form (6.1) with respect to the domain G, and, evidently,

$$F \equiv \Lambda_*(du_*^2 + dv_*^2), \quad \Lambda = \Lambda_* |\Phi\ (w)|^2. \tag{6.5}$$

If we make use of Theorem 2.15 the reduction of the quadratic form (6.1) to the canonical form (6.2) may also be performed in the case in which the coefficients are only measurable bounded functions in a domain G, and $ac - b^2 \geqslant g_0 > 0$. Since $x_u, x_v \in L_p$, $p > 2$, it follows from (6.4) that $\Lambda \in L_{p/2}$.

Let a domain G represent a union of domains $G_1, G_2, \ldots, G_l$, where $G_i G_k = 0$, $i \neq k$. Assume that $a, b, c \in C_\alpha^m(\bar{G}_i)$, $i = 1, 2, \ldots, l$ and that both the functions and their

derivatives up to the mth order may have discontinuities of the first kind on the contours of the domains G_i. Then, according to Theorem 2.12 and Theorem 2.15, there exists a global homeomorphism $W(z)$ of the class $C_\alpha(E)$ belonging to the class $C_\alpha^{m+1}(G_i)$ inside every domain G_i.

6.2. Let us now apply the above results to the solution of the geometric problem of construction on a given surface of a so-called isometric net of lines $x = \text{const.}$, $y = \text{const.}$ defined by the fact that referred to this net the fundamental quadratic form assumes the form

$$\mathrm{I} \equiv \Lambda(dx^2 + dy^2), \quad \Lambda > 0\,. \tag{6.6}$$

Henceforth by a surface we shall understand the union of its interior and boundary points, i.e. the boundary points also belong to the surface.

Let a surface S be mapped homeomorphically onto a domain G of the plane, i.e. there has been established a one to one continuous correspondence between the points of S and G. Then the coordinates x^1, x^2 of a point of the domain G will also be regarded as the coordinates (interior coordinates) of the corresponding point of the surface. It is evident that the domain G is closed. By mapping homeomorphically the domain G onto another (obviously also closed) domain $\widetilde{G}$ we obtain on the surface a new coordinate system $\widetilde{x}^1, \widetilde{x}^2$. Thus, to every one to one mapping of the domain G onto another domain there corresponds a fully determined coordinate system on the surface. Consequently, there exists an infinite set of coordinate systems on the surface S and a change of a coordinate system x^1, x^2 on another system $\widetilde{x}^1, \widetilde{x}^2$ is performed by means of non-singular transformations of the form

$$\widetilde{x}^i = \widetilde{x}^i(x^1, x^2), \quad x^i = x^i(\widetilde{x}^1, \widetilde{x}^2), \quad i = 1, 2\,, \tag{6.7}$$

where $\widetilde{x}^i(x^1, x^2)$ and $x^i(\widetilde{x}^1, \widetilde{x}^2)$ are continuous single-valued functions in G and $\widetilde{G}$ respectively.

By introducing in the space a Cartesian system of coordinates with unit vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$, the position vector $\boldsymbol{r}(x^1, x^2)$ of an arbitrary point of the surface S may be expressed in the form

$$\boldsymbol{r}(x^1, x^2) = X(x^1, x^2)\boldsymbol{e}_1 + Y(x^1, x^2)\boldsymbol{e}_2 + Z(x^1, x^2)\boldsymbol{e}_3,$$

where X, Y, Z are Cartesian coordinates of a point of the surface with interior coordinates x^1, x^2. Let us now assume that there exists a coordinate system x^1, x^2 on the surface S with respect to which the functions $X(x^1, x^2)$, $Y(x^1, x^2)$, $Z(x^1, x^2)$ are continuous and have continuous derivatives of the order $\leqslant k$ in a closed domain G. Then the surface S will be said to belong to the class C^k. In differential geometry it is customary to call a surface regular if $k \geqslant 3$. We shall hereafter also use this term in this sense. We shall also consider surfaces of the classes C^k_α and $D_{k,p}$.

If there exists on a surface a coordinate system x^1, x^2, such that X, Y, Z are analytic functions in the variables x^1, x^2 in the domain G, then the surface S will be called an analytic surface or a surface of the class $\mathfrak{A}$.

We shall henceforth make use of the fundamentals of the theory of surfaces in vectorial and tensorial notation. For references on this topic the reader is advised to consult the book [37]. It should, however, be borne in mind that we shall sometimes use different notations (see also Ch. V, §5).

Let us consider a piecewise smooth surface S. composed of a finite number of surfaces of the class $C^k, k \geqslant 1$, which is homeomorphic to a domain G of the plane (G may coincide with the entire plane). Then the equation of the surface may be represented in the vector form

$$\boldsymbol{r} = \boldsymbol{r}(x^1, x^2),$$

where $\boldsymbol{r}$ is the position vector of a point with (interior) coordinates x^1, x^2. If the vectors

$$\boldsymbol{r}_\alpha = \frac{\partial \boldsymbol{r}}{\partial x^\alpha} \quad (\alpha = 1, 2),$$

are chosen for *the base vectors of the coordinate system* x^1, x^2, then the first fundamental form of the surface has the form

$$\mathrm{I} \equiv ds^2 = a_{\alpha\beta} dx^\alpha dx^\beta, \text{ where } a_{\alpha\beta} = \boldsymbol{r}_\alpha \boldsymbol{r}_\beta . \qquad (6.8)$$

In the case of a piecewise smooth surface $a_{\alpha\beta}$ will be sectionally continuous bounded functions in the domain G. Since $a = a_{11}a_{22} - a_{12}^2 \geqslant a_0 > 0$, according to Theorem 2.15 there exists a global homeomorphism of the form (6.8) with the corresponding isometric coordinate system x, y on the whole surface. It follows that the net of lines $x = \text{const.}$ $y = \text{const.}$ covers the whole surface in a continuous way but it is isometric only on every smooth part of the surface. On the contact lines the coefficients $\Lambda(x, y)$ of the form (6.6) suffer discontinuities of the first kind. On passing through the contact lines of neighbouring smooth parts of the surface the coordinate lines are continuously continued from one part into another, but in general the property of isometry on these lines is violated.

If a surface belongs to the class C_α^{m+1}, $0 < \alpha < 1$, then $a_{\alpha\beta}$ belong to $C_\alpha^m(\overline{G})$ and, consequently, in this case $\Lambda(x, y) \in C_\alpha^m$.

Let us observe that a change of isometric coordinate system—from an isometric system x, y to an isometric system x', y'—is performed by means of a transformation of the form

$$x' + iy' = \Phi(x+iy) \text{ or } x' + iy' = \overline{\Phi(x+iy)} ,$$

where Φ is an analytic function univalent in the domain of the variables x, y. If this domain coincides with the entire plane $z = x+iy$, Φ is a bilinear function

$$\Phi(z = \frac{\alpha z + \beta}{\gamma z + \delta} , \qquad \alpha\delta - \gamma\beta \neq 0 .$$

6.3. We now consider the problem of reduction to the canonical form of the second fundamental quadratic form of the surface,

$$\mathrm{II} = b_{\alpha\beta} dx^\alpha dx^\beta . \qquad (6.9)$$

The coefficients of this form are expressed by the formulae

$$b_{\alpha\beta} = \boldsymbol{n}\boldsymbol{r}_{\alpha\beta}, \quad \boldsymbol{r}_{\alpha\beta} = \frac{\partial^2 \boldsymbol{r}}{\partial x^\alpha \partial x^\beta}, \qquad \boldsymbol{n} = \frac{\boldsymbol{r}_1 \times \boldsymbol{r}_2}{|\boldsymbol{r}_1 \times \boldsymbol{r}_2|}. \tag{6.10}$$

The principal curvature of the surface is given by

$$\mathrm{K} = \frac{b_{11} b_{22} - b_{12}^2}{a_{11} a_{22} - a_{12}^2}. \tag{6.11}$$

It follows that the form II has constant sign on surfaces of positive principal curvature. Let a surface S be composed of pieces of the class $C_\alpha^{m+2} (m \geqslant 0)$ of positive principal curvature. Such being the case, it is seen from the formulae (6.10) that the coefficients of the form II are sectionally continuous functions of the class C_α^m.

As before, we shall assume that the surface is homeomorphic to a domain G of the plane. Moreover, it is assumed that $\mathrm{K} \geqslant \mathrm{K}_0 > 0$ (in $\bar{G}$), $\mathrm{K}_0 = \text{const}$. Accordingly, in view of Theorem 2.15 there exists a homeomorphism of the quadratic form II corresponding to a fully determined coordinate system on the surface. The second fundamental quadratic form referred to this system has the form

$$\mathrm{II} \equiv k_s ds^2 = \Lambda (dx^2 + dy^2), \qquad \Lambda \in C_\alpha^m(G), \tag{6.12}$$

where k_s is the so called normal curvature of the surface in the direction s.

If the coordinate net on the surface is constituted by the lines of curvature $\xi = \text{const.}$, $\eta = \text{const.}$, then the form $\mathrm{II} = A^2 k_1 d\xi^2 + B^2 k_2 d\eta^2$ and the corresponding homeomorphism $z(\zeta) = x(\xi, \eta) + iy(\xi, \eta)$ satisfies Beltrami's equation

$$z_{\bar{\zeta}} - \frac{\mathrm{A}\sqrt{k_1} - \mathrm{B}\sqrt{k_2}}{\mathrm{A}\sqrt{k_1} + \mathrm{B}\sqrt{k_2}} z = 0. \tag{6.13}$$

Since $a\mathrm{K} = \Lambda^2$, either $\Lambda = \sqrt{a\mathrm{K}}$ or $\Lambda = -\sqrt{a\mathrm{K}}$. In what follows we shall always assume that

$$\Lambda = \sqrt{a\mathrm{K}}. \tag{6.14}$$

This can always be achieved by the choice of the orientation of the normal n to the surface, which in our case should be directed towards the concavity of the surface. In fact, according to Meusnier's theorem, $k_s = k\cos\vartheta$. Taking (6.14) into account it is found that this is possible only if the normal to the surface makes an acute angle ϑ with the principal normal of the curve directed always towards the concavity of the surface; this proves our assertion.

Denoting by ds and $d\sigma$ the corresponding elements of arcs of the surface and the plane z we have from (6.12)

$$ds = c_s d\sigma, \quad c_s = \sqrt{\frac{\Lambda}{k_s}}, \quad \sqrt[4]{\frac{ak_2}{k_1}} \leqslant c_s \leqslant \sqrt[4]{\frac{ak_1}{k_2}}. \quad (6.15)$$

Thus, in mapping a surface of positive curvature onto the z-plane by means of the homeomorphism of the second fundamental quadratic form, an element of an arc of the surface suffers an extension inversely proportional to the square root of the normal curvature in the direction of the arc under consideration.

The net of lines on the surface referred to which the second fundamental quadratic form assumes the form (6.12) is named the ***isometric–conjugate system of coordinates.***

Thus, an isometric–conjugate net of lines exists on any surface of the class C_α^{m+2}, $m \geqslant 0$ homeomorphic to a domain of the plane (or to the entire plane). If the surface is homeomorphic to a domain G and composed of a finite number of regular sections of the class C_α^{m+2}, then, as before, there exists on the whole surface a continuous net of lines $x = \text{const.}$, $y = \text{const.}$ which is isometric–conjugate on every regular section. In this case Λ and its derivatives are in general sectionally continuous functions.

If the surface belongs to the class $D_{m+3,p}$, $p > 2$, then $b_{\alpha\beta}$ belong to the class $D_{m+1,p}$ (consequently, in view of Theorem 1.20 $b_{\alpha\beta} \in C_\alpha^m$, $\alpha = \dfrac{p-2}{p}$). According to Theorem

2.5 the homeomorphism $x = x(\xi, \eta)$, $y = y(\xi, \eta)$ of the form II belongs to the class $D_{m+2,p}$ in the corresponding domain. Hence, obviously, $\Lambda \in D_{m+1,p}$.

Since $\mathrm{K} \in D_{m+1,p}$ and $\Lambda = \sqrt{a\mathrm{K}} \in D_{m+1,p}$, it is evident that $\sqrt{a} \in D_{m+1,p}$.

We observe that, similarly to the case of an isometric net of lines, a transformation from an isometric-conjugate coordinate system to another such system is accomplished by means of conformal mappings of the first or of the second kind.

Below we shall investigate in more detail the properties of isometric-conjugate net of lines, for we shall use it in Chapters V and VI in dealing with geometrical and mechanical problems.

6.4. Let us consider on the surface *the reciprocal base of the coordinate system* x^1, x^2

$$\boldsymbol{r}^\alpha = a^{\alpha\beta}\boldsymbol{r}_\beta \quad (\alpha = 1, 2), \tag{6.16}$$

where

$$a^{11} = \frac{a_{22}}{a}, \qquad a^{12} = a^{21} = -\frac{a_{12}}{a}, \qquad a^{22} = \frac{a_{11}}{a}, \tag{6.17}$$

$$a = a_{11}a_{22} - a_{12}^2. \tag{6.18}$$

It should be borne in mind that the covariant and contravariant metric tensors $a_{\alpha\beta}$ and $a^{\alpha\beta}$ are used for raising and lowering indices of components of tensors.

Differentiating the unit normal vector $\boldsymbol{n}$ we have

$$\boldsymbol{n}_\alpha = \frac{\partial \boldsymbol{n}}{\partial x_\alpha} = -b_{\alpha\beta}\boldsymbol{r}^\beta \quad (\alpha = 1, 2). \tag{6.19}$$

Since when the condition $\mathrm{K} \neq 0$ is satisfied the vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are not collinear, we obtain

$$\boldsymbol{r}^\alpha = -d^{\alpha\beta}\boldsymbol{n}_\beta, \qquad d^{\alpha\beta} = \frac{1}{\mathrm{K}} c^{\alpha\lambda} c^{\beta\gamma} b_{\lambda\gamma}, \tag{6.20}$$

where

$$c^{11} = c^{22} = 0 \text{ and } c^{12} = -c^{21} = \frac{1}{\sqrt{a}}. \tag{6.21}$$

Hence

$$a^{\alpha\beta} \equiv \boldsymbol{r}^{\alpha}\boldsymbol{r}^{\beta} = -\bar{d}^{\alpha\lambda}a^{\beta\gamma}\boldsymbol{n}_{\lambda}\boldsymbol{r}_{\gamma} = \bar{d}^{\alpha\lambda}a^{\beta\gamma}b_{\lambda\gamma} = \bar{d}^{\alpha\lambda}b_{\lambda}^{\beta}. \tag{6.22}$$

Considering these relations in the isometric-conjugate system and taking into account that

$$\bar{d}^{11} = \bar{d}^{22} = \frac{1}{\sqrt{a\mathrm{K}}}, \qquad \bar{d}^{12} = \bar{d}^{21} = 0, \tag{6.23}$$

we have

$$a^{11} = \frac{a_{22}}{a} = \frac{b_1^1}{\sqrt{a\mathrm{K}}}, \qquad a^{22} = \frac{a_{11}}{a} = \frac{b_2^2}{\sqrt{a\mathrm{K}}},$$

$$a^{12} = -\frac{a_{12}}{a} = \frac{b_1^2}{\sqrt{a\mathrm{K}}} \equiv \frac{b_2^1}{\sqrt{a\mathrm{K}}},$$

or

$$a^{+} \equiv a_{11} + a_{22} = \frac{b_1^1 + b_2^2}{\sqrt{\mathrm{K}}}\sqrt{a}, \tag{6.24}$$

$$a^{-} \equiv a_{11} - a_{22} + 2ia_{12} = \frac{1}{\sqrt{\mathrm{K}}}(b_2^2 - b_1^1 - 2ib_2^1)\sqrt{a}.$$

Taking into account that

$$b_1^1 + b_2^2 = 2\mathrm{H}, \qquad |b_2^2 - b_1^1 - 2ib_1^2| = \sqrt{(b_2^2 - b_1^1)^2 + 4(b_1^2)^2}$$
$$= |k_1 - k_2| = 2\sqrt{\mathrm{E}}, \tag{6.25}$$

where H is the mean curvature and E is the so called Euler's difference ($\mathrm{E} = \mathrm{H}^2 - \mathrm{K}^2$) we have

$$a^{+} = \frac{2\mathrm{H}}{\sqrt{\mathrm{K}}}\sqrt{a}, \qquad a^{-} = \frac{2\sqrt{a\mathrm{E}}}{\sqrt{\mathrm{K}}}e^{i\varphi}. \tag{6.26}$$

Since

$$a_{12} = \boldsymbol{r}_1\boldsymbol{r}_2 = \sqrt{a_{11}a_{22}}\cos\omega, \quad a = a_{11}a_{22}\sin^2\omega, \tag{6.27}$$

we obtain

$$a_{12} = \sqrt{a}\cot\omega, \tag{6.28}$$

where ω is the angle between the coordinate lines, and $0 < \omega < \pi$.

From (6.28), (6.27), (6.26) we obtain the relation

$$\cot\omega = \sqrt{\frac{\mathrm{E}}{\mathrm{K}}}\sin\psi\,, \tag{6.29}$$

—the expression for ψ in terms of ω. In view of (6.24), (6.26), (6.29) we have

$$a_{11} = \sqrt{\frac{a}{\mathrm{K}}}(\mathrm{H}+ \qquad\qquad a_{22} = \sqrt{\frac{a}{\mathrm{K}}}(\mathrm{H}-\sqrt{\mathrm{E}}\cos\psi),$$

$$a_{12} = a_{21} = \sqrt{\frac{a\mathrm{E}}{\mathrm{K}}}\sin\psi\,. \tag{6.30}$$

Hence, it follows that

$$a^{11} = \frac{\mathrm{H}-\sqrt{\mathrm{E}}\cos\psi}{\sqrt{a\mathrm{K}}}\,, \qquad a^{22} = \frac{\mathrm{H}+\sqrt{\mathrm{E}}\cos\psi}{\sqrt{a\mathrm{K}}}\,, \tag{6.31}$$

$$a^{12} = -a^{21} = -\frac{\sqrt{\mathrm{E}}}{\sqrt{a\mathrm{K}}}\sin\psi\,.$$

In view of these formulae the first fundamental quadratic form is given by

$$ds^2 = a_{\alpha\beta}dx^\alpha dx^\beta = \sqrt{\frac{a}{\mathrm{K}}}\left(\mathrm{H}+\sqrt{\mathrm{E}}\cos(\psi-2\vartheta)\right)d\sigma^2\,, \tag{6.32}$$

where ϑ is the slope of the arc $d\sigma$ on the z-plane corresponding to the element of arc ds of the curve on the surface.

From (6.32), (6.12) and (6.14) it follows that

$$k_s\left(\mathrm{H}+\sqrt{\mathrm{E}}\cos(\psi-2\vartheta)\right) = \mathrm{K}\,. \tag{6.33}$$

This relation holds for an arbitrary tangent direction at a fixed (chosen in an arbitrary way) point of the surface. Writing this relation for the principal directions and taking into account that ψ is independent of the choice of direction at the point under consideration we have

$$\cos(\psi-2\vartheta_1) = -1, \quad \cos(\psi-2\vartheta_2) = 1\,, \tag{6.34}$$

where ϑ_1 and ϑ_2 are slopes of the directions σ_1 and σ_2 on the z-plane, which are images of the principal directions s_1 and s_2 of the surface. The relation (6.34) implies that

$$\psi = 2\vartheta_2, \ \vartheta_2 = \vartheta_1 - \frac{\pi}{2}. \tag{6.35}$$

We have discovered, therefore, the geometric meaning of the function $\psi(x, y)$. *It is equal to twice the angle of slope of the direction on the z-plane corresponding to the principal direction s_2 of the surface. Moreover, to the principal directions s_1 and s_2 of the surface there correspond mutually perpendicular directions σ_1 and σ_2 on the z-plane, i.e. the net of the lines of curvature of the surface is mapped onto an orthogonal net of lines on the z-plane.*

Taking into account that $ds_2^2 = B^2 d\eta^2$, it follows from (6.35) and (6.15) that

$$e^{i\psi} = \left(\frac{dz}{d\sigma_2}\right)^2 = \left(\frac{ds_2}{d\sigma_2}\right)^2 \left(\frac{dz}{ds_2}\right)^2 = \frac{\sqrt{a\mathrm{K}}}{\mathrm{B}^2 k_2} \left(\frac{dz}{d\eta}\right)^2. \tag{6.36}$$

In other words, if the surface $\in D_{m+3,p}$ then $\psi \in D_{m+1,\, p}$.

Since $\mathrm{H} = \frac{1}{2}(k_1 + k_2)$, $\sqrt{\mathrm{E}} = \frac{1}{2}(k_1 - k_2)$ $(k_1 \geqslant k_2)$, then

$$\mathrm{H} - \sqrt{\mathrm{E}} \cos\psi = k_1 \sin^2\frac{\psi}{2} + k_2 \cos^2\frac{\psi}{2} = k''$$

$$\mathrm{H} + \sqrt{\mathrm{E}} \cos\psi = k_1 \cos^2\frac{\psi}{2} + k_2 \sin^2\frac{\psi}{2} = k', \tag{6.37}$$

$$\sqrt{\mathrm{E}} \sin\psi = \frac{k_1 - k_2}{2} \sin 2\frac{\psi}{2} = -\tau',$$

where k' and τ' are the normal curvature and the geodesic torsion of the surface in the direction making an angle $\frac{1}{2}\psi = \vartheta_2$ with the principal direction s_1, and k'' is the normal curvature in the perpendicular direction. Hence

$$a_{11} = k'\sqrt{\frac{a}{\mathrm{K}}}, \quad a_{22} = k''\sqrt{\frac{a}{\mathrm{K}}}, \quad a_{12} = -\tau'\sqrt{\frac{a}{\mathrm{K}}}, \tag{6.38}$$

$$a^{11} = \frac{k''}{\sqrt{a\mathrm{K}}}, \quad a^{22} = \frac{k'}{\sqrt{a\mathrm{K}}}, \quad a^{12} = \frac{\tau'}{\sqrt{a\mathrm{K}}}. \tag{6.39}$$

From (6.29) we have

$$\cos\psi = \pm \frac{\sqrt{\mathrm{H}^2 \sin^2 \omega - \mathrm{K}}}{\sqrt{\mathrm{E}} \sin \omega}. \tag{6.40}$$

The sign should be chosen in accordance with the relation (6.29). This immediately implies the inequality

$$1 \geqslant \sin \omega \geqslant \frac{\sqrt{\mathrm{K}}}{\mathrm{H}}, \tag{6.41}$$

showing that there exist two positive constants $0 < \delta_0 < < \delta_1 < \pi$ depending on the surface but independent of the choice of the isometric–conjugate net of lines, such that $0 < \delta_0 \leqslant \omega \leqslant \delta_1 < \pi$.

Let us now examine the distortion of the angle between two tangent directions in mapping on the plane of a surface of positive curvature, by means of homeomorphisms of the second fundamental quadratic form. Let $\boldsymbol{t}$ and $\boldsymbol{s}$ be two unit vectors tangent to the surface. Denoting by Ω the angle between them we have

$$\cos\Omega = \boldsymbol{ts} = a_{\alpha\beta} t^\alpha s^\beta, \qquad \sin\Omega = \boldsymbol{n}(\boldsymbol{t} \times \boldsymbol{s}) = c_{\alpha\beta} t^\alpha s^\beta, \tag{6.42}$$

where $c_{\alpha\beta}$ is a covariant tensor of rank two defined by the formulae

$$c_{11} = c_{22} = 0, \qquad c_{12} = -c_{21} = \sqrt{a}. \tag{6.43}$$

In deriving the second relation (6.42) we have used the formula

$$\boldsymbol{r}_\alpha \times \boldsymbol{r}_\beta = c_{\alpha\beta} \boldsymbol{n}. \tag{6.44}$$

By virtue of (6.30) and (6.15)

$$\begin{aligned} \cos\Omega &= \sqrt{\frac{k_s}{k_t}} \left\{ \cos(\vartheta_\sigma - \vartheta_\tau) - \frac{\tau_t}{\sqrt{\mathrm{K}}} \sin(\vartheta_\sigma - \vartheta_\tau) \right\}, \\ \sin\Omega &= \sqrt{\frac{k_t k_s}{\mathrm{K}}} \sin(\vartheta_\sigma - \vartheta_\tau), \end{aligned} \tag{6.45}$$

where ϑ_τ and ϑ_σ are slopes of the directions τ and σ on the plane z which are the images of $\boldsymbol{t}$ and $\boldsymbol{s}$ (in the mapping under consideration).

If $\boldsymbol{t}$ and $\boldsymbol{s}$ coincide with the principal directions $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ of the surface, then $k_t = k_1$, $k_s = k_2$ and from (6.45) we again obtain the relations (6.35). If $\boldsymbol{t}$ and $\boldsymbol{s}$ are mutually perpendicular, then from (6.45) after simple transformations we obtain

$$\sin(\vartheta_\sigma - \vartheta_\tau) = \sqrt{\frac{\mathrm{K}}{k_t k_s}}, \qquad \cos(\vartheta_\sigma - \vartheta_\tau) = \frac{\tau_t}{\sqrt{k_s k_t}}. \tag{6.46}$$

These relations are equivalent to the following relation which may also be derived directly:

$$\frac{d\bar{z}}{dt}\frac{dz}{ds} = \frac{i}{\sqrt{a}} + \frac{\tau_t}{\sqrt{a\mathrm{K}}}. \tag{6.47}$$

From (6.45) we have the estimate

$$1 \geqslant \sin(\vartheta_\sigma - \vartheta_\tau) \geqslant \frac{\sqrt{\mathrm{K}}}{\mathrm{H}}. \tag{6.47a}$$

As should be expected, the formulae (6.45), (6.46) and (6.47) are independent of the particular choice of the homeomorphism of the form II.

It is known that two tangent directions $\boldsymbol{t}$ and $\boldsymbol{s}$ are said to be conjugate if they satisfy the relation

$$\boldsymbol{t}\boldsymbol{n}_s = b_{\alpha\beta} t^\alpha s^\beta = 0 .$$

This relation referred to an isometric–conjugate co-ordinate system has the form

$$s^1 t^1 + s^2 t^2 \equiv \mathrm{Re}\left[\frac{dz}{ds}\frac{\overline{dz}}{dt}\right] = 0 .$$

which is equivalent to the relation

$$\cos(\vartheta_\sigma - \vartheta_\tau) = 0 .$$

i.e. the directions σ and τ on the z-plane corresponding to the conjugate directions $\boldsymbol{s}$ and $\boldsymbol{t}$ of the surface, are perpendicular.

Thus, *an isometric–conjugate coordinate system transforms any net of conjugate lines of the surface onto an orthogonal net of lines on the z-plane.*

6.5. In an isometric–conjugate coordinate system we have the conditions $b_{11} = b_{22}$, $b_{12} = 0$, i.e.

$$\boldsymbol{n}\boldsymbol{r}_{11} = \boldsymbol{n}\boldsymbol{r}_{22}, \quad \boldsymbol{n}\boldsymbol{r}_{12} = 0 , \tag{6.48}$$

or, in complex notation,

$$\boldsymbol{n}\boldsymbol{r}_{\bar{z}\bar{z}} \equiv \frac{1}{4}\boldsymbol{n}\,(\boldsymbol{r}_{11}-\boldsymbol{r}_{22}+2i\boldsymbol{r}_{12}) = 0 . \tag{6.49}$$

It follows that the position vector $\boldsymbol{r}$ satisfies the equation

$$\boldsymbol{r}_{\bar{z}\bar{z}}+A\boldsymbol{r}_{\bar{z}}+B\boldsymbol{r}_{z} = 0 . \tag{6.50}$$

Making use of the formulae of Gauss

$$\boldsymbol{r}_{\alpha\beta} = \Gamma^{\lambda}_{\alpha\beta}\boldsymbol{r}_{\lambda}+b_{\alpha\beta}\boldsymbol{n} , \tag{6.51}$$

where $\Gamma^{\lambda}_{\alpha\beta}$ are Christoffel's symbols of the second kind and $b_{\alpha\beta}$ are the coefficients of the second fundamental quadratic form, it is readily found that

$$\begin{aligned} A &= \frac{1}{4}(\Gamma^{1}_{22}-\Gamma^{1}_{11}-2\Gamma^{2}_{12})-\frac{i}{4}(\Gamma^{2}_{22}-\Gamma^{2}_{11}+2\Gamma^{1}_{12}) \\ &\equiv -(\boldsymbol{r}^{1}-i\boldsymbol{r}^{2})\boldsymbol{r}_{\bar{z}\bar{z}} , \\ B &= \frac{1}{4}(\Gamma^{1}_{22}-\Gamma^{1}_{11}+2\Gamma^{2}_{12})+\frac{i}{4}(\Gamma^{2}_{22}-\Gamma^{2}_{11}-2\Gamma^{1}_{12}) \\ &\equiv -(\boldsymbol{r}^{1}+i\boldsymbol{r}^{2})\,\boldsymbol{r}_{\bar{z}\bar{z}} . \end{aligned} \tag{6.52}$$

Equation (6.50) is the complex form of the familiar Laplace's equation for the Cartesian coordinates of a point of the surface, referred to an isometric–conjugate coordinate system.

We have also

$$\boldsymbol{r}_{z\bar{z}}+C\boldsymbol{r}_{\bar{z}}+\bar{C}\boldsymbol{r}_{z}-\frac{1}{2}\Lambda\boldsymbol{n} = 0 , \tag{6.53}$$

where

$$\Lambda = \sqrt{a\mathrm{K}}, \qquad C = -\frac{1}{4}\left(\Gamma^{1}_{11}+\Gamma^{1}_{22}\right)+\frac{i}{4}\left(\Gamma^{2}_{11}+\Gamma^{2}_{22}\right) . \tag{6.54}$$

Making use now of the relations

$$a^+ = 4\boldsymbol{r}_z\boldsymbol{r}_{\bar z}\ , \qquad a^- = 4\boldsymbol{r}_{\bar z}\boldsymbol{r}_{\bar z}\ , \qquad a = \frac{1}{4}\left(a^{+2} - |a^-|^2\right), \tag{6.55}$$

we obtain

$$\frac{\partial a^+}{\partial \bar z} = 4\boldsymbol{r}_{z\bar z}\boldsymbol{r}_{\bar z} + 4\boldsymbol{r}_z\boldsymbol{r}_{\bar z\bar z}\,, \qquad \frac{\partial a^-}{\partial \bar z} = 8\boldsymbol{r}_{\bar z}\boldsymbol{r}_{\bar z\bar z}, \qquad \frac{\partial a^-}{\partial z} = 8\boldsymbol{r}_{\bar z}\boldsymbol{r}_{z\bar z}\,.$$

Hence, by means of the relations (6.50) and (6.53) we have

$$\begin{gathered}\frac{\partial a^+}{\partial \bar z} = -(A+\bar C)a^+ - Ca^- - B\bar a^-,\\ \frac{\partial a^-}{\partial \bar z} = -2Aa^- - 2Ba^+, \qquad \frac{\partial a^-}{\partial z} = -2Ca^- - 2\bar C a^+\,.\end{gathered} \tag{6.56}$$

These relations imply that

$$\begin{aligned} A &= -\frac{\partial \ln\sqrt{a}}{\partial \bar z} + \frac{1}{8a}\left(a^+\frac{\partial a^-}{\partial z} - a^-\frac{\partial \bar a^-}{\partial \bar z}\right) = -\frac{\partial \ln\sqrt{a}}{\partial \bar z} - \bar C\,,\\ B &= \frac{1}{8a}\left(2a^-\frac{\partial a^+}{\partial \bar z} - a^+\frac{\partial a^-}{\partial \bar z} - a^-\frac{\partial a^-}{\partial z}\right),\\ C &= -\frac{1}{8a}\left(a^+\frac{\partial \bar a^-}{\partial \bar z} - \bar a\frac{\partial a^-}{\partial z}\right).\end{aligned} \tag{6.57}$$

5.6. On the basis of the relations (6.19)

$$\boldsymbol{n}_{\bar z} \equiv \frac{1}{2}(\boldsymbol{n}_1 + i\boldsymbol{n}_2) = -\frac{1}{2}\Lambda(\boldsymbol{r}^1 + i\boldsymbol{r}^2) = -\frac{\sqrt{a\mathrm{K}}}{2}(a^{1\lambda} + ia^{2\lambda})\boldsymbol{r}_\lambda\,.$$

or, by virtue of (6.31),

$$\boldsymbol{n}_{\bar z} = -\mathrm{H}\boldsymbol{r}_{\bar z} + \sqrt{\mathrm{E}}\,e^{i\varphi}\boldsymbol{r}_z\,. \tag{6.58}$$

Differentiating (6.50) and (6.53) with respect to z and $\bar z$, respectively, making use of the same relations and the formula (6.58) we obtain

$$\boldsymbol{r}_{z\bar z\bar z} + (A_z - AC - B\bar B)\boldsymbol{r}_{\bar z} + (B_z - A\bar C - \bar A B)\boldsymbol{r}_z + \frac{1}{2}A\Lambda\boldsymbol{n} = 0\,,$$

$$\begin{aligned}&\boldsymbol{r}_{z\bar z\bar z} + \left(C_z - AC - C\bar C + \frac{1}{2}\mathrm{H}\Lambda\right)\boldsymbol{r}_{\bar z} +\\ &\quad + \left(\bar C_{\bar z} - BC - \bar C^2 - \frac{1}{2}\Lambda\sqrt{\mathrm{E}}\,e^{i\varphi}\right)\boldsymbol{r}_z + \frac{1}{2}(C\Lambda - \Lambda_{\bar z})\boldsymbol{n} = 0\,.\end{aligned}$$

The last relations imply the following:

$$A_{\bar z}+(A-\bar C)A=0\,, \tag{6.59}$$

$$A_z-\bar C_{\bar z}-B\bar B+C\bar C-\frac{1}{2}\mathrm{H}A=0\,, \tag{6.60}$$

$$-B_z-\bar C_{\bar z}+(\bar C-A)\bar C+(C-\bar A)B+\frac{1}{2}A\sqrt{\mathrm{E}}\,e^{i\psi}=0\,. \tag{6.61}$$

The equation (6.59) represents the complex form of Codazzi's system of equations. From (6.59) and (6.57) we have

$$A=-\frac{\partial}{\partial\bar z}\ln\sqrt{a\sqrt{\mathrm{K}}}\,, \tag{6.62}$$

$$C=\frac{\partial}{\partial z}\ln\mathrm{K}^{1/4}\,. \tag{6.63}$$

Equations (6.60) and (6.61) are equivalent to the equation of Gauss. Taking into account (6.62), (6.63), and (6.26) we see that these equations may be written in the form

$$\frac{1}{2}\mathrm{H}A\equiv\frac{\mathrm{K}a^+}{4}=\frac{\partial^2\ln A}{\partial z\partial\bar z}+C\bar C-B\bar B\,, \tag{6.64}$$

$$\frac{1}{2}\sqrt{\mathrm{E}}\,Ae^{i\psi}\equiv\frac{\mathrm{K}a^-}{4}=A\frac{\partial}{\partial\bar z}\left(\frac{\bar C}{A}\right)-\frac{1}{A}\frac{\partial}{\partial z}(AB)\,; \tag{6.65}$$

In view of (6.62) we obtain from the second formula (6.57)

$$B=-\frac{a\sqrt{\mathrm{K}}}{2a^+}\frac{\partial}{\partial\bar z}\left(\frac{a^-}{a\sqrt{\mathrm{K}}}\right)\equiv-\frac{\mathrm{K}\sqrt{a}}{2\mathrm{H}}\left(\frac{\sqrt{\mathrm{E}}}{\mathrm{K}\sqrt{a}}e^{-i\psi}\right)_{\bar z}. \tag{6.66}$$

According to (6.63) the third relation (6.57) may be put in the form

$$\frac{\partial}{\partial\bar z}(\sqrt{a\mathrm{E}}\,e^{-i\psi})+\frac{\mathrm{H}\mathrm{K}_z}{2\mathrm{K}}\sqrt{a}=0\,. \tag{6.67}$$

It is easy to verify that the relation (6.57) follows from the relations (6.66) and (6.67).

The relation (6.67) may be regarded as an equation determining the real functions $\sqrt{a}$ and ψ if the mean

curvature H and the principal curvature K of the surface are known. Having determined $\sqrt{a}$ and ψ from (6.67) we can then use the formulae (6.14) and (6.30) in order to calculate the coefficients of the second and the first fundamental quadratic forms and Christoffel's symbols of the second kind corresponding to the isometric-conjugate coordinate system under consideration.

We can obtain the following expression for B by making use of (6.66) and (6.67)

$$B = -\left(\operatorname{Arch}\frac{\mathrm{H}}{\sqrt{\mathrm{K}}}\right)_{\bar{z}} e^{i\psi} - \frac{\mathrm{K}_z}{4\mathrm{K}} e^{2i\psi}. \tag{6.68}$$

By means of the formulae (6.52), (6.54), (6.62), (6.63) and (6.68) it is easy to derive the following formulae for Christoffel's symbols of the second kind:

$$\begin{aligned} &\Gamma^1_{11} = \frac{\partial}{\partial x}\ln\sqrt{a} - \Gamma^2_{12}, \qquad \Gamma^1_{22} = -\frac{\partial}{\partial x}\ln\sqrt{a\mathrm{K}} + \Gamma^2_{12},\\ &\Gamma^2_{11} = -\frac{\partial}{\partial y}\ln\sqrt{a\mathrm{K}} + \Gamma^1_{12}, \qquad \Gamma^2_{22} = \frac{\partial}{\partial y}\ln\sqrt{a} - \Gamma^1_{12},\\ &\Gamma^2_{12} - i\Gamma^1_{12} = \left(\ln\sqrt{a\sqrt{\mathrm{K}}}\right)_z - \left(\operatorname{arch}\frac{\mathrm{H}}{\sqrt{\mathrm{K}}}\right)_{\bar{z}} e^{i\psi} - \frac{\mathrm{K}_z}{4\mathrm{K}} e^{2i\psi}. \end{aligned} \tag{6.69}$$

6.7. Introducing the covariant derivatives of the complex covariant vector $w_* = u_1 + iu_2$ with respect to $\bar{z}$ and z we have

$$\begin{aligned} \nabla_{\bar{z}} w_* &= \frac{1}{2}(\nabla_1 + i\nabla_2)(u_1 + iu_2) \equiv w_{*\bar{z}} + Aw_* + \bar{B}\bar{w}_*,\\ \Delta_z w_* &\equiv \frac{1}{2}(\nabla_1 - i\nabla_2)(u_1 + iu_2) \equiv w_{*z} + Cw_* + \bar{C}\bar{w}_*. \end{aligned} \tag{6.70}$$

If $w^* = u^1 + iu^2$ is a complex contravariant vector, then

$$\nabla_{\bar{z}} w^* \equiv \frac{1}{2}(\nabla_1 + i\nabla_2)(u^1 + iu^2) = w^*_{\bar{z}} - \bar{C}w^* - B\bar{w}^*, \tag{6.71}$$

$$\nabla_z w^* = \frac{1}{2}(\nabla_1 - i\nabla_2)(u^1 + iu^2) = w_z - \bar{A}w^* - \bar{C}\bar{w}^*. \tag{6.72}$$

6.8. Suppose that the surface belongs to the class C_α^{m+2}. Then H, K, E belong to the class C_α^m. Moreover, according to the results proved above, $\Lambda = \sqrt{a\mathrm{K}} \in C_\alpha^m$, i.e. $a \in C_\alpha^m$. Then it follows from the relation (6.67) that $\psi \in C_\alpha^m$. Considering now the formulae (6.62), (6.63) and (6.68) we obtain

$$A, B, C \in C_\alpha^{m-1}, \quad \text{if} \quad m \geqslant 1 . \tag{6.73}$$

We now assume that the surface belongs to the class $D_{m+3,p}$, $p > 2$. Then $\mathrm{H}, \mathrm{K}, \mathrm{E}, a, \psi \in D_{m+1,p}$ and according to (6.62), (6.63) and (6.68) we have

$$A, B, C \in D_{m,p} . \tag{6.74}$$

In particular, the condition (6.73) is satisfied in this case. Also

$$\mathrm{H}, \mathrm{K}, \mathrm{E}, a, \psi \in C_\alpha^m, \qquad \alpha = \frac{p-2}{p} . \tag{6.75}$$

6.9. As a result of a conformal mapping of the first kind $z_* = \Phi(z)$ (or of the second kind $z_* = \overline{\Phi(z)}$) an isometric–conjugate net of lines is transformed into a similar net. In view of (6.55) we have the following formulae of the transformation

$$a^+ = a_*^+ |\Phi'(z)|^2, \quad a^- = a_*^- \Phi'(z)^2 , \tag{6.76}$$

$$a = a_* |\Phi'(z)|^4 , \tag{6.77}$$

$$\arg a^- = \arg a_*^- - 2 \arg \Phi'(z) . \tag{6.78}$$

It follows that

$$\psi_* = \psi + 2\chi, \qquad \text{where} \qquad \chi = \arg \Phi'(z) . \tag{6.79}$$

The last formula follows also from the formula (6.35), since in a conformal mapping the angle ϑ_2 acquires an increment χ.

It follows from (6.76) that

$$a^+ dz d\bar{z} \equiv a_*^+ dz_* dz_*^- , \qquad a^- d\bar{z}^2 = a_*^- d\bar{z}_*^2 . \tag{6.80}$$

Thus, *the first fundamental quadratic form of the surface given by*

$$\mathrm{I} \equiv ds^2 = \frac{1}{2} a^{+} dz\, d\bar{z} + \frac{1}{4} a^{-} d\bar{z}^2 + \frac{1}{4} \overline{a^{-}}\, dz^2 , \qquad (6.81)$$

represents a sum of three quadratic forms invariant with respect to conformal mappings of the first and the second kind. These quadratic forms are independent of the particular choice of the coordinate system.

The arbitrariness in the choice of the isometric–conjugate net of coordinate lines can be used to impose certain additional conditions on the coefficients of the forms I and II. For instance, *one can obtain* $\psi \equiv 0$ *on the boundary curve or on its part.* (In fact, it follows from the relation (6.79) that ψ is determined to within an additive term 2χ which is a harmonic function). Then $a_{12} = 0$ on the boundary curve, i.e. *on this curve the isometric–conjugate net of lines under consideration coincides with the net of lines of curvature of the surface* (it should also be borne in mind that $b_{12} = 0$ everywhere).

If we now consider the formulae (6.62), (6.63), and (6.68), then in view of the formulae (6.69) and (6.71) we obtain the following transformation formulae for the quantities A, B, C:

$$\left.\begin{aligned} A &= A_* \Phi'(z) - \frac{\overline{\Phi''(z)}}{\overline{\Phi'(z)}} , \\ B = B_* \frac{\overline{\Phi'(z)^2}}{\Phi'(z)} ,& \qquad C = C_* \Phi'(z) . \end{aligned}\right\} \qquad (6.82)$$

It follows from the last formulae that *if B and C vanish in one arbitrary isometric–conjugate coordinate system, then they vanish in every such coordinate system.*

Let us observe *that $B \equiv C \equiv 0$ for a spherical surface and only for such a surface.* This result follows immediately form (6.63) and (6.68) if we take into account that $E \equiv 0$ only for a sphere.

In the case of a complete ovaloid (i.e. for a closed surface of positive curvature) the variable z ranges over the

entire plane. In this case only linear fractional functions may be taken for $\Phi(z)$, namely functions of the form

$$\Phi(z)=\frac{\alpha z+\gamma}{\beta z+\delta}, \qquad \alpha\delta-\beta\gamma\neq 0\,. \tag{6.83}$$

Taking into account this result and making use of the formulae (6.76), (6.77), (6.82) we obtain the following asymptotic representations near the point at infinity:

$$a^{+},\, a^{-},\, \sqrt{a}=O(|z|^{-4})\,, \tag{6.84}$$

$$A=O(|z|^{-1}), \qquad B,\, C=O(|z|^{-2}). \tag{6.85}$$

This result implies that

$$B,\, C\in L_{p,2}(E),\ p>2 \tag{6.86}$$

(see Ch. I, §1.5).

6.10. Let the surface S of positive curvature under consideration belong to the class $D_{m+3,p}$, $p>2$. Then it certainly belongs to the class C_{α}^{m+2}, $\alpha=\frac{p-2}{p}$. Let us consider a piecewise smooth curve L on the surface S. The angle between the tangent to the curve L at one of its points and the principal direction $\boldsymbol{s}_1$ at this point will be denoted by φ. Let L' be a smooth arc of the curve L (the ends are assumed to belong to L'); then φ as the function of the length of arc s is, obviously, continuous on L'. Let us assume that $\varphi\in C_{\alpha}^{m'}(L')$ where $m'\leqslant m$. Referring for definitness the surface to the coordinate system of lines of curvature we have

$$\mathrm{I}=A^2d\xi^2+B^2d\eta^2, \qquad \mathrm{II}=k s\, ds^2\equiv A^2k_1d\xi^2+B^2k_2d\eta_2\,, \tag{6.87}$$

$$\frac{d\xi}{ds}=\frac{\cos\varphi}{\mathrm{A}}, \qquad \frac{d\eta}{ds}=\frac{\sin\varphi}{\mathrm{B}} \quad \text{(along } L')\,. \tag{6.88}$$

Since A and $B\in C_{\alpha}^{m+1}$, then it follows at once from (6.88) that $\xi(s),\eta(s)\in C_{\alpha}^{m'+1}(L')$ (we assume that the whole line L' lies inside S).

Introducing on the surface S an isometric–conjugate coordinate system

$$x = x(\xi, \eta), \quad y = y(\xi, \eta), \tag{6.89}$$

corresponding to a global homeomorphism of the quadratic form $\mathrm{II} = A^2 k_1 d\xi^2 + B^2 k_2 d\eta^2$ we obtain

$$\mathrm{II} = k_s ds^2 = \Lambda(dx^2 + dy^2), \quad \Lambda = \sqrt{a\mathrm{K}}. \tag{6.90}$$

Let Γ and Γ' be the images of the curves L and L' on the plane $z = x + iy$. Let us now examine the degree of smoothness of the curve Γ'. Denoting by ds and $d\sigma$ the corresponding arcs of the curves L' and Γ' and taking into account that

$$ds^2 = \frac{\Lambda}{k_s} d\sigma^2 = \frac{\sqrt{a\mathrm{K}}}{\mathrm{H} + \sqrt{\mathrm{E}}\cos 2\varphi} d\sigma^2, \tag{6.91}$$

we have the relation

$$\sigma(s) = \int_{s_0}^{s} \sqrt{\frac{\mathrm{H} + \sqrt{\mathrm{E}}\cos 2\varphi}{\sqrt{a\mathrm{K}}}}\, ds, \tag{6.92}$$

which shows that $\sigma(s)$ (i.e. the length of an arc of the curve Γ') is a function of s (the length of an arc of the curve L') belonging to the class $C_a^{m'+1}$. Regarding now s as a function of σ, $s(\sigma)$, it is readily observed that $s(\sigma)$ is also a function of the class $C_a^{m'+1}$. Denoting by ϑ the angle of slope of the tangent to Γ' we have

$$\begin{aligned} \cos\vartheta &= \frac{dx}{d\sigma} = \left(x_\xi \frac{d\xi}{ds} + x_\eta \frac{d\eta}{ds}\right)\frac{ds}{d\sigma}, \\ \sin\vartheta &= \frac{dy}{d\sigma} = \left(y_\xi \frac{d\xi}{ds} + y_\eta \frac{d\eta}{ds}\right)\frac{ds}{d\sigma}. \end{aligned} \tag{6.93}$$

Therefore $\vartheta \in C_a^{m'}(\Gamma)'$, i.e. $x(\sigma)$ and $y(\sigma) \in C_a^{m'+1}$; it should be taken into account here that $x_\xi, y_\xi, x_\eta, y_\eta, \in C_a^m$, $A, B \in C_a^{m+1}(D_{m+2,p})$. Summing up the results we infer the following:

Let there lie on the interior part of a surface of positive curvature of the class $D_{m+3,p}$, $p > 2$, *a smooth curve* L' *belonging to the class* $C_\alpha^{m'+1}$, *i.e. its coordinates* $\xi(s)$, $\eta(s) \in C_\alpha^{m'+1}$ *where* $\alpha = \dfrac{p-2}{p}$, $m' \leqslant m$. *By introducing on such a surface an isometric–conjugate coordinate system corresponding to a global homeomorphism* $x = x(\xi, \eta)$, $y = y(\xi, \eta)$ *of the form* $II = A^2 k_1 d\xi^2 + B^2 k_2 d\eta^2$, *the curve* L' *is mapped onto a curve* Γ' *of the plane* z, *which also belongs to the class* $C_\alpha^{m'+1}$, *i.e. its coordinates* $x(\sigma)$ *and* $y(\sigma)$ *regarded as functions of the length of the arc of the curve* Γ' *belong to the class* $C_\alpha^{m'+1}$, $m' \leqslant m$.

Let the curve L contain a corner point lying on the interior part of the surface S, the adjacent arcs L' and L'' belonging, as before, to the class $C_\alpha^{m'+1}$, $m' \leqslant m$. Then their images Γ' and Γ'' on the plane z also belong to the class $C_\alpha^{m'+1}$ and constitute a piecewise smooth curve Γ representing the homeomorphic image of the curve L. In this case the angle between the curves Γ' and Γ'' is uniquely determined by the angle between the curves L' and L'', and it is evident that this angle is independent of the choice of the global homeomorphism of the form II which defines the isometric–conjugate net of lines on the surface. This result follows from the fact that a transformation from one such system to another system of this kind is accomplished by means of a conformal mapping of the first or of the second kind. It should be borne in mind that the considered corner point lies strictly inside the surface.

Let us now assume that the whole surface S together with its boundary is a strictly interior part of a surface S_0 of the class $D_{m+3,p}$, $p > 0$, $m \geqslant 0$.

Let the boundary of S, which will be denoted by L, consist of a finite number of piecewise smooth curves $L_0, L_1, \ldots, L_k$, the smooth pieces of the curves belonging to the class $C_\alpha^{m'}$, $m' \leqslant m$.

If we consider on S_0 an isometric–conjugate net of coordinate lines $x = \text{const.}$, $y = \text{const.}$ corresponding to a homeomorphism $x = x(\xi, \eta)$, $y = y(\xi, \eta)$ of the form $\mathrm{II} = A^2 k_1 d\xi^2 + B^2 k_2 d\eta^2$ global with respect to S_0, the surface S together with its boundary L is mapped homeomorphically onto a closed domain $G+\Gamma$, the homeomorphism taking place between the domains S and G, and between their boundaries L and Γ. We have seen already that the nature of the curves L and Γ is the same, i.e. if $L_0, L_1, \ldots, L_k$ are smooth curves of the class $C_\alpha^{m'}$, $m' \leqslant m$, then their images $\Gamma_0, \Gamma_1, \ldots, \Gamma_k$ also belong to the class $C_\alpha^{m'}$. If, now, the curves $L_0, L_1, \ldots, L_k$ possess corner points, then to these corner points (and only to them) there correspond corner points on $\Gamma_0, \Gamma_1, \ldots, \Gamma_k$.

§7. Reduction of equations of elliptic type to the canonical form

In this paragraph we shall apply the results obtained above to the problem of reduction to the canonical form of a system of partial differential equations of the first order of the elliptic type, and of an elliptic equation of the second order, in the case of two independent variables. We shall start from the investigation of an elliptic system of equations.

7.1. Let us consider a system of equations of the form

$$\begin{aligned} a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y} + b_{11}\frac{\partial v}{\partial x} + b_{12}\frac{\partial v}{\partial y} + a_1 u + b_1 v = f_1 \,, \\ a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y} + b_{21}\frac{\partial v}{\partial x} + b_{22}\frac{\partial v}{\partial y} + a_2 u + b_2 v = f_2 \,, \end{aligned} \tag{7.1}$$

where a_{ik}, b_{ik}, a_i, b_i, f_i are known functions of two independent variables in a domain G.

Let us associate with this system the quadratic form

$$F \equiv a\,dx^2 + 2b\,dx\,dy + c\,dy^2 \,, \tag{7.2}$$

where

$$\left.\begin{aligned} a &= \frac{a_{12}b_{22}-a_{22}b_{12}}{\varDelta}, \qquad c = \frac{a_{11}b_{21}-a_{21}b_{11}}{\varDelta}, \\ b &= -\frac{1}{2\varDelta}(a_{11}b_{22}-a_{21}b_{21}+a_{12}b_{21}-a_{22}b_{11}), \end{aligned}\right\} \tag{7.3}$$

$$\begin{aligned} \varDelta = (a_{12}b_{22}-a_{22}b_{12})(a_{11}b_{21}-a_{21}b_{11}) - \\ -\tfrac{1}{4}\,a_{11}b_{22}-a_{12}b_{12}+a_{12}b_{21}-a_{22}b_{11})^2 . \end{aligned} \tag{7.4}$$

The quadratic form F is positive definite if and only if $a > 0$, $\varDelta > 0$. In this case the system of equations (7.1) is said to be of elliptic type. The condition $\varDelta > 0$ implies that $b_{11}b_{22}-b_{12}^2 \neq 0$; in fact, if not, then $b_{11} = \mu b_{12}$, $b_{21} = \mu b_{22}$ and

$$\varDelta = -\tfrac{1}{4}[(a_{11}-\mu a_{12})b_{22}-(a_{21}-\mu a_{22}]^2 \leqslant 0 .$$

Therefore the system (7.1) can always be reduced to the form

$$\begin{aligned} -v_y+a_{11}u_x+a_{12}u_y+a_1u+b_1v &= f_1 , \\ v_x+a_{21}u_x+a_{22}u_y+a_2u+b_2v &= f_2 . \end{aligned} \tag{7.5}$$

In this case the condition of ellipticity has the form

$$a_{11} > 0\,(a_{22} > 0), \qquad \varDelta = a_{11}a_{22}-\tfrac{1}{4}(a_{12}+a_{21})^2 > 0 . \tag{7.6}$$

If $a_{12} = -a_{21}$, $a_{11} = a_{22}$ we have the system

$$\begin{aligned} -v_y+a_{11}u^x+a_{12}u_y+a_1u+b_1v &= f_1 , \\ v_x-a_{12}u_x+a_{11}u_y+a_2u+b_2v &= f_2 . \end{aligned} \tag{7.7}$$

which, after the substitution *

$$U = a_{11}u, \qquad V = v+a_{12}u, \qquad a_{11} > 0 \tag{7.8}$$

is reduced to the canonical form

$$\begin{aligned} U_x-V_y+a_*U+b_*V &= f , \\ U_y+V_x+c_*U+d_*V &= g . \end{aligned} \tag{7.9}$$

* We assume that $a_{11}, a_{12} \in D_{1,p}$, $p > 2$.

We shall now show that a system of the general form (7.5) may also be reduced to the form (7.9) by means of a change of independent variables

$$\xi = \xi(x, y), \qquad \eta = \eta(x, y), \qquad \xi_x\eta_y - \xi_y\eta_x = J \neq 0 \,. \tag{7.10}$$

As a result of such a change the system (7.5) assumes the form

$$\begin{aligned} -v_\xi\xi_y - v_\eta\eta_y + (a_{11}\xi_x + a_{12}\xi_y)\, u_\xi& \\ + (a_{11}\eta_x + a_{12}\eta_y)\, u_\eta + a_1 u + b_1 v &= f_1 \,, \\ v_\xi\xi_x + v_\eta\eta_x + (a_{21}\xi_x + a_{22}\xi_y)\, u_\xi& \\ + (a_{21}\eta_x + a_{22}\eta_y)\, u_\eta + c_1 u + d_1 v &= f_2 \,. \end{aligned}$$

Solving this system with respect to v_ξ and v_η we obtain

$$\begin{aligned} -v_\eta + a'_{11}u_\xi + a'_{12}u_\eta + a'_1 u + b'_1 v &= f'_1 \,, \\ v_\xi + a'_{21}u_\xi + a'_{22}u_\eta + a'_2 u + b'_2 v &= f'_2 \,, \end{aligned}$$

where

$$a'_{11} = \frac{1}{J}\left(a_{11}\xi_x^2 + (a_{12} + a_{21})\,\xi_x\xi_y + a_{22}\xi_y^2\right) ,$$

$$a'_{12} = \frac{1}{J}\,(a_{11}\xi_x\eta_x + a_{12}\xi_x\eta_y + a_{21}\xi_y\eta_x + a_{22}\xi_y\eta_y) \,,$$

$$a'_{21} = \frac{1}{J}\,(a_{11}\xi_x\eta_x + a_{12}\xi_y\eta_x + a_{21}\xi_x\eta_y + a_{22}\xi_y\eta_y) \,,$$

$$a'_{22} = \frac{1}{J}\left(a_{11}\eta_x^2 + (a_{12} + a_{21})\,\eta_x\eta_y + a_{22}\eta_y^2\right) .$$

Suppose now that the considered transformation (7.10) is subject to the condition $a'_{12} = -a'_{21}$, $a'_{11} = a'_{22}$, i.e.

$$\begin{aligned} a_{11}(\xi_x^2 - \eta_x^2) + (a_{12} + a_{12})\,(\xi_x\xi_y - \eta_x\eta_y) + a_{22}(\xi_x^2 - \eta_y^2) &= 0 \,, \\ 2a_{11}\xi_x\eta_x + (a_{12} + a_{21})\,(\xi_x\eta_y + \xi_y\eta_x) + 2a_{22}\xi_y\eta_y &= 0 \,. \end{aligned} \tag{7.11}$$

Introducing the complex function $\zeta = \xi + i\eta$ the last system of equations can be written in the form of one equation

$$a_{11}\zeta_x^2 + (a_{12} + a_{21})\,\zeta_x\zeta_y + a_{22}\zeta_y^2 = 0 \,, \tag{7.12}$$

which is satisfied by the solutions of the equation

$$a_{11}\zeta_x + \left(\tfrac{1}{2}(a_{12}+a_{21}) + i\sqrt{\Delta}\right)\zeta_y = 0 \tag{7.13}$$

or, in the complex notation,

$$\zeta_{\bar z} - q(z)\zeta_z = 0\,, \quad q = \frac{a_{11} - \sqrt{\Delta} + \frac{i}{2}(a_{12}+a_{21})}{a_{11} + \sqrt{\Delta} - \frac{i}{2}(a_{12}+a_{21})}\,. \tag{7.14}$$

Let us assume that in the closed domain under consideration the following condition is satisfied: $\Delta \geqslant \Delta_0 > 0$, $\Delta_0 = \text{const}$. Then $|q(z)| \leqslant q_0 < 1$ in $\bar G$. If, moreover, $q(z) \in D_{m+1,p}(G)$, $p > 2$, $m \geqslant 0$, then there exists a global homeomorphism $\zeta(z) = \xi(x, y) + i\eta(x, y)$ of the equation (7.14) belonging to the class $D_{m+2,p}(G)$.

According to the relation (7.12) we have

$$2a'_{12} = a'_{12} - a'_{21} = \frac{1}{J}(a_{12} - a_{21})(\xi_x\eta_y - \xi_y\eta_x) = a_{12} - a_{21}\,, \tag{7.15}$$

$$2a'_{11} = a'_{11} + a'_{22} = \frac{1}{J}\left[\left(a_{11}\zeta_x + \frac{a_{12}+a_{21}}{2}\zeta_y\right)\bar\zeta_x + \left(a_{22}\zeta_y + \frac{a_{12}+a_{21}}{2}\zeta_x\right)\bar\zeta_y\right]. \tag{7.16}$$

From (7.13) it follows that

$$a_{11}\zeta_x + \frac{a_{12}+a_{21}}{2}\zeta_y = -i\sqrt{\Delta}\zeta_y\,,$$

$$a_{22}\zeta_y + \frac{a_{12}+a_{21}}{2}\zeta_x = i\sqrt{\Delta}\zeta_x\,.$$

Introducing these expressions into (7.16) we obtain

$$a'_{11} = \frac{i\sqrt{\Delta}}{2J}(\zeta_x\bar\zeta_y - \bar\zeta_x\zeta_y) = \sqrt{\Delta}\,. \tag{7.17}$$

Thus, the system (7.5) assumes the form

$$-v_\eta + \sqrt{\Delta}\,u_\xi + \frac{a_{12}-a_{21}}{2}u_\eta + a'_1u + b'_1v = f_1\,,$$

$$v_\xi - \frac{a_{12}-a_{21}}{2}u_\xi + \sqrt{\Delta}\,u_\eta + a'_2u + b'_2v = f_2\,,$$

or, introducing new functions

$$U = \sqrt{\Delta}\, u, \quad V = v - \frac{a_{12} - a_{21}}{2} u, \tag{7.18}$$

we arrive at the canonical system of equations of elliptic type

$$\begin{aligned} U_\xi - V_\eta + a_* U + b_* V &= f, \\ U_\eta + V_\xi + c_* U + d_* V &= g, \end{aligned} \tag{7.19}$$

where

$$\begin{aligned} a_* &= a_1' - \frac{\partial \sqrt{\Delta}}{\partial \xi} - \frac{1}{2}\frac{\partial (a_{12} - a_{21})}{\partial \eta}, \qquad b_* = b_1', \; f = f_1', \\ c_* &= a_2' - \frac{\partial \sqrt{\Delta}}{\partial \eta} + \frac{1}{2}\frac{\partial (a_{12} - a_{21})}{\partial \xi}, \qquad d_* = b_2', \; g = f_2'. \end{aligned} \tag{7.20}$$

If $a_{ik}, b_{ik} \in D_{m+1,p}(G)$, $a_i, b_i, f_i \in D_{m,p}(G)$, $m \geqslant 0, p > 2$, then, evidently, a_*, b_*, c_*, d_*, f and $g \in D_{m,p}(G')$ where G' is the image of G in the homeomorphism $\zeta = \zeta(z)$.

7.2. We now consider an equation of the second order of the elliptic type

$$a(x, y)\frac{\partial^2 u}{\partial x^2} + 2b(x, y)\frac{\partial^2 u}{\partial x \partial y} + c(x, y)\frac{\partial^2 u}{\partial y^2} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0. \tag{7.21}$$

In order to reduce it to the canonical form

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + F_1\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial v}{\partial \eta}\right) = 0, \tag{7.22}$$

we should make use of the homeomorphism of the equation

$$\zeta_{\bar z} - q(z)\zeta_z = 0, \qquad q = \frac{a - \sqrt{\Delta} - ib}{a + \sqrt{\Delta} + ib}, \qquad \Delta = ac - b^2. \tag{7.23}$$

For definitness we shall assume that the condition of uniform ellipticity is satisfied on the entire plane E, i.e. $\Delta \geqslant \Delta_0 > 0$ (in E), $\Delta_0 = \text{const}$. In this case $|q(z)| \leqslant q_0 < 1$,

and, consequently, there exists a complete homeomorphism of the equation (7.23) if a, b, c are measurable bounded functions. This is, however, insufficient in order that the equation (7.21) be reduced to the canonical form; there must exist the first and the second derivatives of the function $\zeta(z)$, at least in the generalized sense. To this end it is sufficient to assume that $a, b, c \in D_{1,p}(E)$, $p > 2$. Then, according to Theorem 2.5 $\zeta(z) \in D_{2,p}(E)$. Generally, if $a, b, c \in D_{m+1,p}(E)$, $m \geqslant 0$, $p > 2$, then $\zeta(z) \in D_{m+2,p}(E)$. If $a, b, c \in C_\alpha^m(E)$, then $\zeta(z) \in C_\alpha^{m+1}(E)$ (Theorem 2.12).

If we are given the linear equation

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu = g\,, \tag{7.24}$$

then as a consequence of the change of variables

$$\xi = \xi(x, y), \quad \eta = \eta(x, y)\,, \tag{7.25}$$

where $\xi + i\eta = \zeta(z)$ is a homeomorphism of the equation (7.23), we obtain

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + p(\xi, \eta)\frac{\partial u}{\partial \xi} + q(\xi, \eta)\frac{\partial u}{\partial \eta} + r(\xi, \eta)u = h(\xi, \eta)\,, \tag{7.26}$$

where, as is easily shown,

$$\begin{aligned} p &= \frac{J}{4\sqrt{\Delta}}(a\xi_{xx} + 2b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y)\,, \\ q &= \frac{J}{4\sqrt{\Delta}}(a\eta_{xx} + 2b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y)\,, \end{aligned} \tag{7.27}$$

$$r = \frac{Jf}{4\sqrt{\Delta}}, \quad h = \frac{Jg}{4\sqrt{\Delta}}\,. \tag{7.28}$$

Here $\Delta = ac - b^2$ and J is the Jacobian of the transformation (7.25).

Let $a, b, c \in D_{m+1,p}(G)$, $d, e, f, g \in D_{m,p}(G)$, $m \geqslant 0$, $p > 2$. Since $\xi(x, y), \eta(x, y) \in D_{m+2,p}(G)$, then $J \in D_{m+1,p}(G)$, and, consequently, $J \in C^m_a(G)$, $a = \frac{p-2}{p}$. Therefore, according to (7.27) and (7.28) we have

$$\begin{gathered} p, q, r, h \in D_{m,p}(G'), \\ \big(G' = \zeta(G)\big). \end{gathered} \tag{7.29}$$

CHAPTER III

FOUNDATIONS OF THE GENERAL THEORY OF GENERALIZED ANALYTIC FUNCTIONS

§1. Basic concepts, terminology and notations

1.1. In this chapter we shall investigate basic properties of solutions of elliptic systems of partial differential equations of the first order, in a two-dimensional domain. First, the following systems written in canonical form will be considered:

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}+au+bv=f, \quad \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+cu+\partial v=g, \tag{1.1}$$

Afterwards more general systems of elliptic equations will be dealt with. A constructive theory will be developed, making possible an investigation of the structural and qualitative nature of solutions.

An equation of the second order of the form

$$\Delta w+pw_x+qw_y=0 \tag{1.2}$$

is equivalent to the system of equations (1.1). In fact, denoting $w_x=u$, $w_y=-v$ we have the system of equations

$$u_x-v_y+pu-qv=0, \quad u_y+v_x=0\,. \tag{1.3}$$

It will be proved below (§9.2) that the converse is also true, i.e. the system of equations (1.1) can always be reduced to a second order equation of the form (1.2).

Introducing the complex function

$$w(z)=u(x,y)+iv(x,y)\,, \tag{1.4}$$

we may write the system of equations (1.1) in the form

$$\mathfrak{C}(w) \equiv \partial_{\bar{z}} w + Aw + B\overline{w} = F\,, \tag{1.5}$$

where

$$\partial_{\bar{z}} w = \tfrac{1}{2}(w_x + i w_y)\,,$$

$$A = \tfrac{1}{4}(a - d + ic + ib), \quad B = \tfrac{1}{4}(a + d + ic - ib)\,,$$

$$F = \tfrac{1}{2}(f + ig)\,.$$

We shall discover below that this form of the system of equations (1.1) has considerable merits in many respects.

In the classical sense by a solution of the system of equations (1.1) we understand a pair of real continuously differentiable functions $u(x, y), v(x, y)$ of the real variables x and y which satisfy this system everywhere in a domain G. Such solutions, however, exist only for a comparatively narrow class of equations. For example, the equation (1.5) with continuous functions A, B and F may possess no solution in the classical sense. The following equation serves as a simple example:

$$w_{\bar{z}} = \frac{e^{2i\varphi}}{\ln\dfrac{1}{r}}, \qquad z = r e^{i\varphi}\,, \tag{$*$}$$

The right-hand side of this equation is, obviously, continuous in the vicinity of the point $z = 0$. Every solution of this equation, continuous inside a neighbourhood of the point $z = 0$ is given by the formula

$$w(z) = -2z \ln\ln\frac{1}{r} + \Phi(z)\,,$$

where $\Phi(z)$ is a function holomorphic in the vicinity of the point $z = 0$. Differentiating the above relation with respect to z we obtain

$$w_{\bar{z}} \equiv \frac{1}{2}(w_x - i w_y) = -2\ln\ln\frac{1}{r} + \frac{1}{\ln\dfrac{1}{r}} + \Phi'(z)\,.$$

It follows that the derivatives with respect to x and y of any solution of the equation $(*)$, continuous in a neighbourhood of the point $z = 0$, possess discontinuities at the point $z = 0$.

Let us consider one more example of an equation with a continuous coefficient

$$\partial_{\bar{z}} w + Aw = 0 , \qquad A = \frac{e^{2i\varphi}}{\ln \frac{1}{r}} .$$

The general solution of this equation is given by the formula

$$w = \Phi(z) e^{-2z \ln \ln \frac{1}{r}} ,$$

where $\Phi(z)$ is an arbitrary function analytic in z. It is now observed that w is continuous at the point $z = 0$ if and only if Φ is continuous inside a neighbourhood of this point. In this case, evidently, $w_{\bar{z}}$ is always continuous inside the same neighbourhood and w_z is continuous only if the condition $\Phi(0) = 0$ is satisfied. If $\Phi(0) \neq 0$ we have solutions of the equation $(*)$ continuous inside the neighbourhood of the point $z = 0$, their first derivatives having discontinuities at this point. Further examples are given in the paper [86].

It will be proved later on that a solution in the classical sense always exists if the coefficients and the right-hand side of the equation (1.5) are continuous in the Hölder sense in the domain under consideration. Nevertheless, for many reasons, it is expedient to investigate more general classes of equations. It should be observed that one encounters numerous problems in applications, which lead to equations with discontinuous coefficients. Below we shall consider a class of equations for which $A, B, F \in L_p$, $p > 2$; therefore it will be necessary to deal with a concept of a solution in a generalized sense. Generalizations of the concept of a solution may be introduced in many ways by imposing certain require-

ments. Henceforth we shall employ a concept which is defined in a natural way by means of the generalized derivatives in the Sobolev sense (Ch. I, §5). Usually such solutions are sought in the class of summable functions; in our case, however, this limitation would be too restrictive. For example, it would exclude from the class of solutions of the Cauchy–Riemann equation $\partial_{\bar{z}} w = 0$ the meromorphic functions. We shall, therefore, introduce below a definition of a generalized solution of the equation $\mathfrak{C}(w) = \partial_{\bar{z}} w + Aw + B\bar{w} = F$ which in the case of the Cauchy–Riemann equation leads to the class of analytic functions which may have a discrete set of isolated singularities inside the domain.

The function $w(z)$ is said to satisfy the equation (1.5) in a neighbourhood G_0 of a point z_0 if $w \in D_{\bar{z}}(G_0)$ and $\mathfrak{C}(w) \equiv \partial_{\bar{z}} w + Aw + B\bar{w} = F$ almost everywhere in G_0. If w satisfies the equation $\mathfrak{C}(w) = F$ in the vicinity of every point of the domain G except, perhaps, points of a set G_w^* discrete with respect to G, it will be said that w is a *generalized solution* of the equation (1.5) in the domain G. The set G_w^* which has only isolated points, in general depends on the choice of w. This set will be called *the set of singularities* or *the set of singular points* of the generalized solution $w(z)$ with respect to the domain G. If G_w^* is an empty set the generalized solution w will be said to be a *regular solution* of the equation (1.5) in the domain G.

In other words, a solution w regular in the domain G belongs to $D_{\bar{z}}(G)$ and satisfies the equation $\mathfrak{C}(w) = F$ almost everywhere in G. It will be established below that equation $\mathfrak{C}(w) = F$ always has both generalized and regular solutions if $A, B, F \in L_p(G)$, $p > 2$. Moreover, the following important property of regular solutions will be proved: *if $A, B, F \in L_p(G)$, $p > 2$, then every solution of the equation* (1.5), *regular in the domain G, belongs to the class $C_\alpha(G)$, $\alpha = \dfrac{p-2}{p}$* (§3).

We now introduce the concepts of generalized and regular solutions of the systems of equations (1.1). The pair of real functions $u(x, y)$ and $v(x, y)$ is said to be a *generalized (regular) solution of the system of equations* (1.1) if the complex function $w = u + iv$ is a generalized (regular) solution of the corresponding complex equation (1.5) in the domain G.

Let us now introduce the following notations. Let $\widetilde{\mathfrak{A}}^*(A, B, F, G)$ $\big(\widetilde{\mathfrak{A}}(A, B, F, G)\big)$ denote the entire class of generalized (regular) solutions of the equation (1.5) in the domain G. If $A, B, F \in L_p(G)$ $\big(A, B, F \in L_{p,2}\big)$ we shall write, respectively, $\widetilde{\mathfrak{A}}_p^*(A, B, F, G)$ and $\widetilde{\mathfrak{A}}_p(A, B, F, G)$ $\big(\widetilde{\mathfrak{A}}_{p,2}^*(A, B, F, G)$ and $\widetilde{\mathfrak{A}}_{p,2}(A, B, F, G)\big)$. A union (a sum from the point of view of the theory of sets) of the classes $\widetilde{\mathfrak{A}}_p^*(A, B, F, G)$ corresponding to all possible functions A, B, F of the class $L_p(G)$, p being fixed, will be denoted by $\widetilde{\mathfrak{A}}_p^*(G)$. In an analogous way the classes $\widetilde{\mathfrak{A}}_p(G)$, $\widetilde{\mathfrak{A}}_{p,2}^*(G)$ and $\widetilde{\mathfrak{A}}_{p,2}(G)$ are defined.

If $F \equiv 0$ we have the homogeneous equation

$$\mathfrak{C}(w) \equiv \partial_{\bar{z}} w + Aw + B\overline{w} = 0 , \tag{1.6}$$

equivalent to the system of homogeneous (real) equations of the form

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + au + bv = 0, \qquad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + cu + dv = 0 . \tag{1.6a}$$

To denote classes of generalized and regular solutions of the equation (1.6) in a domain G we shall employ the symbols $\mathfrak{A}^*(A, B, G)$ and $\mathfrak{A}(A, B, G)$, respectively. The symbols $\mathfrak{A}_p^*(A, B, G)$, $\mathfrak{A}_p(A, B, G)$, $\mathfrak{A}_{p,2}^*(A, B, G)$, $\mathfrak{A}_{p,2}(A, B, G)$, $\mathfrak{A}_p^*(G)$, $\mathfrak{A}_p(G)$, $\mathfrak{A}_{p,2}^*(G)$ and $\mathfrak{A}_{p,2}(G)$ will also be used, their meaning being obvious.

The coefficients A and B of the equation (1.6) will be called *the generating pair* of the class $\mathfrak{A}^*(A, B, G)$.

In view of Theorem 1.15 the class of the generalized solutions in a domain G of the Cauchy–Riemann equa-

tion $\partial_{\bar{z}} w = 0$ coincides with the class of functions analytic in z which may have arbitrary isolated singular points inside G-poles and essential singularities. It was agreed in Chapter I (§1.7) to denote this class of functions by $\mathfrak{A}_0^*(G)$. It is evident that the class of regular solution of the equation $\partial_{\bar{z}} w = 0$ coincides with the class of functions holomorphic in G; the latter will be denoted by $\mathfrak{A}_0(G)$. Obviously, $\mathfrak{A}_0(G) = \mathfrak{A}_0^* C(G)$.

We note the following properties of the classes $\mathfrak{A}(A, B, G)$ and $\mathfrak{A}_p^*(G)$:—

(1) *The class* $\mathfrak{A}^*(A, B, G)$ *constitutes a linear manifold over the field of real numbers, i.e. if* $w_1, w_2 \in \mathfrak{A}^*(A, B, G)$, *then* $c_1 w_1 + c_2 w_2 \in \mathfrak{A}^*(A, B, G)$ (c_1 *and* c_2 *are arbitrary real constants*).

(2) *If* $f(z), \ln f(z) \in D_{1,p}(G)$ *and* $w \in \mathfrak{A}_p^*(G)$, *then the product* $fw \in \mathfrak{A}_p^*(G)$.

The first property is obvious. We prove the second property. We have

$$(fw)_{\bar{z}} = f_{\bar{z}} w + f w_{\bar{z}} = f_{\bar{z}} w - Afw - Bf\bar{w}$$
$$= -[A - (\ln f)_{\bar{z}}] fw - \frac{Bf}{\bar{f}} (\overline{fw}) .$$

Since $A - \partial_{\bar{z}} \ln f$ and $\dfrac{Bf}{\bar{f}} \in L_p(G)$, $fw \in \mathfrak{A}_p^*(G)$.

Below, in §4, we shall examine the structure of a generalized solution of the equation (1.6). It will be established that if $A, B \in L_{p,2}(E)$, $p > 2$, then generalized solutions have the following form in any given domain G:

$$w(z) = \Phi(z) e^{\omega(z)}, \qquad \Phi \in \mathfrak{A}_0^*(G), \qquad \omega \in C_{p-2/p}(E) . \qquad (1.7)$$

This formula indicates that deep connections exist between the classes of generalized solutions of the equations of the form (1.6) and the class of functions analytic in z. We shall see later that the formula (1.7) and a number of relationships which will be derived in the present chapter, enable us to extend many properties of functions

analytic in z to generalized solutions of a very large class of equations of the form (1.6). Therefore, generalized solutions of an equation of the form (1.6), i.e. functions of the class $\mathfrak{A}_p^*(G)$, will be called *generalized analytic functions*. Accordingly, an equation of the form (1.6) will be called a *generalized Cauchy–Riemann equation.*

In this and in the subsequent chapters we shall develop a fairly complete theory of generalized analytic functions; it represents an essential extension of the classical theory preserving at the same time its principal features.

1.2. The form of the equation (1.5) is preserved in conformal mappings of the domain, [58a]. This fact considerably simplifies in many cases an investigation of properties of generalized analitic functions (see also [58a]).

If $A, B, F \in L_p(\bar{G})$, we shall say that the equation $\mathfrak{C}(w) = F$ belongs to the class $L_p(\bar{G})$, the number p being named the order of this class.

Let the function $z = \varphi(\zeta)$ establish a conformal mapping of a domain G_z onto a domain G_ζ of the ζ-plane. Then the equation (1.5) is transformed into the equation

$$\frac{\partial w}{\partial \bar{\zeta}} + \overline{\varphi'(\zeta)} A(\varphi) w + \overline{\varphi'(\zeta)} B(\varphi) \bar{w} = \overline{\varphi'(\zeta)} F(\varphi) . \qquad (1.8)$$

Inside the domain G_ζ this equation evidently belongs to the same class as the original equation, for $\varphi'(\zeta)$ is a function holomorphic inside G_ζ. In the closed domain $\bar{G}_\zeta$, however, this equation may not belong to the original class, depending on the smoothness properties of the boundaries of the domains G_z and G_ζ.

Let the boundary L of the domain G_z belong to the class $C^1_{\mu,\nu_1,\dots,\nu_k}$ (Ch. I, §2.1) and let us assume that G_ζ is a canonical domain bounded by the circles $\Gamma_0, \Gamma_1, \dots, \Gamma_m$, the union of which will be denoted by Γ. Then $\varphi(\zeta)$ is continuous in the closed domain $G_\zeta + \Gamma$, and L is mapped homeomorphically onto Γ. The derivative $\varphi'(\zeta)$ of the function $\varphi(\zeta)$ may have discontinuities at the points of

the boundary corresponding to the corner points of the contour L.

In the vicinity of the point ζ_j corresponding to a corner point z_j with the interior angle $\nu_j\pi$, $0 < \nu_j \leqslant 2$, $\varphi'(\zeta)$ has the form (Ch. I, §2.3).

$$\varphi'(\zeta) = (\zeta - \zeta_j)\varphi_0(\zeta) \quad (j = 1, \ldots, k), \, (\varphi - \zeta_j)^{\nu_j}\zeta$$

where $\varphi_0(\zeta)$ is a function continuous in the vicinity of ζ_j, and $\varphi_0(\zeta_j) \neq 0$. Assume that all $\nu_j \geqslant 1$. Then the coefficients and the free term of the equation (1.8), evidently, belong to the class $L_p(G_\zeta + \Gamma)$, i.e. in this case the class of the original equation is preserved in the conformal mapping of the domain onto the canonical domain. If at least one of the constants $\nu_j < 1$, the coefficients and the free terms of the equation (1.8) belong to the class $L_{p_1}(G_\zeta + \Gamma)$ where p_1 is an arbitrary number satisfying the condition

$$2 > p_1 < 2 + \frac{2\nu(p-2)}{2\nu + p(1-\nu)} < p, \quad \nu = \min(1, \nu_1, \ldots, \nu_k). \quad (1.9)$$

Thus, in a conformal mapping of a domain G_z of the class $C^1_{\mu,\nu_1,\ldots,\nu_k}$ onto the canonical domain G_ζ bounded by circles, the order of the class of the equation (1.5) in general decreases but remains always greater than two.

Let us observe that when $p = 2$ the order of the equation in new variables remains the same.

Henceforth, unless otherwise stated, we shall assume that the following condition is satisfied:

$$A(z), B(z), F(z) \in L_{p,2}(E), \, p > 2\,. \quad (1.10)$$

We note that the function $f_0(z) = |z|^{-2} f\left(\frac{1}{z}\right) \in L_{p,2}(E)$ if $f \in L_{p,2}(E)$, $p > 2$. On this basis it is easy to discover that in the bilinear transformation of the variable

$$z = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}, \quad \alpha\delta - \beta\gamma \neq 0\,.$$

the coefficients and the free term of the new equation (1.8) will also satisfy the condition (1.10). In other words, in a bilinear transformation of the independent variable the class $\tilde{\mathfrak{A}}^*_{p,2}$, $p > 2$, is preserved.

In particular, if $z = \frac{1}{\zeta}$ we have the equation

$$\partial_{\bar{\zeta}} w + A_0(\zeta) w_0 + B_0 \zeta) \overline{w}_0 = F_0, \tag{1.11}$$

where

$$w_0(\zeta) = w\left(\frac{1}{\zeta}\right),$$

$$A_0 = -\frac{1}{\bar{\zeta}^2} A\left(\frac{1}{\zeta}\right), \quad B_0 = -\frac{1}{\bar{\xi}^2} B\left(\frac{1}{\zeta}\right), \quad F_0 = -\frac{1}{\bar{\zeta}^2} F\left(\frac{1}{\zeta}\right).$$

Since in the last transformation the vicinity G_∞ of the point $z = \infty$ is mapped onto the vicinity G_0 of the origin of coordinates, we shall adopt the following definition:

The function $w(z)$ will be said to belong to the class $\tilde{\mathfrak{A}}^*(A, B, F, G_\infty)$ if $w_0(z) \equiv w\left(\frac{1}{z}\right) \in \tilde{\mathfrak{A}}^*(A_0, B_0, F_0, G_0)$.

The last results enable us to reduce the investigation of the behaviour near infinity of a solution of the equation $\partial_{\bar{z}} w + Aw + B\overline{w} = F$ to the investigation of the behaviour of a solution of the equation (1.11) near the origin of coordinates. In particular, if a function $w_0(z)$ of the class $\mathfrak{A}(A_0, B_0, G_0)$ is continuous at the point $z = 0$, then the function $w(z) = w_0\left(\frac{1}{z}\right)$ is, by definition, continuous at the point $z = \infty$ and belongs to $\mathfrak{A}(A, B, G_0)$

§2. Integral equation for functions of the class $\tilde{\mathfrak{A}}(A, B, F, G)$

According to the definition every solution of the equation $\mathfrak{C}(w) = F$ regular in a domain G belongs to the class $D_{\bar{z}}(G)$, i.e. if $w \in \tilde{\mathfrak{A}}(A, B, F, G)$ then $\partial_{\bar{z}} w \equiv -Aw - B\overline{w} + F \in L_1(G)$. Let us now consider the case of $\partial_{\bar{z}} w \in L_1(\overline{G})$, denoting the class of such solutions by

$\mathfrak{A}(A, B, F, \bar{G})$. Obviously, $\tilde{\mathfrak{A}}(A, B, F, \bar{G}) \subset \tilde{\mathfrak{A}}(A, B, F, G)$.

If $w \in \tilde{\mathfrak{A}}(A, B, F, \bar{G})$, then in view of the formula (5.12) of Ch. I, we have

$$w - \mathrm{P}_G w = \Phi(z) + \mathrm{T}_G F, \qquad \Phi \in \mathfrak{A}_0(G), \tag{2.1}$$

where the following notation has been employed:

$$\begin{aligned} \mathrm{T}_G f &= -\frac{1}{\pi} \iint_G \frac{f(\zeta)}{\zeta - z} d\xi \, d\eta, \\ \mathrm{P}_G f &= -\mathrm{T}_G(Af + B\bar{f}). \end{aligned} \tag{2.2}$$

Thus, every function of the class $\tilde{\mathfrak{A}}(A, B, F, \bar{G})$ satisfies the integral equation (2.1) and the analytic function Φ is uniquely determined by the given solution w of the equation $\mathfrak{C}(w) = F$. The converse statement is also true, i.e. *if for a function Φ holomorphic in G the integral equation (2.1) is satisfied by a function w, such that $Aw + B\bar{w} \in L_1(\bar{G})$, then this function will also satisfy the equation $\mathfrak{C}(w) = F$.* In fact, if the operation $\partial_{\bar{z}}$ be applied to both sides of the relation (2.1), then according to the formula (5.8) of Ch. I, we arrive at once at the relation $\mathfrak{C}(w) = F$. When $F \equiv 0$ we have the equation

$$w - \mathrm{P}_G w = \Phi(z), \tag{2.3}$$

where Φ is an arbitrary function analytic in z in the domain G. We have thus obtained an integral equation for generalized functions of the class $\mathfrak{A}(A, B, \bar{G})$.

If the conditions (1.10) are satisfied and, if $w \in C(\bar{G})$, then according to Theorem 1.19 $\mathrm{P}_G w$ and $\mathrm{T}_G F \in C_\alpha(E)$, $\alpha = \frac{p-2}{2}$. Therefore, it follows from (2.1) that $\Phi(z) \in C(\bar{G})$. Let the boundary Γ of the domain G consist of a finite number of rectifiable Jordan curves. Taking into account that $\mathrm{P}_G w$ and $\mathrm{T}_G F$ are holomorphic outside $G + \Gamma$ and vanish at infinity we obtain from (2.3) in view of the Cauchy theorem and Cauchy formula,

$$\Phi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{w(\zeta)}{\zeta - z} d\zeta. \tag{2.4}$$

Thus, the integral of the Cauchy type (2.4) represents a function continuous in the closed domain $\bar{G}$ if the density function coincides with limiting values of a generalized analytic function w of the class $\tilde{\mathfrak{A}}_p(G)$, $p > 2$, which is continuous in $\bar{G}$.

It should be noted that, as is known, an integral of the Cauchy type with an arbitrary continuous density function does not possess this property, i.e. it is not in general a continuous function in the closed domain.

Further, in §5, we shall prove that the integral equation (2.1) has a solution under very general assumptions concerning the functions A, B, F in the domain G. On this basis we shall derive a general representation of the class of functions $\tilde{\mathfrak{A}}(A, B, F, \bar{G})$ by functions analytic in z.

In the following section, making use of the integral equation (2.1) we shall prove several theorems concerning the smoothness and differentiability properties of regular solutions of the equation $\mathfrak{C}(w) = F$.

§3. Continuity and differentiability properties of functions of the class $\tilde{\mathfrak{A}}_p(G)$.

In what follows we shall deal exclusively with these classes of equations of the form (1.5) which inside the considered domain have continuous (generalized) solutions. We shall discover that this class contains all equations for which the condition (1.10) is satisfied. We shall also deal with less general classes of equations the solutions of which have continuous and generalized derivatives up to a certain order; the examination of such cases is important particularly for various geometric and mechanical applications.

3.1. We have the following

THEOREM 3.3. *If* $w \in \tilde{\mathfrak{A}}_{p,2}(A, B, F, G)$, $p > 2$, *then* $w \in C_\alpha(G)$, $\alpha = \dfrac{p-2}{p}$.

PROOF. Since, according to the definition, $w \in D_{\bar z}(G)$ (i.e. $\partial_{\bar z} w \in L_1(G)$), in view of Theorem 1.26 $w \in L_\gamma(G)$ where γ is an arbitrary number satisfying the condition $1 < \gamma < 2$. Therefore we may assume that $\gamma \geqslant \dfrac{p}{p-1}$.

Let us consider two subdomains G_1 and G_2 of the domain G, $G_1 \subset \bar G_1 \subset G_2 \subset \bar G_2 \subset G$. Then $\partial_{\bar z} w \in L_1(\bar G_2)$, $w \in L_\gamma(\bar G_2)$. Therefore, according to the relation (2.1) we have

$$\begin{gathered} w(z) - \mathrm{P}_2 w = h(z), \qquad h = \Phi_0(z) + \mathrm{T}_2 F \\ (\mathrm{P}_2 \equiv \mathrm{P}G_1\,, \qquad \mathrm{T}_2 = \mathrm{T}G_2)\,, \end{gathered} \tag{3.1}$$

where Φ_0 is a function holomorphic inside G_2 and $\mathrm{T}_2 F \in C_{\frac{p-2}{p}}(E)$. Since $w \in L_\gamma(\bar G_2)$, $\gamma \geqslant \dfrac{p}{p-1}$, then in view of Theorem 1.29 $\mathrm{P}_2 w \in L^{a}_{\gamma_1}(\bar G)$, $\dfrac{1}{\gamma_1} = \dfrac{1}{\gamma} + \dfrac{1}{p} - \dfrac{1}{2} + a$ where a is a sufficiently small positive number. Hence, it follows from (3.1) that $h \in L_{\gamma_1}(\bar G_2)$, for $\gamma_1 > \gamma \geqslant \dfrac{p}{p-1}$. By means of iterations we obtain from (3.1) equations of the form

$$w = \mathrm{P}_2^n w + h + \mathrm{P}_2 h + \ldots + \mathrm{P}_2^{n-1} h\,. \tag{3.2}$$

According to Theorem 1.29 there exists an integer n such that $\mathrm{P}_2^{n-1} w \in C(\bar G)$. Therefore, for this n, we have by virtue of Theorem 1.29

$$\mathrm{P}_2^n w = \mathrm{P}_2(\mathrm{P}_2^{n-1} w) \in C_\alpha(E)\,, \qquad \alpha = \frac{p-2}{p}\,.$$

Let G' be a subdomain of G_2 such that $\bar G_1 \subset G'$, $\bar G' \subset G_2$. It is evident that in this case

$$\mathrm{P}_2 = \mathrm{P}' + \mathrm{P}'', \qquad \mathrm{P}' = \mathrm{P}_{G'}, \qquad \mathrm{P}'' = \mathrm{P}_{G_2 - G'}\,,$$

$\mathrm{P}''h$ being, evidently, holomorphic inside G', and consequently continuous in the Lipschitz sense in $\bar G_1$ because $\bar G_1 \subset G'$. Since h is continuous in $\bar G'$, according to Theorem 1.24 $\mathrm{P}'h \in C_{\frac{p-2}{p}}(E)$. Therefore $\mathrm{P}_2 h = \mathrm{P}'h + \mathrm{P}''h \subset$

$\subset C_{\frac{p-2}{p}}(\bar{G}_1)$. By a similar argument we can show that $\mathrm{P}_2^k h \in C_{\frac{p-2}{2}}(\bar{G}_1)$ $(k = 1, 2, \ldots)$.

Thus, for an integer n, the right-hand side of the relation (3.2) belongs to the class $C_{\frac{p-2}{p}}(\bar{G}_1)$. Consequently, $w(z) \in C_{\frac{p-2}{p}}(G)$ because an arbitrary closed subdomain of the domain G may be taken for G_1. This completes the proof of the theorem.

3.2. THEOREM 3.2. *If $A, B, F \in D_{m,p}(G)$ $(m \geqslant 0, p > 2)$, then every function $w(z)$ of the class $\mathfrak{A}(A, B, F, G)$ belongs to the class $D_{m+1,p}(G)$.*

PROOF. If $m = 0$ then $A, B, F \in L_p(G)$, $p > 2$, and in view of the preceding theorem $w \in C_{\frac{p-2}{p}}(G)$. Therefore, it follows from the equation $w_{\bar{z}} + Aw + B\bar{w} = F$ that $w_{\bar{z}} \in L_p(G)$, $p > 2$. In this case, however, in view of Theorem 1.36 $w_z \in L_p(G)$, $p > 2$, i.e. $w \in D_{1,p}(G)$. Let us now consider the case $m \geqslant 1$. Since according to the proved above $w \in D_{1,p}(G)$ we have

$$\begin{aligned} w_{\bar{z}\bar{z}} + A_{\bar{z}} w + B_{\bar{z}} \bar{w} + A w_{\bar{z}} + B \bar{w}_{\bar{z}} &= F_{\bar{z}}\,, \\ w_{z\bar{z}} + A_z w + B_z \bar{w} + A w_z + B \bar{w}_z &= F_z\,. \end{aligned} \tag{3.3}$$

This implies that $w \in D_{2,p}(G)$. Thus, for $m = 1$ the theorem is proved. If $m = 2$, then differentiating both sides of the relations (3.3) with respect to z and $\bar{z}$ we obtain that $w \in D_{3,p}(G)$. Continuing analogous reasoning we find that derivatives of w of the order $m+1$ belong to the class $L_p(G)$, $p > 2$; this completes the proof.

Theorem 3.2 in view of Theorem 1.20 implies the following:

If $A, B, F \in D_{m,p}(G)$ $(m \geqslant 0, p > 2)$ then $w(z) \in C_\alpha^m(G)$, $\alpha = \dfrac{p-2}{2}$.

THEOREM 3.3. *If $A, B, F \in C_\alpha^m(G)$ $(m \geqslant 0, 0 < \alpha < 1)$ then the function $w(z)$ of the class $\mathfrak{A}(A, B, F, G)$ belongs to the class $C_\alpha^{m+1}(G)$.*

PROOF. When $m = 0$ we have $w_{\bar{z}} = -Aw - B\bar{w} + F \in C_\alpha$. Then, according to Theorem 1.32, $w_z \in C_\alpha(G)$ i.e. $w \in C^1_\alpha(G)$. If $m = 1$, it follows from the relations (3.3) that $w \in C^2_\alpha(G)$. It is readily proved by a reasoning analogous to that used above that the theorem holds for an arbitrary integer $m \geqslant 0$.

It is seen from Theorem 3.3. that for the existence of solutions of the equation (1.5) in the classical sense or of the corresponding system of equations (1.1) it is sufficient that $A, B, F \in C_\alpha(G)$, $0 < \alpha < 1$. In this case every regular solution has partial derivatives of the first order continuous in the Hölder, sense inside the domain.

3.3. Theorems 3.1, 3.2, and 3.3 proved above determine the smoothness and differentiability properties of functions of the class $\mathfrak{A}(A, B, F, G)$ in terms of the same properties of functions A, B, F inside the domain G. The smoothness and differentiability properties of these functions in the closed domain $\bar{G}$ depend, evidently, on the same properties of the functions A, B, F in the closed domain and on the degree of smoothness of the boundary of the domain and the limiting (boundary) values of the functions under consideration.

THEOREM 3.4. *Let $G \in C^1$. If $A, B, F \in L_p(G+\Gamma)$, $p > 2$, then a solution of the equation* (1.5) *continuous in $G+\Gamma$ and belonging to the class $C_\alpha(\Gamma)$ on Γ belongs to the class $C_\beta(G+\Gamma)$ where $\beta = \min(\alpha, \gamma)$, $\gamma = \dfrac{p-2}{p}$.*

PROOF. Under these conditions w may be represented in the form $w = \mathrm{P}_G w + \mathrm{T}_G F + \Phi(z)$ where Φ is given by the integral of the Cauchy type (2.4). In view of Theorem 1.10 $\Phi \in C_\alpha(G+\Gamma)$. Since $\mathrm{P}_G w$ and $\mathrm{T}_G F \in C_\gamma(G+\Gamma)$ we obtain at once that $w \in C_\beta(G+\Gamma)$, $\beta = \min(\alpha, \gamma)$.

In §§ 1, 7 of Chapter IV some other conditions will be indicated which guarantee the continuity of generalized analytic functions of the class $\tilde{\mathfrak{A}}_p(G)$, $p > 2$, and their derivatives up to a certain order in the closed domain.

3.4. The case of A, B and F being analytic functions of real variables x and y should be considered separately.

In this case regular solutions of the equation $\mathfrak{C}(w) = F$ are analytic functions in x and y inside the domain and the problem of determination of these functions can be reduced to the solution of an integral equation of the Volterra type in the complex domain. In such a way various integral representations of solutions of the equation $\mathfrak{C}(w) = F$ can be obtained and their properties investigated. An exposition of this method may be found in author's paper [14a] (§10: see also [85c], §3.13). Therefore, this case will not be considered separately in the following sections.

§4. Basic lemma. Generalizations of some classical theorems

In this section we shall derive the formula (1.7) which, as was indicated above, will enable us to prove a number of theorems generalizing properties of functions analytic in z.

4.1. First of all let us prove the following basic lemma.

THE BASIC LEMMA. *Let $w(z)$ be a generalized analytic function of the class $\mathfrak{A}^{*}_{p,2}(A, B, G)$, $p > 2$.*

Let

$$g(z) = \begin{cases} A(z) + B(z)\dfrac{\overline{w(z)}}{w(z)}, & \text{if} \quad w(z) \neq 0, \quad z \in G, \\ A(z) + B(z), & \text{if} \quad w(z) = 0, \quad z \in G. \end{cases} \tag{4.1}$$

Then the function

$$\Phi(z) = w(z) e^{-\omega(z)}, \tag{4.2}$$

where

$$\omega(z) = \frac{1}{\pi} \iint\limits_{G} \frac{g(\zeta)\, d\xi\, d\eta}{\zeta - z} \equiv -\mathrm{T}_G g, \tag{4.2a}$$

belongs to the class $\mathfrak{A}^{*}_0(G)$.

PROOF. Since $|g(z)| \leqslant |A(z)| + |B(z)|$, $g \in L_{p,2}(\bar{G})$, $p > 2$. Therefore, according to Theorem 1.23, the function $\omega(z) \in C_\alpha(E)$, $\alpha = \dfrac{p-2}{p}$. Since $-g = \partial_{\bar{z}}\omega \in L_p(\bar{G})$, $p > 2$, in view of Theorem 1.39 the function $e^{\omega(z)} \in D_{1,p}(G)$, and $\partial_{\bar{z}} e^{-\omega} = -e^{-\omega}\partial_{\bar{z}}\omega = g e^{-\omega}$.

Let G_w^* be the set of singularities of the function w. According to the definition G_w^* is a discrete set with respect to the domain G. Inside the open set $G-G_w^*$ the function w is, evidently, a regular solution of the equation $\partial_{\bar{z}}w+$ $+Aw+B\overline{w}=0$. Consequently, in view of Theorem 3.2, $w \in D_{1,p}(G-G_w^*)$. Therefore, we can apply the formula of differentiation of a product to the function $\Phi = we^{-\omega}$ belonging in view of Theorem 1.48 to the class $D_{\bar{z}}(G-G_w^*)$, i.e.

$$\partial_{\bar{z}}\Phi = e^{-\omega}(\partial_{\bar{z}}w+wg) = e^{-\omega}(-Aw-B\overline{w}+wg)\ .$$

Hence, in view of (4.1) we have

$$\partial_{\bar{z}}\Phi = 0 \text{ almost everywhere in } G-G_w^*\,.$$

This result shows that Φ is holomorphic inside $G-G_w^*$. Since G_w^* contains only isolated points, evidently $\Phi \in \mathfrak{A}_0^*(G)$. This completes the proof of the lemma.

In particular, *if w is a regular solution of the equation $\partial_{\bar{z}}w+Aw+B\overline{w}=0$ in the domain G, then Φ is holomorphic in G.*

The formula (4.2) was proved in the author's paper [14a] (1952) by means of Carleman's uniqueness theorem [38a] (see below, §4.3). The proof given here makes no use of Carleman's theorem; it was stated somewhat later in another author's paper [14e] (1953). It is evident that Carleman's theorem is a simple consequence of the lemma proved above (cf. Theorem 3.5).

In the subsequent articles we shall consider some important consequences of this lemma.

4.2. The formula (4.2) shows that the set of points for which $w(z)=0$ coincides with the set of zeros of the analytic function $\Phi(z)$. Similarly, the set of singular points of $w(z)$ coincides with the set of poles and essentially singular points of $\Phi(z)$. It follows that if w does not vanish identically its zeros and poles are isolated, and the multiplicity of a zero and the order of a pole are positive integers. Besides, it is evident that near a pole

the function $w(z)$ is not bounded and its behaviour in the vicinity of an essential singularity is determined by the Sokhotski–Weierstrass theorem. The first assertion is obvious; we shall prove the second.

If $w(z)$ does not vanish identically the formula (4.2) may be written in the form

$$w(z) = \Phi(z) e^{\omega(z)}, \tag{4.3}$$

where

$$\omega(z) = \frac{1}{\pi} \iint_G \left(A(\zeta) + B(\zeta) \frac{\overline{w(\zeta)}}{w(\zeta)} \right) \frac{d\xi\, d\eta}{\zeta - z}. \tag{4.4}$$

Let z_0 be an essential singularity of the function $w(z)$ and c an arbitrary fixed constant. Then, by virtue of (4.3)

$$|w(z) - c| \leqslant |\Phi(z)| |e^{\omega(z)} - e^{\omega(z_0)}| + |\Phi(z) e^{\omega(z_0)} - c|. \tag{4.5}$$

Since z_0 is an essential singularity of $\Phi(z)$, in view of the Sokhotski–Weierstrass theorem there can be found a sequence z_k converging to z_0 and satisfying the condition $\Phi(z_k) e^{\omega(z_0)}) \to c$ when $k \to \infty$. On this basis, taking into account the continuity of $\omega(z)$ we obtain from the inequality (4.5) that $w(z_z) \to c$ when $z_k \to z_0$. This was to be proved.

The formula (4.3) is named *the representation of the first kind* or *the reciprocal theorem for generalized analytic functions.* * We shall see further that it is of fundamental importance for the construction of the general theory of generalized analytic functions. This formula will frequently be used in the later considerations.

4.3. The formula (4.3) implies immediately the following uniqueness theorem for generalized analytic functions:

THEOREM 3.5. *If a generalized analytic function* $w(z)$ *of the class* $\mathfrak{A}_{p,2}(G)$, $p > 2$, *vanishes in an infinite set* $\mathcal{M}$

* In the papers of Bers [5a, b] this formula is called the principle of similarity.

of the points of a domain G possessing at least one limit point belonging to G, then $w(z) = 0$ everywhere in G.

By means of the formula (4.3) many established limiting uniqueness theorems of the theory of analytic functions may be extended to the generalized analytic functions [72]. For instance, we have

THEOREM 3.6. *Let the boundary of the domain G contain a rectifiable arc γ. Let A and $B \in L_p(G+\gamma)$, $p > 2$. If $w(z) \in \mathfrak{A}(A, B, G)$ and non-tangent limiting values of $w(z)$ on γ vanish, then $w = 0$ everywhere in G.*

PROOF. It follows from (4.2a) that in the interior boundary strip adjoining the arc γ, $\omega(z)$ is continuous. Therefore the non-tangent limiting values of the analytic function $\Phi(z)$ vanish on γ. Consequently, according to a well known theorem, [72] (Ch. IV), $\Phi(z) \equiv 0$, i.e. $w \equiv 0$.

There is an important geometric interpretation of Theorems 3.5 and 3.6 (see Ch. V, §3.5).

Theorem 3.5 was proved by a different method by Carleman in 1933 [38a], under the assumption that $A, B \in C(\bar{G})$ and w is continuous and has piecewise continuous derivatives of the first order. Carleman's reasoning may also be applied to our more general case ($A, B \in L_{p,2}(E)$, $p > 2$) if the Hölder inequality be taken into account.

4.4. If $B \equiv 0$ the formula (4.3) assumes the following form:

$$w(z) = \Phi(z) e^{\frac{1}{\pi}\iint\limits_G \frac{A(\zeta)}{\zeta - z} d\xi d\eta}, \quad \Phi \in \mathfrak{A}_0^*(G). \tag{4.6}$$

This is a representation of a general solution of the equation

$$\partial_{\bar{z}} w + Aw = 0 \tag{4.7}$$

by functions analytic in z. It was first derived by Theodorescu [82a, b] (1931) for the case of a bounded measurable function A.

It should be observed that it is possible to derive the formula (4.3) in the general case by means of the for-

mula (4.6). In fact, the equation $\mathfrak{C}(w) = 0$ may be written in the form

$$\partial_{\bar{z}} w + A_0 w = 0 , \quad A_0 = A + B\frac{\overline{w}}{w} .$$

Taking into account that $A_0 \in L_{p,2}(E)$, $p > 2$, the solution of the last equation may be written in the form (4.6) and hence we immediately arrive at the formula (4.3). In this way the formulae (4.3) and (4.4) were derived in author's paper [14a] (1952), the formula (4.6) being obtained, as it was mentioned before, as a consequence of Carleman's theorem. (At that time the author did not know that this formula had been obtained before Carleman (1933) by Theodorescu [82a].) An analogous proof of the formula (4.3) is given in Bers' paper [5b] (1953) who announced it without proof as early as in 1951 in the paper [5a]. Bers does not mention Theodorescu either. Let us note that in the papers [5b], [6a] the explicit expression (4.4) for ω is not given; it is of essential value in various applications of the formula (4.3).

4.5. The formula (4.3) may be generalized to the case of the class of quasi-summable functions A and B (Ch. I, §1.8), [14f].

THEOREM 3.7. *Let there exist analytic functions $\Phi_A(z)$ and $\Phi_B(z)$ of the class $\mathfrak{A}_0^*(G)$ such that the products $A(z)\Phi_A(z)$ and $B(z)\Phi_B(z) \in L_p(\overline{G})$, $p > 2$.* If $w(z)$ is an analytic function of the class $\mathfrak{A}^*(A, B, G)$ then there can be found an analytic function $\Phi(z)$ of the class $\mathfrak{A}_0^*(G)$, such that*

$$w(z) = \Phi(z) e^{\omega(z)} , \tag{4.8}$$

where

$$\omega(z) = \frac{1}{\pi\Phi_A(z)} \iint_G \frac{\Phi_A(\zeta) A(\zeta)}{\zeta - z} d\xi d\eta + \frac{1}{\pi\Phi_B(z)} \iint_G \frac{\Phi_B(\zeta) B(\zeta) \overline{w(\zeta)}}{(\zeta - z) w(\zeta)} d\xi d\eta . \tag{4.9}$$

* If G is an unbounded domain we shall assume that $A(z)\Phi_A(z)$ and $B(z)\Phi_B(z) \in L_p L_{p'}(G)$, $p > 2$, $1 < p' < 2$...

PROOF. In the trivial case $w \equiv 0$ we shall set $\Phi \equiv 0$. Therefore, in what follows, we shall assume that $w(z)$ does not identically vanish. Since

$$A(z)\Phi_A(z) \in L_p(\bar{G})$$

and

$$B(z)\Phi_B(z)\frac{\overline{w(z)}}{w(z)} \in L_p(\bar{G}) , \qquad p > 2 ,$$

in view of Theorem 1.19 the integrals appearing in the right-hand side of the relation (4.9) belong to the class $C_{\frac{p-2}{p}}(E)$, and the following relation holds everywhere in G:

$$\frac{\partial w}{\partial \bar{z}} = -A(z) - B(z)\frac{\overline{w(z)}}{w(z)} .$$

Hence, according to Theorems 1.38 and 1.39 we have for the function $\Phi(z) = w(z)e^{-\omega(z)}$

$$\frac{\partial \Phi}{\partial \bar{z}} = e^{-\omega(z)}\left(\frac{\partial w}{\partial \bar{z}} + Aw + B\bar{w}\right) = 0 .$$

This relation is satisfied in the whole closed subdomain G' of the domain G, which does not contain zeros or singularities of the analytic functions $\Phi_A(z)$ and $\Phi_B(z)$ and singularities of the function $w(z)$. But the set of such points is discrete inside G. Therefore $\Phi(z) \in \mathfrak{A}_0^*(G)$ i.e. $\Phi(z)$ is an analytic function everywhere in G except for, perhaps, a set of points discrete with respect to G. This completes the proof of the theorem.

The formula (4.8) may easily be generalized to the case of A and B having the form

$$A = A_1 + \ldots + A_n , \qquad B = B_1 + \ldots + B_n , \tag{4.10}$$

where A_j and $B_j \in \mathfrak{A}_0^* \times L_p(\bar{G}), p > 2$ $(j = 1, \ldots, n)$. In this case the formula (4.8) still holds if we take for $\omega(z)$ the function

$$\omega(z) = \sum_{k=1}^{n} \frac{1}{\pi\Phi_k(z)} \iint_G \frac{A_k(\zeta)\Phi_k(\zeta)\,d\xi\,d\eta}{\xi - z} + \\ + \frac{1}{\pi\Psi_k(z)} \iint_G \frac{B_k(\zeta)\Psi_k(\zeta)\bar{w}(\zeta)}{(\zeta - z)w(\zeta)}\,d\xi\,d\eta . \tag{4.11}$$

Thus, if A and B are functions of the form (4.10) where A_j and B_j are quasi-summable functions of the class $\mathfrak{A}_0^* \times L_p(\bar{G})$, then the formulae (4.8) and (4.11) indicate that generalized analytic functions of the class $\mathfrak{A}^*(A, B, G)$ belong to the class $\mathfrak{A}_0^* \times e^{\Sigma \mathfrak{A}_0^* \times C_\alpha(E)}$ where $\alpha = \dfrac{p-2}{p}$ (Ch. I, §1.8). The above generalizations of the reciprocal theorem (4.3) were obtained in author's paper (14f].

4.6. From (4.8) and (4.11) the following inequalities are obtained:

$$e^{-\Omega(z)} \leqslant \left|\frac{w(z)}{\Phi(z)}\right| \leqslant e^{\Omega(z)}, \quad |\omega(z)| \leqslant \Omega(z), \quad (4.11a)$$

where

$$\Omega(z) = \sum_{k=1}^{n} \frac{1}{|\pi\Phi_k(z)|} \iint_G \frac{|A_k(\zeta)||\Phi_k(\zeta)|}{|\zeta - z|} d\xi d\eta + \\ + \frac{1}{\pi|\Psi_k(z)|} \iint_G \frac{|B_k(\zeta)||\Psi_k(\zeta)|}{|\zeta - z|} d\xi d\eta. \quad (4.12)$$

These inequalities hold for all functions of the class $\mathfrak{A}^*(A, B, G)$, $\Omega(z)$ depending only on the generating pair A and B and being independent of the choice of the function w.

If $A, B \in L_{p,2}(E)$, $p > 2$, then $\Phi_A = \Phi_B \equiv 1$ and, according to the inequality (6.14), Ch. I, we have

$$\Omega(z) = \frac{1}{\pi} \iint_G \frac{|A(\zeta)| + |B(\zeta)|}{|\zeta - z|} d\xi d\eta \\ \leqslant M_p L_{p,2}(|A| + |B|). \quad (4.12a)$$

4.7. The functions Φ and ω appearing in the representation of the first kind (4.8) will be named *the analytic divisor* and *the logarithmic difference* of the generalized analytic function $w(z)$; they are not determined uniquely by w. If $\Phi_0(z)$ is a function holomorphic in G we have

$$w(z) = \Phi(z) e^{\omega(z)} = \Phi e^{-\Phi_0(z)} e^{\Phi_0(z) + \omega(z)} = \Phi_*(z) e^{\omega_*(z)}.$$

It follows that the analytic divisor and the logarithmic difference are not expressed uniquely by w. This circumstance makes it possible to impose various additional conditions upon them. In particular, we can select various conditions determining these functions uniquely. For instance, for every function of the class $\mathfrak{A}^*_{p,2}(G)$, $p > 2$, the representation of the first kind (4.3) where $\omega(z)$ is given by (4.4) is unique. This representation is characterized by the fact that $\omega(z) \in C_\alpha(E)$, $\alpha = \frac{p-2}{p}$, is holomorphic outside $G+\Gamma$ and vanishes at infinity (see Theorem 1.19). In fact, if we had two representations of the form (4.3),

$$w(z) = \Phi_1(z) e^{\omega_1(z)}, \qquad w(z) = \Phi_2(z) e^{\omega_2(z)}, \tag{4.13}$$

we would obtain the relation

$$\frac{\Phi_1(z)}{\Phi_2(z)} = e^{\omega_1(z)-\omega_2(z)} \quad (\text{in } G). \tag{4.14}$$

Since $\omega_1(z)$ and $\omega_2(z)$ belong to the class $C_\alpha(E)$, $\alpha = \frac{p-2}{p}$, are holomorphic outside $G+\Gamma$ and vanish at infinity, it is evident that the left-hand side of the relation (4.14) is holomorphic inside G, continuous in $G+\Gamma$ and may be continued outside $G+\Gamma$ by a holomorphic function equal to unity at infinity. Hence, in view of Liouville's theorem, $\Phi_1 \equiv \Phi_2$ and, consequently, $\omega_1 \equiv \omega_2$. Thus, we have established the uniqueness of the representation of functions of the class $\mathfrak{A}^*_{p,2}(G)$, $p > 2$, by means of the formulae (4.3) and (4.4).

In what follows the logarithmic difference $\omega(z)$ given by the formula (4.4) will be called *the normal logarithmic difference* of the function $w(z)$ of the class $\mathfrak{A}^*_{p,2}(A, B, G)$, and the analytic function $\Phi(z)$ appearing in (4.3) *the normal analytic divisor of* $w(z)$.

According to Theorem 1.23 the normal logarithmic differences of functions of the class $\mathfrak{A}^*_{p,2}(A, B, G)$ satisfy

the inequalities

$$|\omega(z)| \leqslant M_p L_{p,2}(|A|+|B|)\,,$$

$$|\omega(z_1)-\omega(z_2)| \leqslant M_p L_{p,2}(|A|+|B|)\,|z_1-z_2|^{\frac{p-2}{p}}\,, \quad (4.15)$$

$$|\omega(z)| \leqslant M_p L_{p,2}(|A|+|B|)\,|z|^{\frac{2-p}{p}} \quad \text{for} \quad |z| \geqslant R > 1\,. \quad (4.16)$$

The inequalities (4.15) hold for all points z, z_1, z_2 of the plane E, and the inequality (4.16) is valid for $|z| \geqslant R > 1$, R being a fixed constant. Let us also observe that the constant M_p in the inequalities (4.15) and (4.16) depends only on p, $p > 2$.

Let $\mathfrak{A}_{p,2}^{M}(G)$ be the set of all generalized analytic functions the generating pairs of which satisfy the condition

$$L_{p,2}(|A|+|B|) \leqslant M\,, \qquad M = \text{const}\,. \qquad (4.17)$$

The inequalities (4.15) immediately imply

THEOREM 3.8. *The set $\{\omega(z)\}$ of the normal logarithmic differences corresponding to the elements of the set $\mathfrak{A}_{p,2}^{M}(G)$ is uniformly bounded and uniformly equi-continuous on the plane, i.e. the set $\{\omega(z)\}$ is compact in the space $C(E)$.*

4.8. Making use of the principle of the maximum modulus of an analytic function we obtain from the formula (4.3) this principle for a generalized analytic function, in the following form.

PRINCIPLE OF THE MAXIMUM MODULUS. *If $w \in \mathfrak{A}_{p,2}(A, B, G)$, $p > 2$, and it is continuous in $\bar{G}$, then*

$$|w(z)| \leqslant \hat{M} \max_{t \in \Gamma} |w(t)|\,, \qquad z \in G+\Gamma\,, \qquad (4.18)$$

where $\hat{M}$ is a positive constant ($\hat{M} \geqslant 1$) which depends only on A, B, p.

By means of the inequality (4.15) we can easily obtain for $\hat{M}$ the inequality

$$1 \leqslant \hat{M} \leqslant e^{2M_p L_{p,2}(|A|+|B|)}\,. \qquad (4.19)$$

If we equate $\hat{M}$ to e^{2MM_p}, then the inequality (4.18) will hold for any function of the class $\mathfrak{A}_{p,2}^{M}(G)$.

By means of the principle of the maximum modulus we can prove the following theorem:

THEOREM 3.9. *Let the sequences A_n and B_n of elements of $L_{p,2}(E)$, $p > 2$, converge in the mean (in the metric $L_{p,2}(E)$) to A and B. Let $w_n \in \mathfrak{A}_{p,2}(A_n, B_n, G)$ $(n = 1, 2, \ldots)$, w_n being continuous in G. If the sequence w_n converges uniformly to a function $w(z)$ on the boundary Γ then w_n converge uniformly in the closed domain to the function $w(z)$ belonging to the class $\mathfrak{A}_{p,2}(A, B, G)$, $p > 2$.*

PROOF. The sequences A_n and B_n evidently satisfy the inequality of the form (4.17). Consequently, $w_n \in \mathfrak{A}^M_{p,2}(G)$ and according to the principle of the maximum modulus the sequence w_n is uniformly bounded. It remains to prove that it converges uniformly in $\bar{G}$ to a function $w(z)$, $w \in \mathfrak{A}(A, B, G)$ where $A = \lim A_n$, $B = \lim B_n$ (in the metric of L_p). It is easy to prove the compactness of $\{w_n\}$ in C by making use of the formulae

$$w_n(z) = \frac{1}{2\pi i}\int_\Gamma \frac{w_n(\zeta)\,d\zeta}{\zeta - z} + \mathrm{P}_G^{(n)} w_n \qquad (n = 1, 2, \ldots).$$

Hence, every convergent subsequence $\{w_{n_k}\}$ converges to a function of the class $\mathfrak{A}(A, B, G)$ satisfying the equation

$$w(z) = \frac{1}{2\pi i}\int_\Gamma \frac{w(\zeta)\,d\zeta}{\zeta - z} - \frac{1}{\pi}\iint_G \frac{A(\zeta)w(\zeta) + B(\zeta)\overline{w(\zeta)}}{\zeta - z}\,d\xi\,d\eta\,.$$

Here the function $w(\zeta) = \lim w_n(\zeta)$, $\zeta \in \Gamma$, is the same for all $\{w_{n_k}\}$. Now, in view of the principle of the maximum modulus all w_{n_k} converge also inside G to the same limit (see also §12). We have incidentally proved the following

THEOREM 3.10. *If the sequence w_n converges uniformly inside G, then the limit belongs to the class $\mathfrak{A}_{p,2}(A, B, G)$.*

We can also easily generalize Schwarz's lemma.

If (1) $w \in \mathfrak{A}_{p,2}(A, B, E_1)$, $p > 2$, $E_1 = \mathcal{E}(|z| < 1)$, (2) $w \in C(\bar{E}_1)$ *and* (3) $w(0) = 0$, *then*

$$|w(z)| \leqslant |z|^k \hat{M} \max_{|t|=1} |w(t)|\,, \qquad z \in E_1\,, \tag{4.20}$$

where k is the multiplicity of the zero $z = 0$.

The above assertion can easily be verified if we take into account that

$$z^{-k}w(z) \in \mathfrak{A}(A, Be^{-2ik\varphi}, E_1), \qquad \varphi = \arg z .$$

It should be observed that the constant $\hat{M}$ in the inequalities (4.18) and (4.20) is the same in both cases.

4.9. We can also generalize Liouville's theorem.

THEOREM 3.11. *If a generalized analytic function $w(z)$ of the class $\mathfrak{A}_{p,2}(E)$, $p > 2$, is continuous on the entire plane, is bounded and vanishes at a fixed point z_0 of the plane (in particular it may occur that $z_0 = \infty$) then $w(z) = 0$ everywhere.*

PROOF. It is evident from (4.3) and (4.4) that the function $\Phi(z)$ is an entire function. It is bounded on the plane (the inequality (4.16) should be taken into account) and vanishes at the point z_0. By virtue of Liouville's theorem we have $\Phi(z) \equiv 0$, i.e. $w \equiv 0$ everywhere.

In Chapters V and VI geometric and mechanical interpretations of Theorem 3.11 will be indicated. For continuous A and B vanishing outside some bounded domain, this theorem was proved in author's paper [14a] (see §3.5: "Basic lemma"). Under more general conditions—in fact equivalent to those in Theorem 3.11—the theorem was proved in another author's paper [14f]. Let us also observe that Theorem 3.11 may be derived as a consequence of the principle of the maximum modulus.

If $w(z)$ is continuous and bounded on the plane, and belongs to the class $\mathfrak{A}_{p,2}(E)$, $p > 2$, then it follows from (4.3) that $\Phi(z) = c = \text{const.}$ *

Thus, we have

THEOREM 3.12. *Every continuous and bounded on the plane function $w(z)$ of the class $\mathfrak{A}_{p,2}(E)$, $p > 2$, has the form*

$$w(z) = ce^{\omega(z)}, \qquad c = \text{const} . \tag{4.21}$$

* Instead of the boundedness it is sufficient to demand that $w(z) = 0(|z|^{\alpha})$, $\alpha < 1$ (near the point $z = \infty$).

where $\omega(z) = -\mathrm{T}_E\left(A + B\dfrac{\overline{w}}{w}\right)$. *Consequently,* $\omega(z)$ *satisfies the conditions* (4.15) *and* (4.16).

It will be seen below, in §6, that these functions are of a special significance in the class of functions $\mathfrak{A}_{p,2}(E)$. They play here roughly the same part as constants in the class of analytic functions $\mathfrak{A}_0(G)$ (for $A \equiv B \equiv 0$, the right-hand side of the relation (4.21) is evidently equal to a constant). They will therefore be named *the generalized constants.* Thus, *by a generalized constant of the class* $\mathfrak{A}^*_{p,2}(A, B, E)$ *we understand any bounded on the entire plane solution of the equation*

$$\partial_{\bar{z}}w + Aw + B\overline{w} = 0\,, \qquad A\,, B \in L_{p,2}(E)\,, \qquad p > 2\,. \qquad (4.22)$$

The generalized constants of the class $\mathfrak{A}^*_{p,2}(A, B, E)$ will also be called *the constant solutions* od the equation (4.22). In other words, *a generalized constant is a function of the class* $\mathfrak{A}^{\cdot}_{p,2}(E)$, $p > 2$, *such that its normal analytic divisor is equal to a constant.*

Let us now introduce the notions of *the generalized polynomial* and *generalized rational function.* A function of the class $\mathfrak{A}_{p,2}(E), p > 2$, is said to be a generalized rational function if it has a finite number of poles on the extended plane. In this case the normal analytic divisor is a rational function the poles of which coincide with the poles of the corresponding generalized rational function. If a generalized analytic function has one pole at $z = \infty$ it will be called a generalized polynomial. In this case the order of the pole will be said to be the degree of the generalized polynomial. Generalized polynomials of degree zero are generalized constants.

It is now easy to generalize Liouville's theorem in the following way:

THEOREM 3.11′. *If* (1) $w \in \mathfrak{A}_{p,2}(E)$, $p > 2$, *and* (2) $w = O(|z|^n)$ *near the point* $z = \infty$, *where* n *is a non-negative integer, then* w *is a generalized polynomial of the class* $\mathfrak{A}_{p,2}(E)$ *of the degree* n.

4.10. In the class of generalized analytic functions the principle of the argument and its corollaries are preserved (in particular Rouche's theorem). This result follows directly from the formula (4.3) (we have in mind the case of A and $B \in L_{p,2}(E)$, $p > 2$). We shall not expose the statements of the theorems here, since they are exactly the same as in the classical case.

§5. Integral representation of the second kind for generalized analytic functions

5.1. Let us now consider the integral equation

$$f - \mathrm{P}f \equiv f(z) - \mathrm{T}_E(Af + B\bar{f}) = g(z)\,, \tag{5.1}$$

and let us prove that this equation is always soluble if $A, B \in L_{p,2}(E)$, $p > 2$, and $g \in L_{q,0}(E)$, $q \geqslant \dfrac{p}{p-1}$. Since P is a completely continuous operator it suffices to show that the corresponding homogeneous equation $f - \mathrm{P}f = 0$ has only the zero solution $f \equiv 0$.

Let $f_0(z)$ be a solution of the class $L_{q,0}(E)$ of the equation $f_0 - \mathrm{P}f_0 = 0$ Then it will also be a solution of all equations $f_0 - \mathrm{P}^n f_0 = 0$ ($n = 1, 2, \ldots$). But, according to Theorem 1.29 an integer n can be found such that $f_0 \equiv \mathrm{P}^n f_0 \in C_\alpha(E)$, $\alpha = \dfrac{p-2}{p}$. Besides, near infinity we have by the inequality (6.22) of Ch. I, $f_0(z) \equiv \mathrm{P}^n f_0(z) = 0(|z|^{-\alpha})$. On the other hand a solution of the equation $f_0 - \mathrm{P}f_0 = 0$ satisfies also the differential equation $\partial_{\bar{z}} f_0 + Af_0 + B\bar{f}_0 = 0$, i.e. $f_0(z)$ is a continuous generalized analytic function of the class $\mathfrak{A}_{p,2}(E)$, $p > 2$, vanishing at infinity. According to Theorem 3.11 $f = 0$ everywhere. Therefore the equation (5.1) has a solution of the class $L_{q,0}(E)$, and this solution is unique for an arbitrary right-hand side $g(z) \in L_{q,0}(E)$, $q \geqslant \dfrac{p}{p-1}$. It admits the representation

$$f(z) = g(z) + Rg\,, \tag{5.2}$$

where R is a completely continuous linear operator in the space $L_{q,0}(E)$, $q \geqslant \frac{p}{p-1}$. This operator is called the *resolvent* of the operator P.

5.2. Let us now return to the equation

$$\mathfrak{C}(e) \equiv \partial_{\bar{z}} w + Aw + B\bar{w} = F \tag{5.3}$$

and let us first investigate the case of $A, B, \in L_p(\bar{G})$, $p > 2$, $F \in L_1(\bar{G})$, G being a bounded domain. For a solution of the equation (5.3) belonging to the class $L_q(\bar{G})$, $q \geqslant \frac{p}{p-1}$, we have the integral equation

$$w(z) - \frac{1}{\pi} \iint_G \frac{A(\zeta) w(\zeta) + B(\zeta) \overline{w(\zeta)}}{\zeta - z} \, d\xi \, d\eta = g \, , \tag{5.4}$$

where $g = \mathrm{T}F + \Phi(z)$ and

$$\mathrm{T}F = -\frac{1}{\pi} \iint_E \frac{F(\zeta) \, d\xi \, d\eta}{\zeta - z}, \qquad \Phi(z) \in \mathfrak{A}_0^* L_q(\bar{G}) \, , \qquad q \geqslant \frac{p}{p-1} .$$

The function Φ may have simple poles in the domain G, since the number q may be taken as less than two.

Since $F \in L_1(\bar{G})$, according to Theorem 1.26 $\mathrm{T}F \in L_\gamma(\bar{G})$ where γ is an arbitrary number smaller than 2. Therefore, in view of the formula (5.2) the equation (5.4) has always a solution of the form

$$w(z) = w_0(z) + w_*(z) \, , \tag{5.5}$$

where

$$w_0(z) = \Phi(z) + R\Phi \, , \tag{5.6}$$

$$w_*(z) = (\mathrm{T} + R\mathrm{T}) F \, . \tag{5.7}$$

The formula (5.6) associates with every function $\Phi(z)$ analytic in the domain G and belonging to $L_q(\bar{G})$, $q \geqslant \frac{p}{p-1}$, a generalized analytic function $w(z)$ of the class $L_q \mathfrak{A}_{p,2}^*(A, B, G)$. This formula will be named *the representation of the second kind* of functions of the class $\mathfrak{A}_{p,2}^*(A, B, G)$.

The formula (5.7) represents a particular solution of the non-homogeneous equation (5.3). If $F \in L_{p,2}(E)$, $p > 2$, then $\mathrm{T}F \in C_{\frac{p-2}{p}}(E)$ and $w_* \in C_{\frac{p-2}{p}}(E)$.

5.3. Let the plane E be divided into a finite number of domains $G_0, G_1, \dots, G_m$ where G_0 is an unbounded domain, the remaining ones—$G_1, \dots, G_m$—being bounded. The more general case of presence of a few unbounded domains may be reduced to the case considered here by means of a bilinear transformation. It was noted already above (§1.2) that the class $\mathfrak{A}_{p,2}(E)$, $p > 2$, is preserved in such a transformation.

Let $\Phi(z)$ be a sectionally holomorphic function satisfying the following conditions: (1) Φ is holomorphic inside every domain $G_0, G_1, \dots, G_m$, continuous up to their boundaries except for a finite number of boundary points, and near every such point $\Phi = O(|z-z_0|^{-a})$, $0 \leqslant a < \frac{2(p-1)}{p}$; (2) Φ is bounded in the vicinity of the point $z = \infty$. Such being the case $\Phi \in L_{q,0}(E)$, $q = \frac{p}{p-1}$ and we may construct a solution of the equation $\mathfrak{C}(w) = 0$ according to the formula (5.6). Since $w = \Phi - \mathrm{T}(Aw + B\overline{w})$ and $\mathrm{T}(Aw + B\overline{w}) \in C_{\frac{p-2}{p}}(E)$ the character of the continuity and the discontinuities of w at the boundary points of the domains G_j will be the same as that of the function Φ. If Φ satisfies on the curves Γ_j bounding the domains G_j the conditions of the form (Hilbert boundary problem, [26], [60a]; Ch. II, §37)

$$\Phi^+(\zeta) = \Phi^-(\zeta) + g_0(\zeta)\,, \qquad \zeta \in \Gamma\,, \tag{5.8}$$

then $w(z)$ will also satisfy the same conditions. Thus, by solving the non-homogeneous Hilbert problem (5.8) for analytic functions we obtain the solution of the analogous problem for generalized analytic functions. The results on this topic were obtained in the paper of Mikhailov [55a, b] (see also [1a]).

5.4. The integral equation (5.4) may be regarded as a device for constructing any solution of the equation (5.3). By means of (5.4) every function $\Phi(z) \in L_q(\bar{G})$, $q \geqslant \frac{p}{p-1}$ analytic in the domain G is associated with a fully determined solution. This equation may be solved by successive approximations according to the following simple scheme:

$$w_{n+1} = \frac{1}{\pi}\iint\limits_G \frac{Aw_n + B\bar{w}_n}{\zeta - z}\, d\xi\, d\eta + \Phi(z) + \mathrm{T}F \qquad (5.9)$$
$$(n = 0, 1, \ldots),$$

where $w_0 = \Phi + \mathrm{T}F$. This scheme may still be simplified by setting $A = 0$ i.e. considering an equation of the form

$$\partial_{\bar{z}} w + B\bar{w} = F\,. \qquad (5.9\text{a})$$

The equation (5.3) may always be reduced to the last equation by means of the substitution

$$w(z) = \hat{w}(z)\, e^{\frac{1}{\pi}\iint\limits_G \frac{A(\zeta)}{\zeta - z} d\xi d\eta}\,. \qquad (5.9\text{b})$$

In future, therefore, if we find it expedient, we shall limit ourselves to the consideration of an equation of the form (5.9a). It should be observed that the substitution (5.9b) leaves the equation in the class $L_{p,2}(E)$.

5.5. Parallel with (5.4) we may consider a number of other integral equations enabling us to construct solutions of the equation (5.3); these equations may so be chosen that the required solutions satisfy certain imposed beforehand conditions. We show, for instance, how an integral equation may be derived making it possible to construct solutions of the equation (5.3) having given values at fixed points $z_1, \ldots, z_n$. To this end we introduce a polynomial of $(n-1)$th degree with respect to z, of the following form:

$$P(z, \zeta, z_1, \ldots, z_n)$$
$$= \sum_{k=1}^{n} \frac{(z-z_1)\ldots(z-z_{k-1})(z-z_{k+1})\ldots(z-z_n)}{(z_k-z_1)\ldots(z_k-z_{k-1})(z_k-z_{k+1})\ldots)(z_k-z_n)}\, \frac{1}{\zeta - z_k},$$

and let us consider the integral equation

$$w(z)+\hat{\mathrm{T}}_G(Aw+B\overline{w}) = \Phi(z)+\hat{\mathrm{T}}_G F\,, \qquad (5.10)$$

where

$$\hat{\mathrm{T}}_G f = -\frac{1}{\pi}\iint_G f(\zeta)\left[\frac{1}{\xi-z}-P(z,\zeta,z_1,\ldots,z_n)\right]d\xi\,d\eta\,.$$

It is readily seen that a solution of this equation satisfies the differential equation (5.3) and, conversely, any solution of the latter equation may be represented in the form (5.10).

We prove that this integral equation is soluble for an arbitrary right-hand side belonging to $L_q(\overline{G})$, $q \geqslant \frac{p}{p-1}$. By considering the appropriate homogeneous equation we discover that its solution $w_0(z)$ satisfies the following conditions: (1) $\mathfrak{C}(w_0) = 0$ (in G), (2) $w_0 \in C_\alpha(E)$, $\alpha = \frac{p-2}{p}$, and (3) $w_0(z)$ is holomorphic outside $G+\Gamma$ and has at infinity a pole of the order $(n-1)$. Representing $w_0(z)$ by the formulae (4.3) we find that the corresponding analytic divisor $\Phi_0(z)$ is holomorphic on the entire plane and has at infinity a pole of the order $(n-1)$. At the same time $\Phi_0(z)$ vanishes at n points $z_1, \ldots, z_n$ at which $w(z)$ vanishes. Therefore $\Phi_0 \equiv 0$, thus proving that the homogeneous integral equation corresponding to (5.10) has no non-zero solutions. Consequently, the latter integral equation has a solution for an arbitrary right-hand side belonging to L_q, $q \geqslant \frac{p}{p-1}$.

If we now take in the right-hand side of (5.10) for $\Phi(z)$ an analytic function satisfying the relations

$$\Phi(z_k) = a_k+ib_k \qquad (k = 1, \ldots, n)\,,$$

where a_k and b_k are given real constants then we shall obtain solutions of the equation (5.3) having at the points z_k the values a_k+ib_k. In particular, in this way we can

construct solutions of the equation (5.3) having fixed beforehand zeros.

On the basis of this result it is readily observed that every solution $w(z)$ of the equation (5.3) can be represented in the form of the sum

$$w(z) = w_0(z) + w_*(z) ,$$

where $w_0(z)$ is a solution of the homogeneous equation $\mathfrak{C}(w) = 0$ having at the points z_k the values $w(z_k)$, and w_* is a solution of the non-homogeneous equation (5.3) vanishing at the points z_k.

§6. Generating pair of functions of the class $\mathfrak{A}_{p,2}(A, B, E)$. Derivative in the Bers sense

6.1. If G coincides with E and $w(z)$ is bounded and belongs to $\mathfrak{A}_{p,2}(A, B, E)$, $p > 2$, then according to the formula (5.4) we have

$$w(z) - \frac{1}{\pi}\iint_E \frac{A(\zeta)w(\zeta) + B(\zeta)\overline{w(\zeta)}}{\zeta - z} d\xi d\eta \equiv w - \mathrm{P}w = c_0 + ic_1 , \qquad (6.1)$$

since $F \equiv 0$ and Φ, evidently, is constant. Consequently, in view of the formula (5.6)

$$w(z) = c_0 w_0(z) + c_1 w_1(z) , \qquad (6.2)$$

where $w_0(z)$ and $w_1(z)$ are solutions of equations

$$w_0 - \mathrm{P}w_0 = 1 , \quad w_1 - \mathrm{P}w_1 = i . \qquad (6.3)$$

According to Theorem 3.12 the following formulae take place:

$$w_0(z) = e^{\omega_0(z)} , \quad w_1(z) = ie^{\omega_1(z)} , \qquad (6.4)$$

where

$$\omega_j(z) = \iint_E \frac{g_j(\zeta)}{\zeta - z} d\xi d\eta , \qquad (6.5)$$

$$g_j(z) = \frac{1}{\pi}\left(A(z) + B(z)\frac{\overline{w_j(z)}}{w_j(z)}\right) \quad (j = 0, 1) .$$

Thus, $w_j \neq 0$ everywhere in E and $w_0(\infty) = 1$, $w_1(\infty) = i$. The formula (6.2) yields the general form of the generalized constant of the class $\mathfrak{A}_{p,2}(A, B, E)$.

By virtue of Theorem 3.12 a generalized constant may vanish at a fixed point of the plane if and only if $c_0 = c_1 = 0$. Hence $w_0(z)$ and $w_1(z)$ satisfy everywhere on the plane the condition

$$\operatorname{Im}[\overline{w_0(z)} \cdot w_1(z)] \geqslant k_0 > 0 ; \quad k_0 = \text{const.} \tag{6.6}$$

Since

$$\partial_{\bar{z}} w_0 + A w_0 + B \overline{w}_0 = 0 , \qquad \partial_{\bar{z}} w_1 + A w_1 + B \overline{w}_1 = 0 ,$$

we have

$$A = \frac{\overline{w}_1 \partial_{\bar{z}} w_0 - \overline{w}_0 \partial_{\bar{z}} w_1}{w_1 \overline{w}_0 - w_0 \overline{w}_1} , \qquad B = \frac{w_0 \partial_{\bar{z}} w_1 - w_1 \partial_{\bar{z}} w_0}{w_1 \overline{w}_0 - w_0 \overline{w}_1} . \tag{6.7}$$

Thus, there is a one-to-one correspondence between the pairs (A, B) and (w_0, w_1). If A and B belonging to $L_{p,2}(E)$ are given, then we can determine uniquely w_0 and w_1 by solving the integral equations (6.3); then also $w_0, w_1 \in C_{\frac{p-2}{p}}(E)$ and possess generalized derivatives with respect to $\bar{z}$ belonging to $L_{p,2}(E)$. Besides, the condition (6.6) is satisfied on the entire plane.

If now the pair of functions w_0, w_1 is prescribed and it satisfies the conditions enumerated above: (1) $w_0, w_1 \in C_{\frac{p-2}{p}}(E)$, (2) $\partial_{\bar{z}} w_0, \partial_{\bar{z}} w_1 \in L_{p,2}(E)$ and (3) the inequality (6.6) is satisfied on the entire plane, then the formulae (6.7) determine uniquely the corresponding pair A, B. Therefore the pair of functions $w_0(z)$, $w_1(z)$, following Bers, will be named *the generating pair* of the class $\mathfrak{A}_{p,2}(A, B, E)$. Such a pair of functions was used by Bers as the basis of his theory of pseudoanalytic functions ([5a, b, c], [7a, b]).

6.2. Every function $w(z)$ may be represented uniquely at any point in the form

$$w(z) = \chi_0(z) w_0(z) + \chi_1(z) w_1(z) , \tag{6.8}$$

where χ_0, χ_1 are real functions. Following Bers we shall define *the derivative with respect to the pair* (w_0, w_1) at the point z_0 as follows:

$$\dot{w}(z_0) = \lim_{z \to z_0} \frac{w(z) - \chi_0(z_0) w_0(z) - \chi_1(z_0) w_1(z)}{z - z_0}. \tag{6.9}$$

It can now readily be proved that it is necessary and sufficient for the existence of $\dot{w}(z_0)$ that the following relation holds at the point z_0

$$\frac{\partial w}{\partial \bar{z}_0} + A(z_0) w(z_0) + B(z_0) \overline{w(z_0)} = 0, \tag{6.10}$$

where A and B are functions of the form (6.7). According to Bers the function $w(z)$ is called *the pseudoanalytic function of the first kind* in the domain G if it is continuous and has almost everywhere in this domain a derivative $\dot{w}(z)$ with respect to the pair (w_0, w_1).

Thus, the class of pseudoanalytic functions in the Bers sense corresponding to the pair (w_0, w_1) coincides with the class of generalized analytic functions $\mathfrak{A}(A, B, G)$ in our sense.

6.3. If a function w satisfies the equation (6.10), then according to the relation (6.8) there is a complex function $\chi = \chi_0 + i\chi_1$ uniquely associated with w, which, following Bers, will be called *the pseudoanalytic function of the second kind*. It is readily observed that it satisfies the equation

$$\partial_{\bar{z}} \chi - q(z) \overline{\partial_z \chi} = 0, \qquad q = \frac{w_0 + i w_1}{w_0 - i w_1}. \tag{6.11}$$

Taking into account (6.6) we easily obtain

$$|q(z)| \leqslant q_0 < 1 \quad (\text{in } E), \qquad q_0 = \text{const}. \tag{6.12}$$

We shall prove below that every solution of the equation (6.11) in the domain G can be represented in the form (§17)

$$\chi(z) = \Phi[W(z)], \tag{6.13}$$

where $W(z)$ is a complete homeomorphism of some Beltrami's equation of the form

$$\partial_{\bar{z}}\chi - \tilde{q}(z)\partial_z\chi = 0\,, \qquad |\tilde{q}| \leqslant q_0 < 1\,, \tag{6.14}$$

and $\Phi(\zeta)$ is a function analytic in ζ in the domain $W(G)$.

The formula (6.13) implies at once the following important theorem proved by Bers [5c]:

Every non-constant pseudoanalytic function of the second kind $\chi(z) = \chi_0(z) + i\chi_1(z)$ *establishes an interior mapping in the Stoiloff sense.*

6.4. Now there arises in a natural way the problem of properties of mappings established by generalized analytic functions. It is easy to prove the following assertion:

For an arbitrary point z_0 *it is possible to find a function in any class* $\mathfrak{A}_{p,2}(E)$, $p > 2$, *which is univalent inside a neighbourhood* G_0 *of the point* z_0.

This condition is satisfied, for instance, by a function of an arbitrary class $\mathfrak{A}_{p,2}(E)$ with the normal divisor $z - z_0$. Thus, there always exist locally univalent solutions of any equation of the form $\mathfrak{C}(w) \equiv \partial_{\bar{z}}w + Aw + B\bar{w} = 0$ $\big(A, B \in L_{p,2}(E)\big)$. Examples may be constructed indicating that in general Riemann theorems for such equations are not preserved. Nevertheless, it was shown by Daniluk [32c] that *it is always possible to construct classes of solutions of an equation of the form* $\mathfrak{C}(w) = 0$, *which establish interior mappings in the Stoiloff sense* (see also [70b]).

§7. Inversion of the non-linear integral equation (4.3)

The formula (5.6) enables us to construct for every function $\Phi(z)$ holomorphic in G the corresponding generalized analytic function $w(z)$. This formula may be applied even if Φ has simple poles in G. However, if Φ has inside the domain or on its boundary singularities of a higher order, then the formula (5.6) in general has no sense. Hence the problem arises in a natural way of

constructing generalized analytic functions corresponding to analytic functions possessing singular points of an arbitrary kind. This problem is solved by

THEOREM 3.13. *Let Φ be a function analytic in G, which may possess arbitrary singularities. Let t be a fixed point of the domain G. Then there exists a function $w(z)$ of the point $z \in G$ which satisfies the following conditions: the function $w_0(z) = \dfrac{w(z)}{\Phi(z)}$ is continuous in G and it is continuously continuable to the entire plane, and*

$$1)\ w_0 \in C_{\frac{p-2}{p}}(E)\,, \quad 2)\ w_0(z) \neq 0 \ (\text{in } E)\,, \quad 3)\ w_0(t) = 1\,. \qquad (7.1)$$

Moreover, at the regular points of $\Phi(z)$ the function w satisfies the equation $\mathfrak{C}(w) \equiv \partial_{\bar{z}} w + Aw + B\overline{w} = 0$.

PROOF. Let us consider the integral equation

$$w_0(z) - \frac{z-t}{\pi} \iint_G \frac{A(\zeta) w_0(\zeta) + B_0(\zeta)\overline{w_0(\zeta)}}{(\zeta - z)(\zeta - t)}\, d\xi\, d\eta = 1\,, \qquad (7.2)$$

$$B_0 = B\frac{\overline{\Phi}}{\Phi}\,.$$

This equation has a unique solution, since the corresponding homogeneous equation has no non-zero solution (this result follows at once from Theorem 3.11). A solution of the equation (7.2) satisfies all three conditions (7.1). The first and the third conditions are obvious. Let us prove that the second condition is also satisfied. Let $w_0(z_0) = 0$ where z_0 is a fixed point. Then it follows from (7.2) that

$$w_0(z) - \frac{z - z_0}{\pi} \iint_G \frac{A(\zeta) w_0(\zeta) + B_0(\zeta)\overline{w_0(\zeta)}}{(\zeta - z(\zeta - z_0)}\, d\xi\, d\eta = 0\,.$$

But this homogeneous equation, as it was already indicated above, has only the zero solution $w_0 \equiv 0$.

By considering the function $w = \Phi w_0$ we readily see that $\mathfrak{C}(w) \equiv 0$ everywhere in G except for the set of singular points of the function $\Phi(z)$. It can easily be

verified that $w = \Phi w_0$ satisfies the non-linear integral equation

$$w(z) = \Phi(z)\exp\left(\frac{z-t}{\pi}\iint_G \frac{A(\zeta)w(\zeta)+B(\zeta)\overline{w(\zeta)}}{(\zeta-z)(\zeta-t)w(\zeta)}d\xi\,d\eta\right). \quad (7.3)$$

The existence of a solution of this equation was proved in a somewhat different way by Bers, [5a]. We have reconstructed the proof given by the author in [14f]. Thus, this equation may be regarded as an operator enabling us to associate with every function $\Phi(z)$ analytic in the domain G and with every fixed point t of the plane, a completely determined solution $w(z, t)$ of the equation $\mathfrak{C}(w) = 0$. In future this (non-linear) operator will be denoted by $\mathcal{R}_t(\Phi)$. The operator $\mathcal{R}_t(\Phi)$ makes possible the construction of solutions of the equation $\mathfrak{C}(w) = 0$ which have singularities of an arbitrary order at given points of the domain or of its boundary. In particular, for $\Phi(z)$ sectionally holomorphic functions may be taken. If, for example, a sectionally holomorphic function $\Phi(z)$ is a solution of the homogeneous Hilbert boundary problem ([60a], Ch. II, §34)

$$\Phi^+(z) = g(z)\Phi^-(z) \qquad (\text{on } \Gamma), \qquad (7.4)$$

then the corresponding sectionally analytic function $w(z) = \mathcal{R}_t[\Phi(z)]$ satisfies the same boundary condition, [55a, b],

$$w^+(z) = g(z)w^-(z) \qquad (\text{on } \Gamma). \qquad (7.5)$$

The formula (7.3) (and, consequently, Theorem 3.13) hold also in the limiting case $t = \infty$. Then the operator $\mathcal{R}_\infty(\Phi)$ associates with a given normal analytic divisor $\Phi(z)$ the corresponding solution of the equation $\mathfrak{C}(w) = 0$.*

* In the paper [23*a] this theorem is unjustly attributed to Mikhailov [55a].

§8. Systems of fundamental generalized analytic functions and fundamental kernels of the class $\mathfrak{A}_{p,2}(A, B, G)$, $p > 2$

8.1. Let $X_j(z, t) = \mathcal{R}_t[\Phi_j(z)]\,(j = 1, 2)$ be solutions of the equation $\mathfrak{C}(w) = 0$ corresponding to the functions

$$\Phi_1(z) = \frac{1}{2(t-z)}, \quad \Phi_2(z) = \frac{1}{2i(t-z)}, \tag{8.1}$$

where t is a fixed point of the plane. Evidently, the following conditions are satisfied:

$$\lim_{z\to t}(t-z)X_1(z, (t) = \frac{1}{2}, \quad \lim_{z\to t}(t-z)X_2(z, t) = \frac{1}{2i}. \tag{8.2}$$

It is readily observed that X_1 and X_2 satisfy the integral equations

$$X_1(z, t) - \frac{1}{\pi}\iint_E \frac{A(\zeta)X_1(\zeta, t) + B(\zeta)\overline{X_1(\zeta, t)}}{\zeta - z}\, d\xi\, d\eta = \frac{1}{2(t-z)}. \tag{8.3}$$

$$X_2(z, t) - \frac{1}{\pi}\iint_E \frac{A(\zeta)X_2(\zeta, t) + B(\zeta)\overline{X_2(\zeta, t)}}{\zeta - z}\, d\xi\, d\eta = \frac{1}{2i(t-z)}. \tag{8.4}$$

Representing X_1 and X_2 in the form (7.3) we have

$$X_1(z, t) = \frac{e^{\omega_1(z,t)}}{2(t-z)}, \quad X_2(z, t) = \frac{e^{\omega_2(z,t)}}{2i(t-z)}, \tag{8.5}$$

where

$$\omega_j(z, t) = \frac{t-z}{\pi}\iint_E \frac{A(\zeta)X_j(\zeta, t) + B(\zeta)\overline{X_j(\zeta, t)}}{(\zeta - z)(t-\zeta)X_j(\zeta, t)}\, d\xi\, d\eta, \tag{8.6}$$
$$j = 1, 2.$$

In view of the inequalities (4.15) and (4.16) it can easily be shown that $\omega_j(z, t)$ satisfy the following conditions:

$$|\omega_j(z, t)| \leqslant M'_p, \quad |\omega_j(z, t)| \leqslant M'_p|z-t|^{\frac{p-2}{p}}, \tag{8.7}$$

$$|\omega_j(z, t)| \leqslant M'_p\left(|z|^{\frac{2-p}{p}} + |t|^{\frac{2-p}{p}}\right) \quad \text{for} \quad |z|, |t| \geqslant R > 1, \tag{8.8}$$

$$|\omega_j(z_1, t) - \omega_j(z_2, t)| \leqslant M'_p|z_1 - z_2|^{\frac{p-2}{p}}, \tag{8.9}$$

$$|\omega_1(z, t) - \omega_2(z, t)| \leqslant M''_p|z-t|^{\frac{p-2}{p}}, \tag{8.10}$$

where

$$M'_p = M_p L_{p,2}(|A|+|B|), \qquad M''_p = 2M_p L_{p,2}(|B|). \quad (8.11)$$

The pair of functions X_1 and X_2 will be called *the system of fundamental generalized analytic functions of the class* $\mathfrak{A}_{p,2}(A, B, E)$ *with a pole at the point* t. These functions are continuous in the Hölder sense everywhere on the plane except for the point t, and satisfy the equations ($z \neq t$)

$$\begin{aligned} &\partial_{\bar z} X_1 + A(z) X_1 + B(z)\overline{X_1} = 0, \\ &\partial_{\bar z} X_2 + A(z) X_2 + B(z)\overline{X_2} = 0. \end{aligned} \quad (8.12)$$

8.2. Let us now introduce the following functions:

$$\begin{aligned} \Omega_1(z, t) &= X_1(z, t) + iX_2(z, t), \\ \Omega_2(z, t) &= X_1(z, t) - iX_2(z, t), \end{aligned} \quad (8.13)$$

which, obviously, satisfy the following system of equations:

$$\begin{aligned} &\partial_{\bar z}\Omega_1 + A(z)\Omega_1 + B(z)\overline{\Omega_2} = 0, \\ &\partial_{\bar z}\Omega_2 + A(z)\Omega_2 + B(z)\overline{\Omega_1} = 0. \end{aligned} \quad (8.14)$$

Since

$$\Omega_1(z, t) = \frac{e^{\omega_1(z,t)} + e^{\omega_2(z,t)}}{2(t-z)}, \qquad \Omega_2(z, t) = \frac{e^{\omega_1(z,t)} - e^{\omega_2(z,t)}}{2(t-z)}, \quad (8.15)$$

in view of the inequalities (8.7) and (8.10) we have

$$\begin{aligned} \Omega_1(z, t) - \frac{1}{t-z} &= O\left(|z-t|^{-\frac{2}{p}}\right), \\ \Omega_2(z, t) &= O\left(|z-t|^{-\frac{2}{p}}\right). \end{aligned} \quad (8.16)$$

If z be fixed and $z \neq \infty$, $t \to \infty$ we have the estimates

$$\Omega_1(z, t) = O(|t|^{-1}), \qquad \Omega_2(z, t) = O(|t|^{-1}). \quad (8.17)$$

It should be observed that, as is seen from (8.6), if $B \equiv 0$, then $\omega_1 = \omega_2$ and, consequently, $\Omega_2 \equiv 0$.

It can easily be proved that the conditions (8.14), (8.16) and (8.17) determine uniquely $\Omega_1(z, t)$ and $\Omega_2(z, t)$. The functions Ω_1 and Ω_2 will be called *the fundamental kernels of the class* $\mathfrak{A}_{p,2}(A, B, E)$, $p > 2$.

§9. Adjoint equation. Green's identity. Equations of the second order

9.1. We consider the equation

$$\mathfrak{C}(w) \equiv \partial_{\bar{z}} w + Aw + B\overline{w} = F \tag{9.1}$$

and its adjoint equation

$$\mathfrak{C}'(w') \equiv \partial_{\bar{z}} w' - Aw' - \overline{B}\overline{w}' = F' . \tag{9.2}$$

If w and w' are continuous in $G+\Gamma$ and belong to the class $D_{1,p}(G)$, $p > 2$, then in view of the integral formula (7.5) of Ch. I, we have

$$\begin{aligned}\frac{1}{2i}\int_\Gamma w(z)w'(z)dz &= \iint_G \frac{\partial ww'}{\partial \bar{z}}\, dx\,dy \\ &= \iint_G \left(w\frac{\partial w'}{\partial \bar{z}} + w'\frac{\partial w}{\partial \bar{z}}\right) dx\,dy \\ &= \iint_G [w\mathfrak{C}'(w') + w'\mathfrak{C}(w) + \overline{B}w\overline{w}' - B\overline{w}w']\,dx\,dy .\end{aligned}$$

We have, therefore, the identity

$$\operatorname{Re}\left[\frac{1}{2i}\int_\Gamma w(z)w'(z)dz\right] = \operatorname{Re}\iint_G \big(w\mathfrak{C}'(w') + w'\mathfrak{C}(w)\big)\,dx\,dy , \tag{9.3}$$

which represents the property of mutual adjointness of the equations (9.1) and (9.2).

If $\mathfrak{C}(w) = 0$ and $\mathfrak{C}'(w') = 0$ we have

$$\operatorname{Re}\left(\frac{1}{2i}\int_\Gamma w(z)w'(z)dz\right) = 0 . \tag{9.4}$$

This formula, which will henceforth be called *Green's identity*, holds for two arbitrarily chosen solutions w and w' of the equations $\mathfrak{C}(w)=0$ and $\mathfrak{C}'(w')=0$, which are continuous in $G+\Gamma$.

In the real notation the mutually adjoint systems of equations have the form

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}+au+bv=0, \qquad \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+cu+dv=0, \tag{9.5}$$

$$\frac{\partial u'}{\partial x}-\frac{\partial v'}{\partial y}+du'-cv'=0, \qquad \frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}-bu'+av'=0. \tag{9.6}$$

The formula (9.4) assumes the form

$$\int\limits_{\Gamma}(uv'+vu')\,dx+(uu'-vv')\,dy=0\,. \tag{9.7}$$

9.2. Let us consider the real function

$$\varphi(z)=\operatorname{Re}\left\{\frac{1}{i}\int\limits_{z_0}^{z}w(\zeta)w'(\zeta)\,d\zeta\right\}, \quad z_0\text{—a fixed point}, \tag{9.8}$$

where w and w' are arbitrary continuous solutions of the equations $\mathfrak{C}(w)=0$ and $\mathfrak{C}'(w')=0$, respectively. By virtue of (9.4) $\varphi(z)$ is single-valued in the domain G which we assume to be simply-connected, and is independent of the path of integration. We have

$$\partial_z\varphi=\frac{1}{2i}\,w(z)w'(z), \qquad \partial_{\bar z}\varphi=-\frac{1}{2i}\,\overline{w(z)}\,\overline{w'(z)}\,. \tag{9.9.}$$

Taking into account that $\partial_{\bar z}w=-Aw-B\overline{w}$, $\partial_{\bar z}w'=Aw'+$ $+\overline{B}\overline{w}'$ we have

$$\frac{\partial^2\varphi}{\partial z\partial\bar z}=\frac{Bw'}{w'}\varphi_{\bar z}+\frac{\overline{B}\overline{w}'}{w'}\varphi_z\,,$$

i.e.

$$\Delta\varphi+a\varphi_x+b\varphi_y=0,\,, \tag{9.10}$$

$$a=-4\operatorname{Re}\left(\frac{Bw'}{\overline{w}'}\right), \qquad b=-4\operatorname{Im}\left(\frac{Bw'}{\overline{w}'}\right). \tag{9.11}$$

Here an arbitrary solution of the equation $\mathfrak{C}(w') = 0$ may be taken for w'. Choosing for w' a fixed solution of the last equation we may, according to the formula (9.8), associate with a solution w of the equation $\mathfrak{C}(w) = 0$ a fully determined solution of the elliptic equation of the second order (9.10), and, conversely, if φ is a solution of the equation (9.10) then the formula

$$w = \frac{2i}{w'}\partial_z\varphi \equiv \frac{i}{w'}(\varphi_x - i\varphi_y) \tag{9.12}$$

yields a solution of the equation $\mathfrak{C}(w) = 0$. Accordingly, *every real solution of the equation* (9.10) *will be called the potential function or simply the potential of the equation* $\mathfrak{C}(w) = 0$. *The equation* (9.10) *will be called the equation of the potential.*

The last results show that the problem of integration of an equation of the second order of the form (9.10) and the problem of integration of a system of equations of the form (9.5), are equivalent.

It should be observed that an equation of the second order of a more general form

$$\Delta\varphi + a\varphi_x + b\varphi_y + c\varphi = 0, \quad c \neq 0, \tag{9.13}$$

in general cannot be reduced to a system of the form (9.5) ([14a], §4.6).

9.3. Let $A \equiv 0, B \not\equiv 0$. Then any solution of the equation $\partial_{\bar z}w + B\overline{w} = 0$ satisfies also the following equation of the second order

$$\frac{\partial}{\partial z}\left(\frac{1}{B}\frac{\partial w}{\partial \bar z}\right) + \frac{\partial \overline{w}}{\partial z} = 0,$$

or, taking into account that $\partial_z\overline{w} = -\bar{B}w$

$$\frac{\partial^2 w}{\partial z\partial \bar z} - \frac{1}{B}\frac{\partial B}{\partial z}\frac{\partial w}{\partial \bar z} - B\bar{B}w = 0. \tag{9.14}$$

If w is a solution of the equation (9.14) it can easily be verified that the functions

$$w_1 = \frac{1}{2}\left(w + \frac{1}{\bar B}\frac{\partial \overline{w}}{\partial z}\right), \quad w_2 = \frac{1}{2i}\left(w - \frac{1}{\bar B}\frac{\partial \overline{w}}{\partial z}\right) \tag{9.15}$$

satisfy the equation $\partial_{\bar{z}} w + B\bar{w} = 0$. But 9.15 implies that

$$w = w_1 + iw_2 \,. \tag{9.16}$$

Thus, any solution of the equation of the second order (9.14) admits a representation in the form (9.16) where w_1 and w_2 are arbitrary solutions of the equation of the first order $\partial_{\bar{z}} w + B\bar{w} = 0$. Conversely, any solution of the equation $\partial_{\bar{z}} w + B\bar{w} = 0$ may be obtained from the formulae (9.15). We note that it is sufficient to make use of one of these formulae in order to obtain all solutions of the equation of the first order $\partial_{\bar{z}} w + \bar{B}w = 0$. In fact, we observe that by replacing w by iw one of the formulae (9.15) transforms into the other one. It is also readily observed that if w is a solution of the equation (9.14), then iw is also its solution.

The solutions of the equation (9.14), *which are in general complex functions, may be called the complex potentials of the equation of the first order* $\partial_{\bar{z}} w + B\bar{w} = 0$.

If $B_z \not\equiv 0$ the equation (9.14) has complex coefficients which have singularities at the points at which B vanishes. Therefore, the integration of an equation of the second order of the form (9.14) with discontinuous coefficients, may be reduced to the integration of the equation of the first order $\partial_{\bar{z}} w + B\bar{w} = 0$ with a continuous coefficient.

§10. Generalized Cauchy formula

10.1. Let us denote by $X_1'(z, t)$ and $X_2'(z, t)$ the system of fundamental solutions of the adjoint equation $\mathfrak{C}'(w') = 0$. Then the kernels of this equation will be given by the formulae (see (8.13))

$$\begin{aligned} \Omega_1'(z, t) &= X_1'(z, t) + iX_2'(z, t) \,, \\ \Omega_2'(z, t) &= X_1'(z, t) - iX_2'(z, t) \,. \end{aligned} \tag{10.1}$$

In this section we shall establish some relationships between the fundamental solutions of adjoint equations. Formulae will be derived representing explicitly $X_j'(z, t)$ in terms of $X_j(z, t)$, and conversely. Moreover, a formula

will be derived which constitutes a generalization of the classical integral Cauchy formula.

Let $t \in G$, where G is a domain bounded by a finite number of simple smooth closed Jordan curves the union of which will be denoted by Γ. Let Γ_ε be the circle $|z-t| = \varepsilon$ where ε is a sufficiently small positive number. Taking for w' the fundamental solutions $X_1'(z, t)$ and $X_2'(z, t)$ of the adjoint equation $\mathfrak{C}'(w') = 0$ and applying the formula (9.4) to the domain bounded by Γ and Γ_ε we have

$$\int_\Gamma w(z)X_k'(z,t)\,dz - \overline{w(z)}\,\overline{X_k'(z,t)}\,d\bar{z}$$
$$= \int_{\Gamma_\varepsilon} w(z)X_k'(z,t)\,dz - \overline{w(z)}\,\overline{X_k'(z,t)}\,d\bar{z}\,, \qquad k = 1, 2\,.$$

Multiplying the second equation ($k = 2$) by i and summing with the first we obtain

$$\int_\Gamma w(z)\Omega_1'(z,t)\,dz - \overline{w(z)}\,\overline{\Omega_2'(z,t)}\,d\bar{z}$$
$$= \int_{\Gamma_\varepsilon} w(z)\Omega_1'(z,t)\,dz - \overline{w(z)}\,\overline{\Omega_2'(z,t)}\,d\bar{z}\,.$$

Hence, when $\varepsilon \to 0$ we obtain in view of the estimates (8.16)

$$\int_\Gamma w(z)\Omega_1'(z,t)\,dz - \overline{w(z)}\,\overline{\Omega_2'(z,t)}\,d\bar{z} = -2\pi i w(t)\,, \qquad t \in G\,.$$

If $t \in \Gamma$ or $t \,\bar{\in}\, G+\Gamma$ we obtain analogous relations in the right-hand sides of which appear the functions $-\alpha\pi i w(t)$ or 0, respectively, $\alpha\pi$ being the interior angle at the point t, and $0 < \alpha \leqslant 2$. In other words the following formula takes place:

$$-\frac{1}{2\pi i}\int_\Gamma \Omega_1'(t,z)w(t)\,dt - \overline{\Omega_2'(t,z)}\,\overline{w(t)}\,d\bar{t}$$
$$= \begin{cases} w(z), & \text{when} \quad z \in G\,, \\ \dfrac{\alpha}{2}w(z), & \text{when} \quad z \in \Gamma\,, \\ 0, & \text{when} \quad z \,\bar{\in}\, \bar{G}\,. \end{cases} \qquad (10.2)$$

Let ζ be a fixed point of the plane. Let Γ_ε and $\Gamma_{1/\varepsilon}$ be the circles $|z-\zeta| = \varepsilon$ and $|z-\zeta| = \dfrac{1}{\varepsilon}$, respectively. Since $X_1(z, \zeta)$ and $X_2(z, \zeta)$ are continuous solutions of the equations (9.1) in the closed domain $\varepsilon \leqslant |z-\zeta| \leqslant \dfrac{1}{\varepsilon}$, according to the formula (10.2) we have the relation

$$X_k(z, \zeta) = -\frac{1}{2\pi i}\int\limits_{\Gamma_{1/\varepsilon}} X_k(t, \zeta)\Omega_1'(t, z)\,dt - \overline{X_k(t, \zeta)}\,\overline{\Omega_2'(t, z)}\,d\bar{t} + \\ + \frac{1}{2\pi i}\int\limits_{\Gamma_\varepsilon} X_k(t, \zeta)\Omega_1'(t, z)\,dt - \overline{X_k(t, \zeta)}\,\overline{\Omega_2'(t, z)}\,d\bar{t}\,. \qquad (10.3)$$

Since

$$\Omega_1'(t, z)X_k(t, \zeta) = O(\varepsilon^2), \quad \Omega_2'(t, z)X_k(t, \zeta) = O(\varepsilon^2)$$

if $|t-\zeta| = \dfrac{1}{\varepsilon}$, we have

$$X_1(\zeta, t) = \frac{1}{2(t-\zeta)}\left[1 + O\left(\varepsilon^{\frac{p-2}{p}}\right)\right],$$

$$X_2(\zeta, t) = \frac{1}{2i(t-\zeta)}\left[1 + O\left(\varepsilon^{\frac{p-2}{p}}\right)\right] \quad \text{if} \quad |t-\zeta| = \varepsilon\,,$$

and passing to the limit $\varepsilon \to 0$ in the relations (10.3) we obtain

$$\begin{aligned} X_1(z, \zeta) &= -\frac{1}{2}[\Omega_1'(\zeta, z) + \overline{\Omega_2'(\zeta, z)}]\,, \\ X_2(z, \zeta) &= -\frac{1}{2i}[\Omega_1'(\zeta, z) - \overline{\Omega_2'(\zeta, z)}]\,. \end{aligned} \qquad (10.4)$$

Since $X_1 + iX_2 = \Omega_1(z, \zeta)$, $X_1 - iX_2 = \Omega_2(z, \zeta)$ we obtain from (10.4)

$$\Omega_1(z, \zeta) = -\Omega_1'(\zeta, z), \qquad \Omega_2(z, \zeta) = -\overline{\Omega_2'(\zeta, z)}\,, \qquad (10.5)$$

i.e.

$$\begin{aligned} X_1'(z, \zeta) &= -\frac{1}{2}\left(X_1(\zeta, z) + \overline{X_1(\zeta, z)}\right) + \\ &\qquad + \frac{1}{2i}\left(X_2(\zeta, z) + \overline{X_2(\zeta, z)}\right), \\ X_2'(z, \zeta) &= -\frac{1}{2}\left(X_2(\zeta, z) - \overline{X_2(\zeta, z)}\right) - \\ &\qquad - \frac{1}{2i}\left(X_1(\zeta, z) - \overline{X_1(\zeta, z)}\right). \end{aligned} \qquad (10.5a)$$

The formulae (10.5) were established in author's paper [14f].

10.2. On the basis of the last relations the formula (10.2) may now be written in the form

$$\frac{1}{2\pi i}\int_{\Gamma} \Omega_1(z,\zeta)\,w(\zeta)\,d\zeta - \Omega_2(z,\zeta)\,\overline{w}(\zeta)\,\overline{d\zeta}$$
$$= \begin{cases} w(z), & \text{when} \quad z \in G, \\ \frac{a}{2}w(z), & \text{when} \quad z \in \Gamma, \\ 0, & \text{when} \quad z \,\bar{\in}\, \overline{G}. \end{cases} \qquad (10.6)$$

If $A = B = 0$, then $\Omega_1 \equiv (\zeta - z)^{-1}$, $\Omega_2 \equiv 0$ and the formula (10.6) becomes the classical Cauchy formula

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{w(\zeta)\,d\zeta}{\zeta - z} = \begin{cases} w(z), & \text{when} \quad z \in G, \\ \frac{a}{2}w(z), & \text{when} \quad z \in \Gamma, \\ 0, & \text{when} \quad z \,\bar{\in}\, \overline{G}. \end{cases} \qquad (10.7)$$

Therefore the formula (10.6) will be called *the generalized Cauchy formula.*

Making use of the relations (10.5) the generalized Cauchy formula for the adjoint equation $\mathfrak{C}'(w') = 0$ may be written in the form

$$= \begin{cases} w'(z), & \text{when} \quad z \in G, \\ \frac{a}{2}w'(z), & \text{when} \quad z \in \Gamma, \\ (0, & \text{when} \quad z \,\bar{\in}\, G. \end{cases} \qquad (10.8)$$

It can easily be verified that the formulae (10.6) and (10.8) are valid also in the case of an unbounded domain the boundary of which consists of a finite number of rectifiable Jordan curves, if we impose the additional requirement that w and w' vanish at infinity

$$w(z) = O\left(|z|^{-1}\right), \quad w'(z) = O\left(|z|^{-1}\right) \ \text{(near the point } z = \infty).$$

If $w(z)$ is continuous in $G+\Gamma$ and satisfies the non-homogeneous equation

$$\mathfrak{C}(w) \equiv \partial_{\bar{z}} w + Aw + B\overline{w} = F(z)\,,$$

we have the formula

$$\frac{2}{1\pi i}\int_{\Gamma} \Omega_1(z,\zeta)\,w(\zeta)\,d\zeta - \Omega_2(z,\zeta)\,\overline{w(\zeta)}\,\overline{d\zeta} - $$
$$-\frac{1}{\pi}\iint_{G} \big(\Omega_1(z,\zeta)F(\zeta) + \Omega_2(z,\zeta)\overline{F(\zeta)}\big)\,d\xi\,d\eta$$
$$= \begin{cases} w(z), & \text{when} \quad z \in G\,, \\ \dfrac{a}{2}\,w(z), & \text{when} \quad z \in \Gamma\,, \\ 0, & \text{when} \quad z \,\overline{\in}\, G+\Gamma\,. \end{cases} \tag{10.9}$$

If now $w'(z)$ is continuous in $G+\Gamma$ and satisfies the adjoint non-homogeneous equation

$$\mathfrak{C}'(w') \equiv \partial_{\bar{z}} w' - Aw' - \overline{B}\overline{w}' = F'(z)\,,$$

then we have

$$-\frac{1}{2\pi i}\int_{\Gamma} \Omega_1(\zeta,z)\,w'(\zeta)\,d\zeta - \overline{\Omega_2(\zeta,z)}\,\overline{w'(\zeta)}\,\overline{d\zeta} + $$
$$+\frac{1}{\pi}\iint_{G} \big(\Omega_1(\zeta,z)F'(\zeta) + \overline{\Omega_2(\zeta,z)}\,\overline{F'(\zeta)}\big)\,d\xi\,d\eta$$
$$= \begin{cases} w'(z), & \text{when} \quad z \in G\,, \\ \dfrac{a}{2}\,w'(z), & \text{when} \quad z \in \Gamma\,, \\ 0, & \text{when} \quad z \,\overline{\in}\, G+\Gamma\,. \end{cases} \tag{10.9$'$}$$

The above derivation of the formulae (10.6) and (10.8) was indicated in the author's paper [14a]. These formulae were obtained in a somewhat different way by Bers [5a]. In a particular case, namely for the system of equations of the form $u_x - pv_y = 0$, $u_y + pv_x = 0$ analogous formulae were obtained even earlier by Polozhyi [70a]. For an elliptic system of the form

$$au_x + bu_y - v_y = 0$$
$$du_x + cu_y + v_x = 0$$

the Cauchy formula was generalized by Shabat [92c] (see also [70c]).

10.3. The generalized Cauchy formula implies immediately the following

THEOREM 3.14. *If the sequence w_n of functions belonging to the class $\mathfrak{A}_{p,2}(A, B, G)$ and continuous in $G+\Gamma$, strongly converges on Γ to a function $\varphi(\zeta) \in L_{p'}(\Gamma)$, $p' \geqslant 1$, then it converges uniformly inside G to the function*

$$w(z) = \frac{1}{2\pi i} \int_{\Gamma} \Omega_1(z, \zeta)\varphi(\zeta)\, d\zeta - \Omega_2(z, \zeta)\overline{\varphi(\zeta)}\, \overline{d\zeta}\,, \quad (10.10)$$

which, evidently, belongs to the class $\mathfrak{A}_{p,2}(A, B, G)$.

The above theorem can easily be proved by means of the Hölder inequality and Theorem 3.10.

§11. Continuous continuations of generalized analytic functions. Generalized principle of symmetry

Making use of the uniqueness theorem 3.5 and the generalized Cauchy formula (10.6) we can generalize the concept and many criteria of analytic continuation to the class of generalized analytic functions, replacing the term "analytic continuation" by the term "continuous continuation" We shall confine ourselves to the consideration of the class $\mathfrak{A}_{p,2}(G)$, $p > 2$. The functions of this class are continuous in the Hölder, sense with the index equal to $\dfrac{p-2}{p}$.

THEOREM 3.15. *Let $w_1 \in \mathfrak{A}_{p,2}(A, B, G_1)$, $w_2 \in \mathfrak{A}_{p,2}(A, B, G_2)$, $p > 2$, G_1 and G_2 being domains the boundaries of which have a common rectifiable Jordan arc γ. If $w_1 \in C(G_1+\overline{\gamma})$, $w_2 \in C(G_2+\overline{\gamma})$ and $w_1 = w_2$ on $\overline{\gamma}$, then the function*

$$w(z) = \begin{cases} w_1(z), & \text{when} \quad z \in G_1\,, \\ w_2(z), & \text{when} \quad z \in G_2\,, \end{cases} \quad (11.1)$$

belongs to $\mathfrak{A}_{p,2}(A, B, G_1+G_2+\gamma)$, $p > 2$.

The proof of this theorem is carried out with the help of the generalized Cauchy formula (10.6). It suffices to reconstruct literally the well known argument concerning the case of analytic functions [57], (Ch. II, §26).

The last theorem implies immediately the following theorems.

THEOREM 3.16. (*Generalized Riemann–Schwarz principle of symmetry*). *Let G be a domain lying in the upper semi-plane and adjacent to a segment γ of the real axis. Let G_* be the mirror image of G in γ. If* (1) $w \in \mathfrak{A}_{p,2}(A, B, G)$, $p > 2$, (2) $w \in C(G+\gamma)$ *and* (3) $\mathrm{Re}(w) = 0$ *on γ, then the function*

$$w_*(z) = \begin{cases} w(z), & \text{when} \quad z \in G, \\ -\overline{w(\bar{z})}, & \text{when} \quad z \in G_*, \end{cases} \tag{11.2}$$

belongs to the class $\mathfrak{A}_{p,2}(A_*, B_*, G+G_*+\gamma)$ *where A_* and B_* are defined in $G+G_*$ by formulae of the form* (11.2).

THEOREM 3.17. *Let G be a domain lying inside the circle $|z| < 1$ and adjacent to the arc γ of the circumference $|z| = 1$. Let G_* be the mirror image of G in γ. If* (1) $w \in \mathfrak{A}_{p,2}(A, B, G)$, $p > 2$, (2) $w \in C(G+\gamma)$ *and* (3) $Re(w) = 0$ *on γ, then the function*

$$w_*(z) = \begin{cases} w(z), & \text{when} \quad z \in G, \\ -\overline{w\left(\dfrac{1}{\bar{z}}\right)}, & \text{when} \quad z \in G_*, \end{cases} \tag{11.3}$$

belongs to the class $\mathfrak{A}_{p,2}(A_0, B_0, G+G_*+\gamma)$, $p > 2$, *where*

$$A_0(z), B_0(z) = \begin{cases} A(z), B(z), & \text{when} \quad z \in G, \\ -\dfrac{1}{\bar{z}^2}\overline{A\left(\dfrac{1}{\bar{z}}\right)}, -\dfrac{1}{\bar{z}^2}\overline{B\left(\dfrac{1}{\bar{z}}\right)}, & \text{when} \quad z \in G_*. \end{cases} \tag{11.4}$$

PROOF. We may restrict ourselves to the proof of the last theorem. It can be established by a direct verification that $\partial_{\bar{z}} w_* + A_0 w_* + B_0 \overline{w}_* = 0$ in G_*. Moreover, $w_*^+ = w_*^-$ on γ which easily follows from the condition $w(z) + \overline{w(z)} = 0$ on γ.

The last theorem can easily be stated also for an arbitrary circle $|z-z_0| < R$.

§12. Compactness

12.1. *The set of functions $\{w(z)\}$ belonging to the class $\mathfrak{A}(A, B, G)$ is said to be compact in the domain G if from an arbitrary infinite subsequence of the elements of the set a subsequence uniformly convergent inside G can be extracted.*

Let us observe that the uniform convergence of a sequence of generalized analytic functions inside a domain G is ensured by the weak convergence in a $L_p(\bar{G})$, $p \geqslant 1$. We have the following

THEOREM 3.18. *Let $w_i \in \mathfrak{A}_{p,2}(A, B, G)$, $p > 2$, $i = 1, 2, \ldots$. If the sequence w_i is weakly convergent in $L_p(\bar{G})$, $p \geqslant 1$, then it is strongly and uniformly convergent inside G to a function w of the class $\mathfrak{A}_{p,2}(A, B, G)L_p(\bar{G})$.*

PROOF. Let us assume that G is a bounded domain. It can easily be proved that the sequence of functions $\Phi_i = w_i - \mathrm{P}_G w_i$ holomorphic in G is weakly convergent in $L_p(\bar{G})$ to the function $\Phi \equiv w - \mathrm{P}_G w$ where w is the weak limit of the sequence w_i. But it was shown in Ch. I, §1.7 that Φ is holomorphic in G. Consequently, $w \in \mathfrak{A}(A, B, G)$. Besides, Φ_i is strongly convergent to Φ in L_p and also uniformly inside G. Hence $w_i \equiv \Phi_i + R_G\varphi_i$ is strongly convergent in L_p and uniformly convergent inside G to w. This completes the proof.

THEOREM 3.19. *The set $\{w\}$ of functions of the class $\mathfrak{A}_{p,2}(A, B, G)$ is compact in the domain G if and only if the set $\{\Phi\}$ of the corresponding normal analytic divisors is compact.*

PROOF. Let us associate with the set $\{w\}$ the sets $\{\Phi\}$ and $\{\omega\}$ according to the formulae

$$w(z) = \Phi(z)e^{\omega(z)}, \qquad \Phi \in \mathfrak{A}_0(G),$$

$$\omega(z) = \frac{1}{\pi}\iint\limits_E \frac{A(\zeta)w(\zeta)+B(\zeta)\overline{w(\zeta)}}{(\zeta-z)w(\zeta)}\,d\xi\,d\eta\,. \qquad (12.1)$$

The inequalities (4.15) imply that the set $\{\omega\}$ is uniformly bounded and uniformly equi-continuous on the entire plane E. Accordingly, we obtain immediately from (12.1) that the set $\{w\}$ is uniformly bounded and uniformly equi-continuous on every closed set on which the set $\{\Phi\}$ is uniformly bounded and uniformly equi-continuous, and conversely, thus completing the proof.

The last theorem immediately implies

THEOREM 3.20. *The set $\{w\}$ of functions of the class $\mathfrak{A}_{p,2}(A, B, G)$ is compact in the domain G if it is uniformly bounded inside G.*

We observe that the compactness of the set $\{w\}$ of functions of the class $\mathfrak{A}_{p,2}(A, B, G)$ is ensured by the condition of weak compactness in $L_p(\bar{G})$. In fact, if $\{w\}$ is weakly compact in L_p, then $L_p(w, \bar{G}) \leqslant M$, where M is a constant independent of the elements of the set $\{w\}$ (Theorem 1.3). Then the set $\{\Phi\}$ of normal analytic divisors of functions of the set $\{w\}$ is also weakly compact in L_p, i.e. $L_p(\Phi, \bar{G}) \leqslant M'$. But the last inequality implies at once that the set $\{\Phi\}$ is uniformly bounded inside G, i.e. $\{\Phi\}$ is compact (Ch. I, §1.7). Thus we have completed the proof.

12.2. We are in a position now to extend the criteria of compactness to wider classes of generalized analytic functions.

The set $\{w\}$ elements of $\mathfrak{A}_{p,2}(G)$ will be said to be compact in G if every infinite sequence contains a subsequence uniformly convergent inside G to an element of $\mathfrak{A}_{p,2}(G)$.

THEOREM 3.21. *Let $\mathfrak{M}$ be a compact set of functions of the class $L_{p,2}(E)$. Let $\mathfrak{A}^{\mathfrak{M}}_{p,2}(G)$ be a set of generalized analytic functions the generating pairs (A, B) of which belong to $\mathfrak{M}$, i.e. $A, B \in \mathfrak{M}$.*

In this case the infinite set $\{w\}$ of elements of $\mathfrak{A}^{\mathfrak{M}}_{p,2}(G)$ is compact if and only if the set $\{\Phi\}$ of the corresponding normal analytic divisors is compact.

PROOF. If a set $\{w\}$ is given, then we associate with it the sets $\{\Phi\}$ and $\{\omega\}$ uniquely defined by the formulae (12.1). If the set $\{\Phi\}$ is prescribed, the set $\{w\}$ is defined in the following way: we associate with every element of Φ a pair (A, B) belonging to the compact set $\mathfrak{M}$ and then we construct a function w of the class $\mathfrak{A}(A, B, G)$ the normal analytic divisor of which is Φ. Thus, every set $\{\Phi\}$ can be associated with the infinite sets $\{w\}$ and $\{\omega\}$.

In view of the compactness of $\{A\}$ and $\{B\}$, (4.15) implies that the set $\{\omega\}$ is uniformly bounded and uniformly equi-continuous on the entire plane. Consequently, if one of the sets $\{w\}$ and $\{\Phi\}$ is uniformly bounded and uniformly equi-continuous in a closed domain, then the other set possesses this property as well. Hence, evidently, if $\{w\}$ is compact, $\{\Phi\}$ is also compact. It remains to prove the converse statement. The compactness of $\{\Phi\}$ implies that from an arbitrary infinite sequence of elements of $\{w\}$ a subsequence $w_1, w_2, \ldots$ uniformly convergent inside G can be extracted; obviously

$$w_n(z) = \Phi_n(z)\, e^{\omega_n(z)} \qquad (n = 1, 2, \ldots),$$

where

$$\omega_n(z) = \frac{1}{\pi} \iint_E \left(A_n(\zeta) + B_n(\zeta) \frac{\overline{w_n(\zeta)}}{w_n(\zeta)}\right) \frac{d\xi\, d\eta}{\zeta - z},$$

$$A_n, B_n \in \mathfrak{M}.$$

The sequences w_n, Φ_n, ω_n are uniformly convergent on every closed subset of the domain G to the functions w, Φ, ω, respectively. The latter functions evidently satisfy the relation

$$w(z) = \Phi(z)\, e^{\omega(z)}, \tag{12.2}$$

$\Phi(z)$ being holomorphic inside G and $\omega(z)$ being continuous on the entire plane. If $\Phi \equiv 0$ then $w \equiv 0$ and the theorem is proved, since the zero is an element of $\mathfrak{A}^{\mathfrak{M}}_{p,2}(G)$. Let us assume that $\Phi(z) \not\equiv 0$. Then $w(z)$ has only isolated zeros inside G.

In view of the compactness of $\mathfrak{M}$, from the sequences $\{A_n\}$ and $\{B_n\}$ we can extract subsequences, denoted again by $\{A_n\}$ and $\{B_n\}$, convergent in the mean to some A and B belonging to $\mathfrak{M}$:

$$L_{p,2}(A_n - A) \to 0, \quad L_{p,2}(B_n - B) \to 0 \quad \text{for} \quad n \to \infty .$$

Now let

$$(\omega_0 z) = \frac{1}{\pi} \iint_E \left(A(\zeta) + B(\zeta) \frac{\overline{w(\zeta)}}{w(\zeta)} \right) \frac{d\xi \, d\eta}{\zeta - z} .$$

It is readily observed that the sequence $g_n = A_n + B_n \frac{\overline{w}_n}{w_n}$ is strongly convergent (in the metric $L_{p,2}$) to $g = A + B\frac{\overline{w}}{w}$. Therefore $\omega_n \to \omega_0$ when $n \to \infty$, i.e. $\omega = \omega_0$ and the formula (12.2) assumes the form

$$w(z) = \Phi(z) \exp\left[\frac{1}{\pi} \iint_E \left(A(\zeta) + B(\zeta) \frac{\overline{w(\zeta)}}{w(\zeta)} \right) \frac{d\xi \, d\eta}{\zeta - z} \right] .$$

This proves that $w \in \mathfrak{A}^{\mathfrak{M}}_{p,2}(G)$. Thus, Theorem 3.21 has been proved.

12.3. The last theorem immediately implies

THEOREM 3.22. *The set $\{w\}$ of elements of the set $\mathfrak{A}^{\mathfrak{M}}_{p,2}(G)$ is compact in the domain G if it is uniformly bounded inside G.*

THEOREM 3.23. *Let $A_n, B_n \to A, B$ (in the metric $L_{p,2}$). If the sequence $\{\Phi_n\}$ of the normal analytic divisors of the sequence $\{w_n\}$ of functions of the class $\mathfrak{A}(A_n, B_n, G)$ is uniformly convergent inside G to the function $\Phi(z)$, then the sequence w_n is uniformly convergent inside G to a function w of the class $\mathfrak{A}(A, B, G)$ with the normal analytic divisor Φ.*

In fact, since the set $\{w_n\}$ is bounded, then, by Theorem 3.22, it is compact. But every subsequence $\{w_{n_k}\}$ of the sequence $\{w_n\}$ which is uniformly convergent inside G is, obviously, convergent to a function w of the class $\mathfrak{A}(A, B, G)$ whose normal divisor is Φ. This means that $w_n \to w$ which completes the proof of the theorem.

Two corollaries follow from this theorem.

Corollary 1. *If* $A_n, B_n \to A, B$ *(in the metric* $L_{p,2}$*), then for every function* w *of the class* $\mathfrak{A}(A, B, G)$ *a sequence of functions* w_n *of the class* $\mathfrak{A}(A_n, B_n, G)$ *can be found such that it is uniformly convergent to* w *inside* G.

In fact, if Φ is the normal analytic divisor of the function w then for w_n a function of the class $\mathfrak{A}(A_n, B_n, G)$ having the same normal divisor Φ may be taken.

Corollary 2. *If* $A_n, B_n \to A, B$ *(in the metric* $L_{p,2}$*) then for* $z \neq \zeta$ *the sequences of the fundamental solutions* $X_{jn}(z, \zeta)$ *and the kernels* $\Omega_{jn}(z, \zeta)$ $(j = 1, 2)$ *of the equations* $\partial_{\bar z} w + A_n w + B_n \bar w = 0$ *converge to the fundamental solutions* $X_j(z, \zeta)$ *and the kernels* $\Omega_j(z, \zeta)$ *of the equation* $\partial_{\bar z} w + Aw + B\bar w = 0$, *the convergence being uniform with respect to both arguments* z *and* ζ *if* $z \in G'$ *and* $\zeta \in G''$ *where* G' *and* G'' *are arbitrary closed sets on the plane having no points in common.*

In fact, it is sufficient to note that the fundamental solutions X_{jn} and X_j have the same normal analytic divisors $\frac{1}{2}(\zeta - z)^{-1}$ and $\frac{1}{2i}(\zeta - z)^{-1}$.

§13. Representation of resolvents by means of kernels

13.1. Let us denote by $\Omega_1(z, t, G)$ and $\Omega_2(z, t, G)$ the kernels of the class $\mathfrak{A}_{p,2}(A, B, G), p > 2$, if $A = B = 0$ outside $\bar G$. Such kernels will be called *the normalized kernels with respect to the domain* G.

According to (8.4)

$$\partial_{\bar z}\Omega_1(z, t, G) = 0\,, \quad \partial_{\bar z}\Omega_2(z, t, G) = 0 \quad (\text{outside } \bar G)\,.$$

If $t \in G$, then, obviously, $\Omega_1(z, t, G)$ and $\Omega_2(z, t, G)$ are holomorphic with respect to z outside G and vanish at infinity. In an analogous way we discower that $\Omega_1(t, z, G)$ and $\overline{\Omega_2(t, z, G)}$ are holomorphic with respect to z outside G and vanish at infinity. It should be taken into account here that

$$\begin{aligned} \Omega_1(z, t, G) &= -\Omega_1'(t, z, G)\,, \\ \Omega_2(z, t, G) &= -\overline{\Omega_2'(t, z, G)}\,. \end{aligned} \tag{13.1}$$

Hence, if $\Phi(z)$ is holomorphic outside $\bar{G}$, continuous up to the boundary and vanishes at infinity, then according to the Cauchy theorem we have (for $z \in G$)

$$\frac{1}{2\pi i}\int_{\Gamma} \Omega_1(z, t, G)\Phi^-(t)\,dt - \Omega_2(z, t, G)\overline{\Phi^-(t)}\,d\bar{t} = 0 . \quad (13.2)$$

Let $w(z) \in \mathfrak{A}_{p,2}(A, B, G)$, $p > 2$, and let it be continuous in $\bar{G}$. Then for $z \in G$ we have

$$w(z) = \frac{1}{2\pi i}\int_{\Gamma} \Omega_1(z, t, \Omega)w(t)\,dt - \Omega_2(z, t, G)\overline{w(t)}\,d\bar{t} . \quad (13.3)$$

We introduce the analytic function

$$\Phi(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{w(t)\,dt}{t-z} , \quad (13.4)$$

which, as it was shown in §2, is continuous in the closed domain $\bar{G}$. In view of the relations (8.25), Ch. I, we have

$$w(t) = \Phi^+(t) - \Phi^-(t) , \qquad t \in \Gamma . \quad (13.5)$$

Since $w(t)$ and $\Phi(t) \equiv \Phi^+(t)$ are continuous on Γ it follows from (13.5) that $\Phi^-(t)$ is also continuous on Γ. Substituting (13.5) into the right-hand side of (13.3) and taking into account the relation (13.2) we obtain

$$w(z) = \mathcal{K}(\Phi, G)$$
$$\equiv \frac{1}{2\pi i}\int_{\Gamma} \Omega_1(z, t, G)\Phi(t)\,dt - \Omega_2(z, t, G)\overline{\Phi(t)}\,d\bar{t} . \quad (13.6)$$

The last formula containing the kernels $\Omega_1(z, \zeta, G)$ and $\Omega_2(z, \zeta, G)$ of the equation $\mathfrak{C}(w) = 0$ normalized with respect to the domain G, associates with every function $\Phi(z)$ holomorphic in G and continuous in $G+\Gamma$ a definite function $w(z)$ of the class $\mathfrak{A}_{p,2}(A, B, G)$, $p > 2$, continuous in $G+\Gamma$. The function Φ is given in terms of w by the formula (13.4). Consequently, the formula (13.6) may be considered as a device enabling us to construct all functions of the class $\mathfrak{A}_{p,2}(A, B, G)$.

In view of (13.1) and (13.6) the general representation of solutions of the adjoint equation

$$\partial_{\bar{z}} w' - A w' - B \overline{w'} = 0 \qquad (13.7)$$

is given by the formula

$$w'(z) \equiv \mathcal{K}'(\Phi, G)$$
$$\equiv -\frac{1}{2\pi i} \int_{\Gamma} \Omega_1(\zeta, z, G) \Phi(\zeta) d\zeta - \overline{\Omega_2(\zeta, z, G)} \overline{\Phi(\zeta)} \overline{d\zeta} . \qquad (13.8)$$

13.2. Applying Green's identity to the right-hand side of the relation (13.6) we may write

$$w(z) = \lim_{\varepsilon\to 0} \left\{ \frac{1}{\pi} \iint_{G_\varepsilon} \partial_{\bar{\zeta}} \Omega_1(z, \zeta, G) \Phi(\zeta) d\xi d\eta + \right.$$
$$\left. + \frac{1}{\pi} \iint_{G_\varepsilon} \partial_{\bar{\zeta}} \Omega_2(z, \zeta, G) \overline{\Phi(\zeta)} d\xi d\eta \right\} +$$
$$+ \lim_{\varepsilon\to 0} \left\{ \frac{1}{2\pi i} \int_{|\zeta - z| = \varepsilon} \Omega_1(z, \zeta, G) \Phi(\zeta) d\zeta - \Omega_2(z, \zeta, G) \overline{\Phi(\zeta)} d\bar{\zeta} \right\},$$

where G_ε is the intersection of the domains G and $|\zeta - z| > \varepsilon$. Taking into account the formulae (8.16) we have

$$w(z) = \mathcal{K}(\Phi, G) \equiv \Phi(z) + \iint_G \Gamma_1(z, \zeta, G) \Phi(\zeta) d\xi d\eta +$$
$$+ \iint_G \Gamma_2(z, \zeta, G) \overline{\Phi(\zeta)} d\xi d\eta , \qquad (13.9)$$

where

$$\Gamma_1(z, \zeta, G) = \frac{1}{\pi} \partial_{\bar{\zeta}} \Omega_1(z, \zeta, G) ,$$
$$\Gamma_2(z, \zeta, G) = \frac{1}{\pi} \partial_{\zeta} \Omega_2(z, \zeta, G) . \qquad (13.10)$$

Since by (13.1) $-\Omega_1(\zeta, z, G)$ and $-\overline{\Omega_2(\zeta, z, G)}$ are the kernels of the adjoint equation (13.7), then, according to the equations (8.14), we have

$$\partial_{\bar{\zeta}} \Omega_1(z, \zeta, G) - A(\zeta) \Omega_1(z, \zeta, G) - \overline{B(\zeta)} \Omega_2(z, \zeta, G) = 0 ,$$
$$\partial_{\zeta} \Omega_2(z, \zeta, G) - \overline{A(\zeta)} \Omega_2(z, \zeta, G) - B(\zeta) \Omega_1(z, \zeta, G) = 0 \qquad (13.11)$$

or, according to (13.10)

$$\begin{aligned} \Gamma_1(z,\zeta,G) &= \frac{1}{\pi}A(\zeta)\Omega_1(z,\zeta,G)+\frac{1}{\pi}\overline{B(\zeta)}\Omega_2(z,\zeta,G)\,, \\ \Gamma_2(z,\zeta,G) &= \frac{1}{\pi}\overline{A(\zeta)}\Omega_2(z,\zeta,G)+\frac{1}{\pi}B(\zeta)\Omega_1(z,\zeta,G)\,. \end{aligned} \tag{13.12}$$

It should be borne in mind that $\Phi(z)$ is a function holomorphic in G connected with $w(z)$ by the formula (13.4). Therefore, the formula (13.9) represents the solution of the integral equation

$$w(z)-\frac{1}{\pi}\iint\limits_G \frac{A(\zeta)w(\zeta)+B(\zeta)\overline{w(\zeta)}}{\zeta-z}\,d\xi\,d\eta = \Phi(z) \tag{13.13}$$

for an arbitrary right-hand side $\Phi(z)$ which is holomorphic in G and continuous in $\overline{G}$. Moreover, this formula holds for an arbitrary right-hand side belonging to $L_q(\overline{G})$, $q \geqslant \dfrac{p}{p-1}$ (see §5). In other words, the resolvent of the integral equation (13.13) is given by the formula

$$\begin{aligned} Rg \equiv \iint\limits_G \Gamma_1(z,\zeta,G)g(\zeta)\,d\xi\,d\eta\, + \\ + \iint\limits_G \Gamma_2(z,\zeta,G)\overline{g(\zeta)}\,d\xi\,d\eta\,, \end{aligned} \tag{13.14}$$

$$g \in L_q(\overline{G})\,, \qquad q \geqslant \frac{p}{p-1}\,.$$

Thus, all solutions of the equation

$$\partial_{\bar z}w + Aw + B\overline{w} = 0\,, \qquad A, B \in L_{p,2}(E)\,, \qquad p>2\,, \tag{13.15}$$

belonging to the class $L_q(\overline{G})$, $q \geqslant \dfrac{p}{p-1}$ are represented by the formula (13.19) where $\Phi(z)$ is an arbitrary function analytic inside G and belonging to $L_q(\overline{G})$, $q \geqslant \dfrac{p}{p-1}$. If $q<2$ Φ can have simple poles. Thus, the formula (13.9) enables us to construct solutions of the equation (13.15) possessing prescribed beforehand simple poles.

If the formulae (13.1) and (13.12) are taken into account we obtain the following expressions for the resolvents of the adjoint equation (13.7):

$$\begin{aligned} \pi\Gamma_1'(z,\zeta,G) &= A(\zeta)\Omega_1(\zeta,z,G)+\overline{B(\zeta)}\,\overline{\Omega_2(\zeta,z,G)} \\ &\equiv -\partial_{\bar{\zeta}}\Omega_1(\zeta,z,G)\,, \\ \pi\Gamma_2'(z,\zeta,G) &= \overline{A(\zeta)}\,\overline{\Omega_2(\zeta,z,G)}+B(\zeta)\Omega_1(\zeta,z,G) \\ &\equiv -\partial_{\zeta}\overline{\Omega_2(\zeta,z,G)}\,. \end{aligned} \tag{13.16}$$

Consequently, the formula (13.8) may be written in the form

$$w'(z) = \mathcal{K}'(\Phi,G) \equiv \Phi(z)+\iint_G \Gamma_1'(z,\zeta,G)\Phi(\zeta)\,d\xi\,d\eta + \iint_G \Gamma_2'(z,\zeta,G)\overline{\Phi(\zeta)}\,d\xi\,d\eta\,, \tag{13.17}$$

where $\Phi(z)$ is an arbitrary function analytic in G belonging to $L_q(\bar{G})$, $q \geqslant \frac{p}{p-1}$. This formula yields all solutions of the class $L_q(\bar{G})$ of the adjoint equation (13.7).

13.3. The particular solution of the non-homogeneous equation

$$\mathfrak{C}(w) = \partial_{\bar{z}}w + Aw + B\bar{w} = F \tag{13.18}$$

according to the formula (5.7) can be constructed in the form $w_1 = TF + RTF$. This formula can easily be transformed to the form

$$w_1 = -\frac{1}{\pi}\iint_G \Omega_1(z,\zeta,G)F(\zeta)\,d\xi\,d\eta - \frac{1}{\pi}\iint_G \Omega_2(z,\zeta,G)\bar{F}(\zeta)\,d\xi\,d\eta\,. \tag{13.19}$$

The particular solution of the adjoint non-homogeneous equation

$$\mathfrak{C}'(w') \equiv \partial_{\bar{z}}w' - Aw' - \overline{Bw'} = F' \tag{13.20}$$

has the form

$$w_1' = \frac{1}{\pi}\iint_G \Omega_1(\zeta, z, G)F'(\zeta)\,d\xi\,d\eta + \frac{1}{\pi}\iint_G \overline{\Omega_2(\zeta, z, G)}\,\overline{F'(\zeta)}\,d\xi\,d\eta\,. \tag{13.21}$$

The formulae and relations presented in this section were established in the author's paper [14a].

§14. Representation of generalized analytic functions by means of generalized integrals of the Cauchy type

14.1. If Γ is a union of rectifiable Jordan curves and $\varphi(t)$ is a function summable on Γ, then the integral

$$w(z) = \frac{1}{2\pi i}\int_\Gamma \Omega_1(z, t)\varphi(t)\,dt - \Omega_2(z, t)\overline{\varphi(t)}\,d\bar{t}\,, \tag{14.1}$$

where Ω_1 and Ω_2 are normal kernels of the class $\mathfrak{A}_{p,2}(A, B, E)$, will be called the generalized integral of the Cauchy type. It constitutes a solution of the equation (13.15) regular outside Γ, and $w = O(|z|^{-1})$ near infinity. These assertions can easily be verified by means of the relations (8.14) and (8.15).

Let Γ contain an arc γ of the class C^1. If $\zeta \in \gamma$ and $\varphi \in C_\alpha(\gamma)$, $0 < \alpha \leqslant 1$, then the following formulae hold:

$$\begin{aligned} w^+(\zeta) &= +\frac{1}{2}\varphi(\zeta) + w(\zeta)\,, \\ w^-(\zeta) &= -\frac{1}{2}\varphi(\zeta) + w(\zeta)\,, \qquad \zeta \in \gamma\,, \end{aligned} \tag{14.2}$$

where

$$w(\zeta) = \frac{1}{2\pi i}\int_\Gamma \Omega_1(\zeta, t)\varphi(t)\,dt - \Omega_2(\zeta, t)\overline{\varphi(t)}\,d\bar{t}\,, \tag{14.3}$$

the first integral in the right-hand side of the last relation being understood as the Cauchy principal value, and the second integral converging in the ordinary sense.

It has already been indicated above that if $A \equiv B \equiv 0$ then $\Omega_1 = (\zeta - z)^{-1}$, $\Omega_2 = 0$ and we obtain the well known formulae for the integral of the Cauchy type

$$
\begin{aligned}
w^+(\zeta) &= +\frac{1}{2}\varphi(\zeta) + \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(t)\,dt}{t-\zeta}\,, \\
w^-(\zeta) &= -\frac{1}{2}\varphi(\zeta) + \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(t)\,dt}{t-\zeta}\,.
\end{aligned}
\tag{14.4}
$$

The formulae (14.2) can be obtained by means of the last relations if the relations (8.16) be taken into account.

14.2. If φ represents the limiting value of a function $w(z)$ continuous in $G+\Gamma$ and satisfying inside G the equation $\mathfrak{C}(w) \equiv \partial_{\bar z} w + Aw + B\overline{w} = 0$, then in view of (10.6) we have the relation

$$
\frac{1}{2\pi i}\int_\Gamma \Omega_1(z,t)\varphi(t)\,dt - \Omega_2(z,t)\overline{\varphi(t)}\,\overline{dt} = 0\,, \tag{14.5}
$$

which is satisfied for any point z lying outside $G+\Gamma$. In view of (14.2) this relation is equivalent to the following one ($\Gamma \in C^1$)

$$
\varphi(\zeta) - \frac{1}{\pi i}\int_\Gamma \Omega_1(\zeta,t)\varphi(t)\,dt - \Omega_2(\zeta,t)\overline{\varphi(t)}\,\overline{dt} = 0\,,\quad \zeta \in \Gamma\,. \tag{14.6}
$$

If φ assumes the limiting values of a function $w(z)$ continuous outside G and satisfying outside $G+\Gamma$ the equation $\mathfrak{C}(w) = 0$, and if it vanishes at $z = \infty$, then we have the relation (14.5) valid for any point z lying inside G. In this case it is equivalent to the relation

$$
\varphi(\zeta) + \frac{1}{\pi i}\int_\Gamma \Omega_1(\zeta,t)\varphi(t)\,dt - \Omega_2(\zeta,t)\overline{\varphi(t)}\,\overline{dt} = 0\,,\quad \zeta \in \Gamma\,. \tag{14.7}
$$

It can easily be proved that the above conditions are also sufficient. For instance, if $\varphi \in C_\alpha(\Gamma)$, $0 < \alpha < 1$, and it satisfies the relation (14.5) or (14.6), then at every boundary point φ assumes a value equal to the limiting

value of the generalized integral of the Cauchy type (14.1) from the inside of the domain G. Consequently, *the fulfilling of the relation* (14.5) *or the equivalent relation* (14.6) *is a necessary and sufficient condition for* $\varphi(t)$ *to assume limiting values of a function of the class* $\mathfrak{A}_{p,2}(A, B, G)$, $p > 2$, *continuous in* $G+\Gamma$.

Analogously, *the fulfilment of the relation* (14.7) *on the contour* Γ *or the relation* (14.5) *for points belonging to* G *is a necessary and sufficient condition for* $\varphi(t)$ *to assume limiting values of a function of the class* $\mathfrak{A}_{p,2}(A, B, G')$, $p > 2$, *continuous in* $G'+\Gamma$ *and vanishing at infinity,* G' *being the supplement of* $G+\Gamma$ *to the entire plane.*

Similar statements may be made for the adjoint equation $\mathfrak{C}'(w') = 0$. In this case the conditions (14.6) and (14.7) have the form

$$\varphi(\zeta)+\frac{1}{\pi i}\int_{\Gamma} \Omega_1(t, \zeta)\varphi(t)\,dt-\overline{\Omega_2(t, \zeta)}\,\overline{\varphi(t)}\,\overline{dt} = 0 \,, \quad (14.8)$$

$$\varphi(\zeta)-\frac{1}{\pi i}\int_{\Gamma} \Omega_1(t, \zeta)\varphi(t)\,dt-\overline{\Omega_2(t, \zeta)}\,\overline{\varphi(t)}\,\overline{dt} = 0 \,. \quad (14.9)$$

We have made use here of the relations (13.1).

14.3. THEOREM 3.24. *Let* $A, B \in L_{p,2}(E)$, $G \in C^1$. *If* $\varphi(t) \in C_\alpha(\Gamma)$, $0 < \alpha < 1$, *then the generalized Cauchy type integral* (14.1) *belongs to* $C_\beta(\overline{G})$ *where* $\beta = \min\left(\alpha, \frac{p-2}{p}\right)$.

PROOF. The relation (14.1) can be written in the form

$$w(z) = \Phi(z)+\iint_G \Gamma_1(z, \zeta)\Phi(\zeta)\,d\xi\,d\eta+ \\ +\iint_G \Gamma_2(z, \zeta)\overline{\Phi(\zeta)}\,d\xi\,d\eta \,, \quad (14.10)$$

where

$$\Phi(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{\varphi(\zeta)\,d\zeta}{\zeta-z} \,. \quad (14.11)$$

Consequently, w satisfies the integral equation $w = -$ $-\mathrm{T}_G(Aw+B\overline{w})+\Phi(z)$ and since in view of Theorem 1.10 $\Phi \in C_\alpha(G+\Gamma)$, w is continuous in $G+\Gamma$. Hence $\mathrm{T}_G(Aw+$ $+B\overline{w}) \in C_\nu(E)$, $\nu = \dfrac{p-2}{p}$ and, evidently, $w \in C_\beta(G+\Gamma)$ where $\beta = \min(\alpha, \nu)$. This completes the proof.

14.4. Integral representations of generalized analytic functions including the generalized Cauchy type integrals, have many applications. In particular they are used in the investigation of boundary value problems. These representations can be obtained from the formula (13.6) using for this purpose various integral representations of holomorphic functions, [14g], [60a]. Consequently, the forms of the representations may be varied in a wide range in an effort to adapt them to the concrete conditions of the problem under investigation. As an example we shall consider a form of the integral representation of generalized analytic functions suitable for the investigation of the boundary value problem which will be dealt with in the following chapter ([14a]).

THEOREM 3.25. *Let* $G \in C_\alpha^1$, $0 < \alpha \leqslant 1$. *If the function* $w(z) \in \mathfrak{A}_{p,2}(G)$, $p > 2$, *and it is continuous in* $\overline{G}$, *then a continuous real function* $\mu(t)$ *of the point* t *of the contour* Γ *and real constant* c *can be found, such that*

$$w(z) = \int_\Gamma \mu(t)\, M(z, t)\, ds + c w_1(z)\,, \qquad z \in G\,, \quad (14.12)$$

where

$$M(z, t) = \frac{t'(s)}{\pi i}\Omega_1(z, t, G) - \frac{\overline{t'(s)}}{\pi i}\Omega_2(z, t, G)\,, \quad (14.13)$$

$$w_1(z) = K(i, G)\,.$$

Moreover, if G *is a simply-connected domain then there exist a one-to-one correspondence between* $w(z)$ *and the pair* $(\mu(t), c)$; *if* G *is an* $(m+1)$-*connected domain, then the constant* c *is uniquely expressed by* w *and the real function* $\mu(t)$ *is determined to within an additive factor of the form*

$c_1\mu_1+\dots+c_m\mu_m$ where $c_1,\dots,c_m$ are arbitrary constants and $\mu_1,\dots,\mu_m$ are functions defined by the relations

$$\mu_j(t)=\begin{cases}1\,, & \text{if}\quad t\in\Gamma_j \qquad\qquad (j=1,\dots,m)\\ 0\,, & \text{if}\quad t\in\Gamma_k\,,\quad k\neq j\quad (k=0,1,\dots,m)\,,\end{cases}\tag{14.14}$$

where $\Gamma_1,\dots,\Gamma_m$ are "interior" boundary contours of the domain G and Γ_0 is the "exterior" contour (the positive direction is so chosen that the domain lies on the left-hand side).

If w is continuous in the Hölder sense in $G+\Gamma$, then μ also satisfies the Hölder condition on Γ, and conversely.

Proof. The theorem is known for the class of holomorphic functions ([14g], [60a], Ch. III, 2). In this case the formula (14.12) assumes the form

$$\Phi(z)=\frac{1}{\pi i}\int\limits_{\Gamma}\frac{\mu(t)\,dt}{t-z}+ic\,.$$

Substituting this expression into the right-hand side of the formula (13.6) we arrive at the formula (14.12). If, moreover, we take into account the fact that Φ is related to w by the formula (13.14) in a unique way, then we obtain the full proof of the theorem.

§15. Complete systems of generalized analytic functions. Generalized power series

15.1. Let there be given a system of particular solutions w_n $(n=1,2,\dots)$ of the equation

$$\partial_{\bar z}w+Aw+B\overline{w}=0\,,\qquad \big(A\,,B\in L_{p,2}(E)\big)\,.\tag{15.1}$$

The above system will be said to be *a complete system of solutions* with respect to the domain G if every solution w of this equation regular in G can be uniformly approximated inside G by means of linear expressions of the form $c_1w_1+\dots+c_nw_n$ with real coefficients. Complete systems of solutions can be constructed for example by

means of the formula (13.6). It can be proved that this formula associates with every system of analytic functions Φ_n complete with respect to the domain G and continuous in $G+\Gamma$, a complete system of solutions $w_n = \mathcal{K}(\Phi_n, G)$ of the equation (15.1). Let us for instance consider the following system of rational functions in the variable z:

$$(z-z_0)^{n-1}, \qquad (z-z_j)^{-k} \tag{15.2}$$

$$(j = 1, \ldots, m; \qquad n, k = 1, 2, \ldots),$$

where $z_0, z_1, \ldots, z_m$ are fixed points. It is known, [87], that this system is complete (in the class of holomorphic functions) with respect to an arbitrary domain G which is bounded by simple Jordan curves $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$, Γ_0 containing the curves $\Gamma_1, \ldots, \Gamma_m$, and the points z_j being situated inside Γ_j $(j = 1, \ldots, m)$. The formula (13.6) associates with the system of functions (15.2) the following system of particular solutions of the equation (15.1), which is complete with respect to G:

$$\begin{aligned}
w_{2n}(z, z_0) &= \mathcal{K}\big((z-z_0)^n, G\big), \\
w_{2n+1}(z, z_0) &= \mathcal{K}\big(i(z-z_0)^n, G\big)(n = 0, 1, \ldots), \\
w_{-2n+1}(z, z_j) &= \frac{\partial^{n-1}\Omega_1(z, z_j, G)}{\partial z_j^{n-1}} + \frac{\partial^{n-1}\Omega_2(z, z_j, G)}{\partial \bar{z}_j^{n-1}} \\
w_{-2n}(z, z_j) &= i\frac{\partial^{n-1}\Omega_1(z, z_j, G)}{\partial z_j^{n-1}} - i\frac{\partial^{n-1}\Omega_2(z, z_j, G)}{\partial \bar{z}_j^{n-1}} \\
&(n = 1, 2, \ldots).
\end{aligned} \tag{15.3}$$

We have taken into account here that for $z \in G$ the kernels $\Omega_1(z, \zeta, G)$ and $\overline{\Omega_2(z, \zeta, G)}$ are holomorphic with respect to ζ outside $G+\Gamma$.

It is readily seen that $w_n(z, z_0)$ is a generalized polynomial of degree $\left[\frac{n}{2}\right]$ and $w_{-n}(z, z_j)$ is a generalized rational function its only pole z_j being of the order $\left[\frac{n+1}{2}\right]$; this

function vanishes at infinity. We shall see later on that $w_n(z, z_0)$ constitute the coefficients of the expansion of the functions $\Omega_j(z, \zeta, G)$ with respect to ζ in the vicinity of infinity.

In fact, if $z \in G$ and ζ lies outside $G+\Gamma$, then in view of the Cauchy formula

$$\left.\begin{aligned}\Omega_1(z, \zeta, G) &= -\frac{1}{2\pi i}\int_\Gamma \frac{\Omega_1(z, t, G)}{t-\zeta}\,dt\,,\\ \Omega_2(z, \zeta, G) &= \frac{1}{2\pi i}\int_\Gamma \frac{\Omega_2(z, t, G)}{\bar{t}-\bar{\zeta}}\,d\bar{t}\,.\end{aligned}\right\} \tag{15.4}$$

Expanding now the right-hand sides of the last relations for sufficiently large ζ with respect to the negative powers of $(\zeta-z_0)$ and $(\bar{\zeta}-\bar{z}_0)$ we obtain according to the formula (13.6)

$$\left.\begin{aligned}&\Omega_1(z, \zeta, G)\\ &\quad= \frac{1}{2}\sum_{k=0}^{\infty}[w_{2k}(z, z_0)-iw_{2k+1}(z, z_0)](\zeta-z_0)^{-k-1}\,,\\ &\Omega_2(z, \zeta, G)\\ &\quad= \frac{1}{2}\sum_{k=0}^{\infty}[w_{2k}(z, z_0)+iw_{2k+1}(z, z_0)](\bar{\zeta}-\bar{z}_0)^{-k-1}\,.\end{aligned}\right\} \tag{15.5}$$

Suppose now that G is the circle $|z-z_0|<\varrho$; then the above series are uniformly convergent with respect to z and ζ inside and outside the circle.

If the conditions $|z-z_0|>\varrho$ and $|\zeta-z_0|<\varrho$ are satisfied we have the expansions

$$\begin{aligned}&\Omega_1(z, \zeta, G)\\ &\quad= -\frac{1}{2}\sum_{k=0}^{\infty}[w'_{2k}(\zeta, z_0)-iw'_{2k+1}(\zeta, z_0)](z-z_0)^{-k-1}\,,\\ &\Omega_2(z, \zeta, G)\\ &\quad= -\frac{1}{2}\sum_{k=0}^{\infty}[\overline{w'_{2k}(\zeta, z_0)}-\overline{iw'_{2k+1}(\zeta, z_0)}](z-z_0)^{-k-1}\,,\end{aligned} \tag{15.6}$$

where $w_n'(z, z_0)$ are generalized polynomials of the degree $\left[\frac{n}{2}\right]$ satisfying the adjoint equation

$$\partial_{\bar{z}} w' - Aw' - \bar{B}\bar{w}' = 0: \tag{15.7}$$

$$\begin{aligned} w_{2n}'(z, z_0) &= \mathcal{K}'[(z-z_0)^k, G], \\ w_{2k+1}'(z, z_0) &= \mathcal{K}'[i(z-z_0)^k, G]. \end{aligned} \tag{15.8}$$

The expansions (15.6) are obtained from the formulae

$$\Omega_j(z, \zeta, G) = -\frac{1}{2\pi i}\int_\Gamma \frac{\Omega_j(t, \zeta, G)}{t-z}\,dt \qquad (j = 1, 2), \tag{15.9}$$

which hold if $\zeta \in G$ and z lies outside $G+\Gamma$.

15.2. By means of the system of generalized rational functions $w_n(z, z_0)$ $(n = 0, \pm 1, \pm 2, \ldots)$ it is possible to obtain expansions of arbitrary solutions of the equation (15.1), which constitute generalizations of the Taylor and Laurent series for analytic functions [14a], [5a].

Let G be the circle $|z-z_0| < \varrho$ and Γ the circumference $|z-z_0| = \varrho$. In this case the corresponding generalized polynomials $w_n(z, z_0)$ will be denoted by $w_n(z, z_0, \varrho)$. If $w(z)$ satisfies inside G the equation (15.1) and if it is continuous in $G+\Gamma$, then it is representable by the formula

$$\begin{aligned} w(z) &= \mathcal{K}(w, G) \\ &\equiv \frac{1}{2\pi i}\int_\Gamma \Omega_1(z, \zeta, G)w(\zeta)\,d\zeta - \Omega_2(z, \zeta, G)\overline{w(\zeta)}\,d\bar{\zeta} \end{aligned} \tag{15.10}$$

or

$$\begin{aligned} w(z) &\equiv \mathcal{K}(\Phi, G) \\ &= \frac{1}{2\pi i}\int_\Gamma \Omega_1(z, \zeta, G)\Phi(\zeta)\,d\zeta - \Omega_2(z, \zeta, G)\overline{\Phi(\zeta)}\,d\bar{\zeta}, \end{aligned} \tag{15.11}$$

where

$$\Phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{w(\zeta)}{\zeta - z}\,d\zeta. \tag{15.12}$$

Expanding the right-hand side of the last relation into the series with respect to the powers of $(z-z_0)^n$ we obtain

$$\Phi(z) = \sum_{k=0}^{\infty} (c_{2k} + ic_{2k+1})(z-z_0)^k , \qquad (15.13)$$

where c_k are real constants which are to be determined from the relations

$$c_{2k} + ic_{2k+1} = \frac{1}{2\pi i} \int_{\Gamma} \frac{w(\zeta)}{\zeta^{k+1}} d\zeta \qquad (k = 0, 1, \ldots) . \qquad (15.14)$$

Let us now consider the following system of solutions of the equation (15.1):

$$\hat{w}_n(z) = \sum_{k=0}^{n} c_k w_k(z, z_0, \varrho) \equiv \mathcal{K}(\Phi_n, G) , \qquad (15.15)$$

where

$$\Phi_n(z) = \sum_{k=0}^{\left[\frac{n}{2}\right]}{}' (c_{2k} + ic_{2k+1})(z-z_0)^k .$$

Since $w(z) - \hat{w}_n(z) = \mathcal{K}(\Phi - \Phi_n, G) \equiv \mathcal{K}(w - \Phi_n, G)$ we have the inequality

$$|w(z) - \hat{w}_n(z)|$$
$$\leqslant \frac{1}{2\pi} \int_{\Gamma} (|\Omega_1(z, \zeta, G)| + |\Omega_2(z, \zeta, G)|) |w(\zeta) - \Phi_n(\zeta)| ds .$$

If z belongs to a closed subset G' of the circle G we have

$$|w(z) - \hat{w}_n(z)| \leqslant M(G') L_1(w - \Phi_n, \Gamma) . \qquad (15.16)$$

Since on the circumference Γ the series (15.13) is by (15.14) the Fourier series of a continuous function $w(\zeta)$, the series is convergent to w on Γ in the metric of an arbitrary $L_p(\Gamma)$, $p \geqslant 1$. Hence $L_1(w - \Phi_n, \Gamma) \to 0$ when $n \to \infty$. Consequently, (15.16) implies that the sequence $\hat{w}_n(z)$ is uniformly convergent to $w(z)$ inside G. Thus, it has been proved that $w(z)$ can be expanded into the series

$$w(z) = \sum_{k=0}^{\infty} c_k w_k(z, z_0, \varrho) , \qquad (15.17)$$

which is uniformly convergent inside the circle G. The coefficients of this series are given by the formulae (15.14) which are identical with the well known integral formulae for the coefficients of the Taylor series of an analytic function. Therefore, the series (15.17) will be called *the generalized Taylor series.*

Let now G be the ring $0 \leqslant \varrho < |z-z_0| < \varrho_1$ bounded by the circles $|z-z_0| = \varrho_0$ and $|z-z_0| = \varrho_1$ which will be denoted by Γ_0 and Γ_1. The generalized rational functions $w_n(z, z_0)$ $(n = 0, \pm 1, \ldots)$ corresponding to this domain will be denoted by $w_n(z, z_0, \varrho_0, \varrho_1)$.

If $w(z)$ satisfies inside G the equation (15.1) and it is continuous in $\bar{G}$ then it can be expanded into the following series:

$$w(z) = \sum_{-\infty}^{+\infty} c_k w_k(z, z_0, \varrho_0, \varrho_1) , \tag{15.18}$$

where c_k are real constants which are to be calculated by the formulae

$$\begin{aligned} c_{2k} + ic_{2k+1} &= \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{w(\zeta)\,d\zeta}{\zeta^{k+1}} \qquad (k = 0, 1, \ldots) , \\ c_{2k} + ic_{2k+1} &= \frac{1}{2\pi i} \int\limits_{\Gamma_0} w(\zeta)\zeta^k d\zeta \qquad (k = -1, -2, \ldots) . \end{aligned} \tag{15.19}$$

The series (15.18) converges uniformly inside the ring $0 \leqslant \varrho_0 < |z-z_0| < \varrho_1$. This assertion can be proved by almost literally repeating the reasoning concerning the series (15.17).

15.3. With every power series of the form

$$\sum_{-\infty}^{+\infty} (c_{2k} + ic_{2k+1})(z-z_0)^k \tag{15.20}$$

we can associate the series

$$\sum_{-\infty}^{+\infty} c_k w_k(z, z_0, \varrho_0, \varrho_1) , \tag{15.21}$$

where $0 \leqslant \varrho_0 < |z-z_0| < \varrho_1$ is the ring of convergence of the series (15.20). This series will be called *the generalized power series of the first kind.* Such series have the following important property:

THEOREM 3.26. *The series* (15.21) *is uniformly convergent inside the domain (circle, ring) of convergence of the power series* (15.20).

PROOF. We shall restrict ourselves to the proof of the theorem for the case in which (15.20) contains only positive powers of $z-z_0$. The extension to the general case offers no essential difficulties.

Let $|z-z_0| < \varrho$ be the circle of convergence of the power series

$$\sum_0^\infty (c_{2k}+ic_{2k+1})(z-z_0)^k . \tag{15.22}$$

We have to prove the uniform convergence of the generalized power series of the form

$$\sum_0^\infty c_k w_k(z, z_0, \varrho) , \tag{15.23}$$

in any circle $|z-z_0| \leqslant \varrho' < \varrho$.

Let A_n and B_n be functions coinciding with A and B inside the circle G_n—$|z-z_0| < \dfrac{n}{n+1}\varrho$—and vanishing outside this circle. Let G' be a closed set of points of the circle $|z-z_0| < \varrho$. Then an integer n_0 can be found such that for $n > n_0$ the set G' belongs to all circles G_n. Let G'' be a closed set which does not intersect G'. In this case, according to Corollary 2 of Theorem 3.23, the sequences of kernels $\Omega_{jn}(z, \zeta, G) \equiv \Omega_j(z, \zeta, G_n)$ of the equation $\partial_{\bar z} w + A_n w + B_n \bar w = 0$ converge uniformly to the kernels $\Omega_j(z, \zeta, G)$ of the equation $\partial_{\bar z} w + A_n w + B_n \bar w = 0$ with respect to the both arguments z and ζ, if $z \in G'$ and $\zeta \in G''$. Since for $z \in G'$ the functions $\Omega_{1n}(z, \zeta, G)$ and $\overline{\Omega_{2n}(z, \zeta, G)}$

are holomorphic with respect to the argument ζ outside G we have:

$$w_{n,p}(z) = \frac{1}{2\pi i}\int_{\Gamma} \Omega_1(z,\zeta,G)\Phi_{n,p}(\zeta)\,d\zeta - \Omega_2(z,\zeta,G)\overline{\Phi_{n,p}(\zeta)}\,d\bar{\zeta}$$

$$= \lim_{m\to\infty}\frac{1}{2\pi i}\int_{\Gamma_m} \Omega_1(z,\zeta,G_m)\Phi_{n,p}(\zeta)\,d\zeta - \Omega_2(z,\zeta,G_m)\overline{\Phi_{n,p}(\zeta)}\,d\bar{\zeta}, \tag{15.24}$$

where Γ_m is the circle $|z-z_0| = \dfrac{m}{m+1}\varrho$,

$$w_{n,p}(z) = \sum_{k=n}^{n+p} c_k w_k(z, z_0),$$

$$\Phi_{n,p}(z) = \sum_{k=n}^{\left[\frac{n+p}{2}\right]} (c_{2k} + ic_{2k+1})(z-z_0)^k.$$

According to the principle of the maximum modulus (p. 152) there exist a positive constant M and a point $\hat{z}$ on the circumference Γ_{n_0} such that

$$\operatorname*{Max}_{z\in G'}|w_{n,p}(z)| < M|w_{n,p}(\hat{z})|. \tag{15.25}$$

Since in view of (15.24)

$$|w_{n,p}(\hat{z})| \leqslant \frac{1}{2\pi}\overline{\lim}\int_{\Gamma_m}(|\Omega_1(\hat{z},\zeta,G_m)| + |\Omega_2(\hat{z},\zeta,G_m)|)|\Phi_{n,p}(\zeta)|\,ds,$$

an integer $m_0 > n_0$ can be found such that because of (15.25) we have the inequality

$$\operatorname*{Max}_{z\in G'}|w_{n,p}(z)| \leqslant \frac{M}{2\pi}\int_{\Gamma_{m_0}}(|\Omega_{1m_0}(\hat{z},\zeta)| + |\Omega_{2m_0}(\hat{z},\zeta)|)\Phi_{n,p}(\zeta)|\,ds.$$

Since the series (15.22) converges uniformly inside the circle $|z-z_0| < \varrho$, $\Phi_{n,p}(\zeta)\to 0$ uniformly on Γ_{m_0}. Hence, $w_{n,p}(z)\to 0$ uniformly in G', thus completing the proof.

15.4. Series of generalized analytic functions analogous to the Laurent and Taylor series can be also obtained in the following way.

Let $\tilde{w}_{2n}(z, z_0)$ and $\tilde{w}_{2n+1}(z, z_0)$ be generalized polynomials of the class $\mathfrak{A}_{p,2}(A, B, E)$, $p > 2$, corresponding to the normal analytic divisors $(z-z_0)^n$ and $i(z-z_0)^n$ (§7):

$$\begin{aligned} \tilde{w}_{2n}(z, z_0) &= \mathfrak{K}_{z_0}\big((z-z_0)^n\big) , \\ \tilde{w}_{2n+1}(z, z_0) &= \mathfrak{K}_{z_0}\big(i(z-z_0)^n\big) . \end{aligned} \tag{15.26}$$

These functions are determined uniquely and they satisfy the following inequalities (§4.6):

$$e^{-\Omega_0(z)} \leqslant \frac{|\tilde{w}_n(z, z_0)|}{|z-z_0|^n} \leqslant e^{\Omega_0(z)} . \tag{15.27}$$

In view of the inequality (6.15) of Ch. I and the formula (7.3) we have for Ω_0 the estimate

$$0 \leqslant \Omega_0(z) \leqslant M_p L_p(|A| + |B|)|z-z_0|^{\frac{p-2}{p}} . \tag{15.28}$$

The functions $\tilde{w}_n(z, z_0)$ will be called *the generalized power functions* of the class $\mathfrak{A}_{p,2}(A, B, E)$, $p > 2$.

Let us now associate with the power series

$$\sum_{n=-\infty}^{+\infty} (c_{2n} + ic_{2n+1})(z-z_0)^n \tag{15.29}$$

the series

$$\sum_{n=-\infty}^{+\infty} c_n \tilde{w}_n(z, z_0) , \tag{15.30}$$

which will be called *the generalized power series of the second kind.*

With the help of the inequalities (15.27) it is easy to prove

Theorem 3.27. (*Generalized Abel theorem*). *The series* (15.29) *and* (15.30) *have identical domains (ring, circle) of convergence and divergence, and inside the domain of con-*

vergence they converge absolutely and uniformly. In particular, if the series

$$\sum_{n=0}^{\infty} c_n \widetilde{w}_n(z, z_0) \tag{15.31}$$

is convergent at a point $z' \neq z_0$, *then it is convergent absolutely and uniformly inside the circle* $|z-z_0| < |z'-z_0|$. *If the series* (15.31) *is divergent at the point* $z' \neq z_0$, *then it is divergent for* $|z-z_0'| > |z'-z_0|$.

Since, according to Theorem 3.10, in the domain of convergence of the series (15.31) its sum is a function of the class $\mathfrak{A}_{p,2}(A, B, G)$, $p > 2$, this series may be regarded as a (linear) operator associating with every function analytic in this domain (or, which is equivalent, with the corresponding power series) a function of the class $\mathfrak{A}_{p,2}(A, B, G)$, $p > 2$.

Let $w(z)$ be the sum of the series (15.31). Then the coefficients of this series are calculated by the following recursive formulae:

$$c_0 + ic_1 = w(z_0),$$

$$c_{2n} + ic_{2n+1} = \lim_{z \to z_0} \frac{w(z) - \sum\limits_{k=0}^{n-1} c_{2k}\widetilde{w}_{2k}(z, z_0) + c_{2k+1}\widetilde{w}_{2k+1}(z, z_0)}{(z-z_0)^n}$$

$$(n = 1, 2, 3, \ldots). \tag{15.32}$$

Obviously, it is possible to prove the following assertion.

Let $w(z)$ *be a function of the class* $\mathfrak{A}_{p,2}(A, B, G)$, $p > 2$, *where* G *is the circle* $|z-z_0| < R$. *Let* c_n *be real constants calculated by means of the recursive formulae* (15.32). *Then we have the following expansion*:

$$w(z) = \sum_{n=0}^{\infty} c_n \widetilde{w}_n(z, z_0), \tag{15.33}$$

which is uniformly and absolutely convergent inside the circle $|z-z_0| < R$.

Expanding $\widetilde{w}_n(z, z_0)$ into the generalized Taylor series of the form (15.17) and, consequently, calculating the

coefficients of expansion by means of the formulae (15.14) we obtain

$$\widetilde{w}_n(z, z_0) = a_{n0}w_0(z, z_0) + \ldots + a_{nl_n}w_{l_n}(z, z_0), \quad (15.34)$$

where a_{nj} are real constants, $l_n = n+1$ for even n and $l_n = n$ for odd n. It follows that

$$\widetilde{w}_n(z, z_0') = \mathcal{K}\left[P_{\frac{l_n-1}{2}}(z-z_0), E\right], \quad (15.35)$$

where

$$P_{\frac{l_n-1}{2}}(z) = \sum_{k=0}^{\frac{1}{2}(l_n-1)} (a_{n,2k} + ia_{n,2k+1})z^k. \quad (15.36)$$

In conclusion we observe that the theory of generalized power series has been constructed in a somewhat different way in the papers of Bers and Agmon, [6], and Bers, [5a, c].

§16. Integral equations for the real part of a generalized analytic function

The construction of the general solution of the equation

$$\mathfrak{E}(w) \equiv \partial_{\bar{z}}w + Aw + B\overline{w} = F \quad (16.1)$$

can also be carried out by means of an integral equation with a real kernel which contains only either the real or the imaginary part of the unknown function $w = u+iv$.

Writing the equation (16.1) in the form

$$\partial_{\bar{z}}w + (A-B)w + 2Bu = F \quad (16.2)$$

and introducing the function

$$\omega(z) = -\frac{1}{\pi}\iint_G \frac{A(\zeta)-B(\zeta)}{\zeta - z}\,d\xi\,d\eta + \Phi_0(z), \quad (16.3)$$

where Φ_0 is an arbitrary function analytic in G and continuous in $G+\Gamma$, the relation (16.2) can be put in the form

$$\frac{\partial}{\partial\bar{z}}[e^{\omega(z)}w] + 2Be^{\omega(z)}u = F(z)e^{\omega(z)}.$$

Hence we have

$$w(z)e^{\omega(z)} = \frac{2}{\pi}\iint_G \frac{B(\zeta)e^{\omega(\zeta)}u(\zeta)}{\zeta-z}d\xi d\eta + F_0(z), \quad (16.4)$$

where

$$F_0(z) = \Phi(z) - \frac{1}{\pi}\iint_G \frac{F(\zeta)e^{\omega(\zeta)}}{\zeta-z}d\xi d\eta. \quad (16.5)$$

Here Φ is an arbitrary function analytic with respect to z, which is related to w by the formula

$$\Phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{w(\zeta)e^{\omega(\zeta)}}{\zeta-z}d\zeta. \quad (16.6)$$

Multiplying both sides of the relation (16.4) by $e^{-\omega(z)}$ and separating the real and imaginary parts we have

$$u(z) - \frac{2}{\pi}\iint_G u(\zeta)\,\mathrm{Re}\left[\frac{e^{\omega(\zeta)}B(\zeta)}{(\zeta-z)e^{\omega(z)}}\right]d\xi d\eta = f(z), \quad (16.7)$$

$$v(z) - \frac{2}{\pi}\iint_G u(\zeta)\,\mathrm{Im}\left[\frac{e^{\omega(\zeta)}B(\zeta)}{(\zeta-z)e^{\omega(z)}}\right]d\xi d\eta = g(z), \quad (16.8)$$

where

$$f(z) = \mathrm{Re}[e^{-\omega(z)}\Phi(z)] - \mathrm{Re}\left[\frac{1}{\pi}\iint_G \frac{e^{\omega(\zeta)}F(\zeta)}{e^{\omega(z)}(\zeta-z)}d\xi d\eta\right], \quad (16.9)$$

$$g(z) = \mathrm{Im}[e^{-\omega(z)}\Phi(z)] - \mathrm{Im}\left[\frac{1}{\pi}\iint_G \frac{e^{\omega(\zeta)}F(\zeta)}{e^{\omega(z)}(\zeta-z)}d\xi d\eta\right]. \quad (16.10)$$

Thus, we have obtained for the real part $u(z)$ of the function $w(z)$ a real integral equation of the Fredholm type (16.7). We shall show that this equation is soluble for an arbitrary continuous right-hand side. To this end we have to prove that the corresponding homogeneous equation

$$u(z) - \iint_G \mathrm{Re}\left[\frac{2B(\zeta)e^{\omega(\zeta)}}{\pi(\zeta-z)e^{\omega(z)}}\right]u(\zeta)d\xi d\eta = 0$$

has no non-trivial solution. This equation may be put in the form

$$u(z) = \operatorname{Re}\{e^{-\omega(z)} w_1(z)\}\,,$$

where

$$w_1(z) = \frac{2}{\pi} \iint_G u(\zeta) \frac{B(\zeta)\, e^{\omega(\zeta)}}{\zeta - z}\, d\xi\, d\eta\,. \tag{16.11}$$

It is readily observed that w_1 satisfies the equation

$$\partial_{\bar{z}} w_1 + B w_1 + B e^{\omega - \bar{\omega}} \overline{w}_1 = 0\,.$$

On the other hand the relation (16.11) indicates that w_1 can be continuously continued to the entire plane, it is holomorphic outside $G + \Gamma$ and vanishes at infinity. According to the generalized Liouville Theorem 3.11 $w_1 \equiv 0$ i.e. $u = 0$ which was to be proved.

Having found from the equation (16.7) the real part u of the function $w(z)$ we can find its imaginary part v by making use of the formula (16.8)

We note that the kernel of the equation (16.7) contains one arbitrary holomorphic function $\Phi_0(z)$ the choice of which is entirely up to us. By a suitable choice of Φ_0 we can satisfy additional special conditions for the function $\omega(z)$.

§17. Properties of solutions of elliptic systems of equations of the general form

Many properties of the solutions of the equation $\partial_{\bar{z}} w + Aw + B\overline{w} = F$ can be extended to the solutions of elliptic systems of partial differential equations of the first order not reduced to the canonical form. This, obviously, can be carried out by the reduction of the system to the canonical form; such a way, however, has a disadvantage, namely that certain restrictions must be imposed on the coefficients of the system, which are somewhat stronger than is really necessary. Generalizations in this respect have been obtained independently in the

papers of Bojarski [11d] and Bers and Nirenberg [7a]. Further we shall state the basic results following in principle the paper of Bojarski, [11d] (§§ 2, 3, 4, 6).

17.1. The system of equations will be written in the form (Ch. II, §7.1)

$$\begin{aligned} -v_y + a_{11}u_x + a_{12}u_y + a_0 u + b_0 v &= f\,, \\ v_x + a_{21}u_x + a_{22}u_y + c_0 u + d_0 v &= g\,. \end{aligned} \tag{17.1}$$

We shall assume that the coefficients a_{ik} are bounded measurable functions on the plane E, satisfying the following conditions of ellipticity of the system:

$$a_{11} > 0\,, \quad \Delta = a_{11}a_{22} - \frac{1}{4}(a_{12} + a_{21})^2 > \Delta_0 > 0 \text{ on } E)\,, \tag{17.2}$$

$$\Delta_0 = \text{const.}$$

If a_{ik} are given in a bounded domain in which the conditions (17.2) are satisfied it is sufficient to continue them outside G according to the rule $a_{11} = a_{22} = 1$, $a_{12} = a_{21} = 0$. With respect to the other coefficients and the free terms it will be assumed that they belong to $L_{p,2}(E)$, $p > 2$.

Introducing the complex function $w = u + iv$ the system (17.1) can be written in the following form:

$$(\tilde{p}+1)\partial_{\bar z}w + (\tilde{p}-1)\partial_{\bar z}\overline{w} + \tilde{q}(\partial_z w + \partial_{\bar z}\overline{w}) + \\ + \tilde{A}w + \tilde{B}\overline{w} = \tilde{F}\,, \tag{17.3}$$

where

$$\tilde{p} = \frac{a_{11} + a_{22} - i(a_{12} - a_{21})}{2}\,, \qquad \tilde{q} = \frac{a_{11} - a_{22} + i(a_{12} + a_{21})}{2}\,.$$

Taking into account the equation complex conjugate to (17.3) and eliminating from these equations $\partial_z\overline{w}$ we arrive at the equation

$$\partial_{\bar z}w - q_1(z)\partial_z w - q_2(z)\partial_{\bar z}\overline{w} + Aw + B\overline{w} = F\,, \tag{17.4}$$

where

$$q_1 = \frac{2\tilde{q}}{|\tilde{q}|^2 - |1+\tilde{p}|^2}, \qquad q_2 = \frac{|\tilde{q}|^2 + (1+\tilde{p})(1-\overline{\tilde{p}})}{|1+\tilde{p}|^2 - |\tilde{q}|^2}. \tag{17.5}$$

A simple calculation shows that

$$|1+\tilde{p}|^2 - |\tilde{q}|^2 = 1 + \Delta + a_{11} + a_{22} + \frac{1}{4}(a_{21} - a_{12})^2 \geqslant 1 + \Delta,$$

$$|q_1(z)| + |q_2(z)| = \frac{\sqrt{(a_{11}+a_{22})^2 - 4\Delta} + \sqrt{(1+\delta)^2 - 4\Delta}}{1 + a_{11} + a_{22} + \delta}, \tag{17.6}$$

where

$$\delta = \Delta + \frac{1}{4}(a_{21} - a_{12})^2 .$$

Taking into account the condition $\Delta > \Delta_0 > 0$ (on E) the relation (17.6) implies at once that

$$|q_1(z)| + |q_2(z)| \leqslant q_0 < 1 \qquad (q_0 = \text{const}). \tag{17.7}$$

Thus, we shall further deal with the complex equation (17.4) which, in view of (17.7), is entirely equivalent to the original system of equations (17.1). The coefficients q_1 and q_2 are measurable functions on E satisfying the condition (17.7) and A, B, F are functions belonging to the class $L_{p,2}(E)$, $p > 2$.

The solution of the equation (17.4) will be sought for in a class $D_{1,p}(G)$, $p \geqslant 2$, without stating this assumption every time.*

In the conformal mapping of the domain—$z = \varphi(\zeta)$—the equation (17.4) is transformed into the equation

$$\partial_{\bar{\zeta}} w - q_{1*}\partial_{\zeta} w - q_{2*}\partial_{\bar{\zeta}}\overline{w} + A_* w + B_* \overline{w} = F_*, \tag{17.8}$$

* It can be proved that if the solution $w(z)$ of the equation (17.4) belongs to $D_{1,p'}$, $2-\varepsilon \leqslant p'$, we have always for a sufficiently small ε $w(z) \in D_{1,p}(G)$ for a certain $p > 2$.

where

$$q_{1*} = q_1 \frac{\overline{\varphi'(\zeta)}}{\varphi'(\zeta)}, \qquad q_{2*} = q_2 ,$$

$$A_* = A\overline{\varphi'}, \qquad B_* = B\overline{\varphi'}, \qquad F_* = F\overline{\varphi'} .$$

Thus the condition (17.7) is satisfied for the new equation as well, the constant q_0 being the same. In particular, this enables us to reduce the investigation of the behaviour of the solution of the equation (17.4) near the point $z = \infty$ to the investigation of the behaviour of the solution of the equation (17.8) in the vicinity of the point $\zeta = 0$. To this end it is sufficient to take for $\varphi(\zeta)$ the function $\frac{1}{\zeta}$.

17.2. We shall now investigate the properties of solutions of the homogeneous equation of the form

$$\partial_{\bar{z}} w - q(z)\partial_z w + Aw + B\bar{w} = 0, \qquad |q(z)| \leqslant q_0 < 1 . \qquad (17.9)$$

Together with the solutions of the equation (17.9) we shall consider the solutions of the corresponding Beltrami's equation which in this paragraph will be denoted by the letter f:

$$\partial_{\bar{z}} f - q(z)\partial_z f = 0 . \qquad (17.10)$$

These equations will be considered in a bounded domain G. We shall prove the following theorem which is a generalizations of the basic lemma (§4.1).

THEOREM 3.28. *Let $w = w(z)$ be a solution (which may possess isolated singular points) of the equation* (17.9) *in a domain G. Then $w(z)$ is representable in the form*

$$w(z) = f(z)e^{T\omega} = f(z)e^{\varphi(z)} , \qquad (17.11)$$

where $f(z)$ is a solution of the equation (17.10), $\omega \in L_p(\bar{G})$, $p > 2$,

$$\varphi(z) = \mathrm{T}\omega = -\frac{1}{\pi}\iint\limits_G \frac{\omega(\zeta)}{\zeta - z}\, d\xi\, d\eta ;$$

the function $\varphi(z) \in C_\alpha(E)$, $\alpha = \dfrac{p-2}{p}$, *is holomorphic outside* $\bar{G}$ *and vanishes at infinity.*

PROOF. Let $w(z)$ be the solution under consideration. Assume that

$$h(z) = -\left(A + B\frac{\overline{w}}{w}\right) \text{ in } G, \text{ where } w \neq 0, w \neq \infty, \quad (17.12)$$

$h(z) = 0$ at all other points of the plane.

Consider the integral equation

$$\begin{gathered} \omega - q\Pi\omega = h, \\ \Pi\omega = \partial_z \mathrm{T}\omega \equiv -\frac{1}{\pi}\iint_E \frac{w(\zeta)}{(\zeta - z)^2}\, d\xi\, d\eta\,, \end{gathered} \quad (17.13)$$

where we assume that $q = 0$ and $h = 0$ outside G. Specifying $p > 2$ so that

$$q_0 \Lambda_p < 1, \quad \text{where} \quad \Lambda_p = L_p(\Pi) \quad (\Lambda_2 = 1)\,, \quad (17.14)$$

the equation (17.13) will have the unique solution $\omega \in L_p(\bar{G})$ for an arbitrary $h \in L_p(\bar{G})$ and, evidently, $\omega \equiv 0$ outside $\bar{G}$.

For the function $f(z) = w(z) e^{-\varphi(z)}$ we have

$$\begin{aligned} [f_{\bar{z}} - q(z) f_z] e^{\varphi} &= w_{\bar{z}} - w[\omega - q\Pi\omega] - q w_z \\ &= w_{\bar{z}} - wh - q w_z = 0\,. \end{aligned}$$

Thus, $f(z)$ is a solution of the equation (17.10); but $w = f e^{\varphi}$, which proves (17.11). The remaining statements of the theorem follow from the formulae given for $\varphi(z)$.

Under the conditions for the function φ stated in the theorem the representation (17.11) is unique.

In fact, assuming that $w = f(z) e^{\varphi} = f_1(z) e^{\varphi_1(z)}$ we observe that the ratio $\dfrac{f(z)}{f_1(z)} = e^{\varphi - \varphi_1}$ is a function holomorphic in the domain G, it is analytically continuable to the entire plane and equal to unity at infinity. According to Liouville's theorem such a function is identically equal to unity. Thus, the uniqueness has been proved.

It should be observed that $\varphi(z)$ in the formula (17.11) depends on the solution being represented if $B \not\equiv 0$. But if $B \equiv 0$ then φ depends only on the coefficients q and A and the formula (17.11) yields the general solution of the equation

$$\partial_{\bar{z}} w - q(z) \partial_z w + Aw = 0 . \tag{17.15}$$

Then in the formula (17.11) $f(z)$ is an arbitrary solution of Beltrami's equation (17.10) which may possess arbitrary singularities (poles, essentially singular points, branch points, lines of discontinuity, etc). According to Theorem 2.15 of Ch. II, $f(z)$ has the form

$$f(z) = \Phi(W(z)) . \tag{17.16}$$

where $W(z)$ is a basic homeomorphism of the equation (17.10) and Φ is an arbitrary function analytic in the domain $W(G)$.

We recall that $W(z) \in D_{1,p}^{\sim}(E)$, $p > 2$, and, consequently, $W \in C_\alpha(E)$, $\alpha = \dfrac{p-2}{p}$.

In the formula (17.16), for $W(z)$ we may also take arbitrary global homeomorphisms of the equation (17.10) with respect to the domain G.

The representation (17.11) is not the only one possible. It is characterized by the fact that $\varphi(z)$ is continuable in a continuous way to the entire plane, is holomorphic outside $\bar{G}$ and vanishes at infinity. If these properties be abandoned, other representations of the form (17.11) can be derived. For instance we may state the following theorem (in the particular case of G being the unit circle).

THEOREM 3.29. *Let G be the unit circle $|z| < 1$. Then every solution of the equation* (17.9) *in G can be represented in the form*

$$w(z) = f(z) e^{\psi(z)} , \tag{17.17}$$

where

$$\psi(z) = \tilde{\mathrm{T}}\omega = -\frac{1}{\pi} \iint_G \left[\frac{\omega(t)}{t-z} + \frac{z\overline{\omega(t)}}{1-z\bar{t}}\right] dG_t , \tag{17.18}$$

$$\omega \in L_p(G), \; p > 2 ,$$

and $Re\psi(z) = 0$ *for* $|z| = 1$, $f(z)$ *being a solution of the equation* (17.10).

This representation is unique.

The proof is the same as that for the preceding theorem, the only difference being that instead of the integral equation (17.13) we have to solve an equation of the form

$$\omega - q\tilde{\Pi}\omega = h ,$$

where the operator $\tilde{\Pi}$ is defined by the formula

$$\tilde{\Pi}\omega = \partial_z\tilde{\mathrm{T}}\omega = -\frac{1}{\pi}\iint\limits_G \left(\frac{\omega(\zeta)}{(\zeta-z)^2} + \frac{\overline{\omega(\zeta)}}{(1-z\bar{\zeta})^2}\right) d\xi\, d\eta .$$

Continuing $\omega(z)$ outside G according to the formula

$$\omega(\zeta) = \frac{1}{\bar{\zeta}}\,\overline{\omega\left(\frac{1}{\bar{\zeta}}\right)}, \quad |\zeta| \geqslant 1 , \tag{17.19}$$

we have

$$\tilde{\Pi}\omega = -\frac{1}{\pi}\iint\limits_E \frac{\omega(\zeta)\, d\xi\, d\eta}{(\zeta-z)^2} .$$

Continuing by means of this integral $\tilde{\Pi}\omega$ to the entire plane and setting $g(z) = \tilde{\Pi}\omega$ we easily discover that

$$g(z) = \frac{1}{z^2}\,\overline{g\left(\frac{1}{\bar{z}}\right)} . \tag{17.20}$$

By virtue of the relations (17.20) and (17.19) we have (see §9.1)

$$\iint\limits_G \tilde{\Pi}\omega\overline{\tilde{\Pi}\omega}\, dx\, dy = \frac{1}{2}\iint\limits_E g(z)\overline{g(z)}\, dx\, dy$$
$$= \frac{1}{2}\iint\limits_E \omega\bar{\omega}\, dx\, dy = \iint\limits_G \omega\bar{\omega}\, dx\, dy .$$

This implies that the norm of the operator $\tilde{\Pi}$ in L_2 is equal to unity. Hence, fixing so $p > 2$ that $q_0 L_p(\tilde{\Pi}) < 1$

we easily find that the equation $\omega - q\tilde{\Pi}\omega = h$ has the unique solution $\omega \in L_p(\bar{G})$ for an arbitrary $h \in L_p(\bar{G})$.

17.3. The theorems proved above will in this section be generalized to the case of a more general equation

$$\partial_{\bar{z}}w - q_1(z)\partial_z w - q_2(z)\partial_{\bar{z}}\bar{w} + Aw + B\bar{w} = 0 \,. \qquad (17.21)$$

Besides (17.21) we shall consider the equation

$$\partial_{\bar{z}}w - q_1(z)\partial_z w - q_2(z)\partial_{\bar{z}}\bar{w} = 0 \,. \qquad (17.22)$$

If $w = w(z)$ is its solution, then $w(z)$ satisfies the equation

$$\partial_{\bar{z}}w - q(z)\partial_z w = 0 \,, \qquad (17.23)$$

where

$$q(z) = q_1(z) + q_2\frac{\overline{w_z}}{w_z} \quad \text{in } G, \text{ and} \quad w_z \neq 0 \,,$$

and

$$q(z) = 0, \quad \text{where} \quad w_z = 0 \quad \text{also outside } G \,.$$

Obviously, $|q(z)| \leqslant |q_1(z)| + |q_2(z)| \leqslant q_0 < 1$, q_0 being a constant. Hence Theorem 2.15 of Ch. II, at once implies

THEOREM 3.30. *Any solution of the equation* (17.22) *is representable in the form*

$$w(z) = f\big(W(z)\big) \,, \qquad (17.24)$$

where $f(W)$ is a function analytic in $W(G)$ and $W(z)$ is a basic homeomorphism of the equation (17.23).

For $W(z)$ we may also take an arbitrary homeomorphism of the equation (17.23) global with respect to the domain G.

In an analogous way Theorem 3.28 implies

THEOREM 3.31. *Any solution of the equation* (17.21) *is representable in the form*

$$w(z) = f\big(W(z)\big)e^{\varphi(z)} \,, \qquad (17.25)$$

where $f(W)$ is a function analytic in $W(G)$, $W(z)$ is a basic homeomorphism of the equation (17.23) *and $\varphi(z) \in D_{1,p}(E)$, $p > 2$, is holomorphic outside G and vanishes at infinity.*

The conditions imposed on W and φ ensure the uniqueness of the representation (17.25). They may, however, be varied, i.e. φ and W may be subjected to other conditions without affecting the uniqueness of the representation (17.25).

The representations (17.24) and (17.25) are also valid in the case of solutions possessing isolated singularities. Then all the singularities are transferred to the analytic function $f(W)$.

Theorem 3.30 and Theorem 3.31 differ from Theorem 3.28 and Theorem 3.29; in fact, now the homeomorphism $W(z)$ cannot be regarded as fixed, the same for all solutions. In fact, $W(z)$ satisfies the equation (17.23) in which the coefficient $q(z)$ depends on the solution to be represented. Therefore the formula (17.25) cannot be regarded as a device for constructing solutions of the equation (17.21). Nevertheless, the formula (17.25) is important for the investigation of the properties of the solutions of the equation (17.21). It enables us to transfer a number of properties of analytic functions to the solutions of equations of the form (17.21). Thus, for the system of equations of the form (17.21) the following principles and theorems are valid literally: the principle of the maximum modulus, the principle of the argument, the theorem on the unique continuous continuation, the theorem on the isolation of zeros, analogues of the theorems on removable singularities, behaviour of solutions near poles or essentially singular points, criteria of the univalency of a mapping, etc.

The restriction $q_1 = q_2 = A = B = 0$ outside a bounded domain G is irrelevant for the validity of the foregoing results. It is easily observed that it is sufficient to assume for instance that $A, B \in L_{p,2}(E)$ and $|q_1| + |q_2| \leqslant q_0 < 1$, $q_0 = \text{const.}$, on the entire plane.. These restrictions imply only certain limitations on the order of the magnitude of A and B as $z \to \infty$ without requiring the identical vanishing of these functions.

In conncetion with this problem let us consider the following theorem which will be made use of further; the theorem is valid under the condition $|q_1|+|g_2| \leqslant \leqslant q_0 < 1$, $q_0 = \text{const.}$ on the entire plane E

THEOREM 3.32. *A solution of the equation*

$$\partial_{\bar z} w - q_1 \partial_z w - q_2 \partial_{\bar z} \overline{w} = 0, \qquad |q_1| + |q_2 \leqslant q_0 < 1, \tag{17.25a}$$

bounded on the entire plane E and vanishing at one point $z_0 \in E$ vanishes identically if its derivative $w_{\bar z} \in L_{p'}, L_p(E)$, $p \geqslant 2, 1 \leqslant p' < 2$.

In fact, representing the solution $w(z)$ in the form

$$w(z) = -\frac{1}{\pi} \iint_E \frac{\omega(t)}{t-z} dE_t + \Phi(z), \qquad \omega = \partial_{\bar z} w, \tag{17.25b}$$

where $\Phi(z)$ is an entire function, we have in view of Theorem 1.21 and the conditions of Theorem 3.32 $\Phi(z) = = \text{const.}$ Substituting (17.25b) into the equation (17.25a) we obtain $\omega - q\Pi\omega = 0$, $|q| \leqslant q_0 < 1$. Hence $\omega = 0$ i.e. $w = \text{const.} = 0$.

REMARK. Theorem 3.30 and Theorem 3.31 hold also for the solutions of the inequalities

$$\begin{gathered} |w_{\bar z}| \leqslant |q_1| \cdot |w_z|, \qquad |q_1| \leqslant q_0 < 1, \\ |w_{\bar z} - q_1 w_z - q_2 \overline{w}_z| \leqslant A|w|, \qquad A \in L_p, p > 2, \\ |q_1| + |q_2| < q_0 < 1. \end{gathered} \tag{17.26}$$

17.4. Presently, we shall indicate a method of construction of solutions of the equation (17.4). Seeking the solutions in a bounded domain G we assume that the coefficients and the free term of the equation vanish outside G. In other words, outside G, the Cauchy–Riemann equation holds, i.e. $\partial_{\bar z} w = 0$. We have

THEOREM 3.33. *Under the indicated conditions the equation (17.4) has always a solution $w = w(z)$ continuable analytically outside G, such that*

$$w(z) \sim \Phi(z) \quad \text{for} \quad z \to \infty,$$

where $\Phi(z)$ is an arbitrary given beforehand entire function.

This solution is unique.

PROOF. The solutions of the equation (17.5) satisfying the conditions of the theorem will be sought for in the form

$$w(z) = \Phi(z) - \frac{1}{\pi}\iint_E \frac{\omega(\zeta)}{\zeta - z} d\xi\, d\eta \equiv \Phi(z) + \mathrm{T}\omega\,, \quad (17.27)$$

We assume that $\omega = 0$ outside G. Substituting (17.27) into (17.5) we obtain the equation for ω:

$$\omega - q_1 \Pi\omega - q_2 \overline{\Pi}\overline{\omega} + A\mathrm{T}\omega + B\overline{\mathrm{T}\omega} = F_*\,, \quad (17.28)$$

where

$$F_* = A\Phi + B\overline{\Phi} + F + q_1\Phi' + q_2\overline{\Phi'}\,, \qquad z \in G\,, \quad (17.28\text{a})$$

and $F_* = 0$ if z lies outside G.

We may apply Fredholm's theory to the equation (17.28). In fact, denoting by R the operator inverse to the operator $\mathrm{I} - q_1\Pi - q_2\overline{\Pi}$ which exists in a $L_p(E)$, $p > 2$, we observe that the equation (17.28) is equivalent to the equation $\omega = -RA\mathrm{T}\omega - RB\mathrm{T}\overline{\omega} + RF_* \equiv R_1\omega + RF_*$. Since T is completely continuous R_1 is also completely continuous. Therefore the Fredholm theorems may be applied to the equation (17.28). Let us consider the homogeneous equation

$$\omega - q_1\Pi\omega - q_2\overline{\Pi}\overline{\omega} + A\mathrm{T}\omega + B\overline{\mathrm{T}\omega} = 0\,. \quad (17.29)$$

Let $\omega \in L_p$, $p > 2$, be a solution of this equation. Then $w_1(z) = \mathrm{T}\omega \in C_\alpha(E)$, $\alpha = \dfrac{p-2}{p}$, is a solution of the homogeneous equation (17.28), it is holomorphic outside G and vanishes at infinity. By virtue of (17.25) $w_1(z) = \Phi_0\big(W(z)\big)e^{\varphi(z)}$ where $W(z)$ is a basic homeomorphism of the equation (17.23), Φ_0 is a function holomorphic in the domain $W(G)$; $\varphi(z) \in D_{1,p}(E)$, $p > 2$, is holomorphic outside $\overline{G}$ and vanishes at infinity. Changing the variable z on the variable W and taking into account that $W(z)$ is holomorphic outside $\overline{G}$ with respect to z, we observe that

$\Phi_0(W)$ is analytically continuable to the entire plane and $\Phi_0(W) = 0$ for $W = \infty$. Thus, $\Phi_0(W) \equiv 0$ i.e. $w_1(z) \equiv 0$ whence $\omega(z) = \partial_{\bar{z}} w_1 = 0$.

Thus, we have proved that the homogeneous equation (17.29) has only a trivial solution. We infer, therefore, that the non-homogeneous equation (17.28) has always the unique solution ω. Then the formula (17.27) yields the required solution of the equation (17.4). The uniqueness follows from the fact that according to Theorem 1.16 every solution satisfying the conditions of Theorem 3.33 is representable in the form (17.27). If $\Phi(z)$ is not an entire function but satisfies the condition $\Phi'(z) \in L_p(G)$, $p > 2$, then the above method yields also solutions of the equation (17.5); these solutions, however, in general are not analytically continuable to the entire plane. It is readily seen that the equations (17.28) can be solved by the method of successive approximations, according to the scheme

$$\omega_{n+1} - q_1 \Pi \omega_{n+1} - q_2 \overline{\Pi \omega_{n+1}} = -A \mathrm{T} \omega_n - B \overline{\mathrm{T} \omega_n} + F_*. \quad (17.30)$$

Thus, the integral equation (17.28) makes possible the construction of all solutions of the equation (17.4) belonging to the class $D_{1,p}$, $p \geqslant 2-\varepsilon$ (where ε is a sufficiently small positive number). Hence, an arbitrary function $\Phi(z)$ analytic in G and belonging to the class $D_{1,p}(G)$, $p \geqslant 2-\varepsilon$, is associated with a definite solution of the equation (17.4); evidently, in this way we can obtain all solutions of the class $D_{1,p}$ in the domain G under consideration. It should, however, be borne in mind that the above is not the only method of constructing solutions of the equation (17.4). Varying the form of the representation of the solution (17.27) we can obtain various integral equations which make it possible to construct solutions satisfying some prescribed conditions.

17.5. We shall consider, for instance, a method of construction of solutions acquiring at given points $z_1, \ldots, z_n$

some prescribed values. Let us introduce, similarly to §5.4, the polynomial of the $(n-1)$-th degree with respect to z:

$$P(z,\zeta;z_1,\ldots,z_n)$$
$$=\sum_{k=1}^{n}\frac{(z-z_1)\ldots(z-z_{k-1})(z-z_{k+1})\ldots(z-z_n)}{(z_k-z_1)\ldots(z_k-z_{k-1})(z_k-z_{k+1})\ldots(z_k-z_n)}\,\frac{1}{\zeta-z_k}$$

and let us seek the solution of the equation (17.4) in the form

$$w(z)=\Phi(z)-\frac{1}{\pi}\iint\limits_{E}\frac{\omega(\zeta)d\xi\,d\eta}{\zeta-z}+$$
$$+\frac{1}{\pi}\iint\limits_{E}\omega(\zeta)P(z,\zeta;z_1,\ldots,z_n)d\xi\,d\eta\,, \qquad (17.31)$$

where Φ is a function analytic with respect to z in G; we assume that $\omega\in L_p(\overline{G})$, $p>2$, and $\omega\equiv 0$ outside G. We recall that, according to the assumption made above, the coefficients and the free term of the equation (17.4) also vanish outside G. It is readily observed that the functions $w(z)$ and $\Phi(z)$ have the same values at the points z_j. Inserting the expression (17.31) into the equation (17.4) we obtain for ω the integral equation

$$\omega-q_1\Pi\omega-q_2\overline{\Pi\omega}+\mathrm{T}_0\omega=F_0\,, \qquad (17.32)$$

where T_0 is a completely continuous operator which can easily be written explicitly. Let us note the important fact that for any function $\omega\in L_p(\overline{G})$, $p>2$, $\omega=0$ outside G, $\mathrm{T}_0\omega\in C_\alpha\,(E)$, $\alpha=\dfrac{p-2}{p}$, it is holomorphic outside G and is of the order $(n-1)$ near infinity. The right-hand side F_0 has the form (17.28a) and vanishes identically outside G. By a reasoning similar to that used above for the proof of solubility of the equation (17.28) we can establish that the equation (17.32) has a solution for an arbitrary right-hand side $F_0\in L_p$, $p>2$, $F_0\equiv 0$ outside G. In fact,

considering the solution ω_0 of the corresponding homogeneous equation ($\Phi \equiv F \equiv 0$) and making use as before of the formula (17.25), $w_0 = \Phi_0(W(z))e^{\varphi(z)}$, we discover that Φ_0 is analytically continuable to the entire plane and at infinity has a pole of the order $(n-1)$. Therefore Φ_0 is a polynomial of the $(n-1)$-th degree. Since Φ_0 and w_0 vanish at the points $z_1, \ldots, z_n$, obviously $\Phi_0 \equiv 0$. This completes the proof of our assertion.

Consequently, besides (17.28) we may also make use of the integral equation (17.32) for the construction of solutions of the equation (17.4). The latter integral equation makes it possible to construct solutions assuming prescribed values at fixed points; this is a great advantage over the equation (17.28). In order to construct such a solution it suffices to take in the formula (17.31) for $\Phi(z)$ an analytic function assuming at the points z_k the values equal to those of the required solution. In particular, by applying this procedure we can construct solutions having prescribed zeros.

We now note a corollary following from this result.

Every solution $w(z)$ of the equation (17.5) can be represented as the sum

$$w(z) = w_0(z) + w_*(z)\,, \tag{17.33}$$

where w_0 is a solution assuming at the fixed points $z_1, \ldots, z_n$ the values of the function $w(z)$, i.e. $w_0(z_k) = w(z_k)$ $(k = 1, \ldots, n)$, and w_* is a solution vanishing at these points.

In fact, taking $w(z)$ in the form (17.31) the analytic function Φ can be represented as a sum of two analytic functions

$$\Phi(z) = \Phi_0(z) + \Phi_*(z)\,, \tag{17.34}$$

where $\Phi_0(z_k) = w(z_k)$ and $\Phi_*(z_k) = 0$ $(k = 1, \ldots, n)$. Since Φ constitutes an additive term in the right-hand side of the integral equation (17.32) its solution has the form $\omega = \omega_0 + \omega_*$.

Inserting this result into the right-hand side of (17.31) we obtain the representation of w in the form of the sum (17.33).

17.6. There are some other theorems for the equation (17.4), enabling us to construct solutions with specified properties. We state without proof the following theorem similar to Theorem 3.31.

THEOREM 3.34. *Let G be the unit circle $|\zeta| < 1$ and $\Phi(\zeta)$, $\zeta \in G$, an arbitrary function analytic in G, which may possess arbitrary isolated singularities inside and on the boundary of G. Then there exist two functions $W_0(z)$ and $\varphi(z)$ of the class $D_{1,p}(G)$, $p > 2$, continuous in the Hölder sense in the closed circle, such that the formula*

$$w(z) = \Phi\big(W_0(z)\big)\, e^{\varphi(z)}$$

yields a solution of the equation (17.21). *The function $\zeta = W_0(z)$ establishes a homeomorphic mapping of the circle G onto itself; this mapping may be normalized by specifying three points of the boundary of G and their images, or one interior point and one boundary point and their images. It is required that φ be analytically continuable to the entire plane as a continuous, holomorphic outside G and vanishing at infinity function of the variable z. If $A = B \equiv 0$ the function φ is chosen as the identical zero.*

All statements of Theorem 3.34 remain valid if instead of the restrictions imposed upon φ it is required that $Re\varphi = 0$, $\varphi(1) = 0$ on the boundary of the unit circle G. In Theorem 3.34 the circle may be replaced by an arbitrary simply-connected domain; in particular the domain may coincide with the entire plane of the variable z.

The proof of Theorem 3.34 is given in the paper of Bojarski [11d]. It is based on the Schauder principle of the fixed point.

Theorem 3.34 makes possible the construction of solutions of the equation (17.21) in simply-connected do-

mains of the plane z, having many prescribed properties. For instance, there exists a solution of the equation (17.21) in an arbitrary simply-connected domain having (in the qualitative sense) a prescribed distribution of zeros, poles and essentially singular points. Any simply-connected domain turns out to be a domain of existence of a regular generalized solution of the equation (17.21) which cannot be continued to any wider domain. Theorem 3.34 can be used for the construction of solutions of the equation (17.21) of the type of powers $z^n (n = \pm 1, \pm 2, \ldots)$; in particular in this way we can obtain the global "formal powers" introduced by Bers, [5a, b], [6]. Theorem 3.34 can also be employed for the construction of Green's function, fundamental solution of the equation (17.21) and of an equation of the second order, on the plane.

In the particular case when $A = B \equiv 0$ Theorem 3.34 contains the basic theorems of existence of quasi-conformal mappings with two pairs of characteristics, under very general assumptions in respect of the characteristics of the mapping ([11c]).

If we abandon the assumption of simple-connectedness of the domain of definition of the analytic function $\Phi(\zeta)$, then in order that Theorem 3.34 be valid its statement must be changed. In the case $q_2 \equiv 0$ it is easy to obtain the appropriate formulation. In the general case this problem has not so far been explored in detail.

Remarks. Constructions aimed at various generalizations of analytic functions of one complex argument are encountered in the works of many authors (see for example [67]). Most significant constructions are naturally connected with some of the main problems of analysis, geometry and mechanics. Important results in this topic were first obtained by Lavrentyev, [45a, b, c], in his investigations on quasi-conformal mappings which are also connected with problems of the gas dynamics. In these papers many geometric and analytic properties of solutions of the Cauchy-Riemann equations were generalized to a very large class of linear and nonlinear equations of elliptic type. The papers [46a], [93a, b], [94], [19] also belong to this group of problems.

Papers by Polozhyi [70a, b, c] are devoted to the investigation of properties of solutions of some classes of linear systems of the form (17.1).

Recently Bojarski generalized many results treated in this chapter to elliptic systems of equations with $2n\,(n>1)$ unknown functions [11i, j].

We also mention generalizations to the case of eliptic equations of the second and higher orders. The papers of Bergman, Vekua, Bitsadse, Lopatinski, Khalilov and others are devoted to these problems (bibliographic references may be found in the papers [4*], [14b], [49], [90*]).

CHAPTER IV

BOUNDARY VALUE PROBLEMS

IN THIS chapter we shall investigate some boundary value problems for an elliptic system of equations of the first order and for an elliptic equation of the second order, in a two-dimensional domain. In view of their nature the problems under consideration cannot be regarded as classical problems; we have in mind the fact that the celebrated Fredholm alternatives are in general not valid in this case. A very typical example of such a problem is the so-called generalized Riemann–Hilbert problem which will be investigated extensively in the present chapter. It is of a particular interest also because it has a great number of applications in various problems of analysis, geometry and mechanics. Comparatively weak restrictions are imposed on the coefficients and other data of the problems under investigation; therefore the solutions are to be considered in a generalized sense. We shall also examine the differential properties of the solutions in terms of the differential properties of the data. It should be observed that the problems investigated in this chapter constitute particular cases of general boundary value problems considered in the paper [14b], for elliptic equations with analytic coefficients.

§1. Formulation of the generalized Riemann-Hilbert problem. Continuity properties of the solution of the problem

1.1. To begin with let us consider the following boundary value problem.

PROBLEM A. *It is required to find in the domain G the solution $w(z) = u + iv$ of the equation*

$$\mathfrak{C}(w) \equiv \partial_{\bar{z}} w + A(z) w + B(z) \overline{w} = F(z) \qquad (\text{in } G), \tag{1.1}$$

satisfying the boundary condition

$$\alpha u + \beta v \equiv \mathrm{Re}[\overline{\lambda(z)}w] = \gamma(z) \quad (\text{or } \Gamma), \quad \lambda = \alpha + i\beta. \tag{1.2}$$

If $A \equiv B \equiv F \equiv 0$ we have the familiar Riemann–Hilbert problem for analytic functions.* Therefore the above problem (1.1)–(1.2) will be called *the generalized Riemann–Hilbert problem* or, briefly, *Problem* **A**. If $F \equiv 0$, $\gamma \equiv 0$ we have the homogeneous Problem **Å**.

In what follows A and B will as usually be called the coefficients of the equation (1.1), α, β and $\lambda = \alpha + i\beta$—the coefficients of the boundary condition (1.2), and F and γ—the free terms or the right-hand sides of the equation (1.1) and the boundary condition (1.2) or, briefly, of Problem **A**.

We shall see later, in Chapter V and Chapter VI, that the boundary condition of the form (1.2) appears in numerous problems of the theory of infinitesimal bendings of surfaces of a positive curvature, and also in statical problems of the membrane theory of shells.

1.2. In respect of the data of the Problem **A** we shall make the following assumptions, the set of which will be hereafter called the conditions **I**.

Conditions **I**.

(1) A, B and $F \in L_{p,2}(E)$, $p > 2$.

(2) $F \in C^1_{\mu,\nu_1,\dots,\nu_k}$, $0 < \mu \leqslant 1$, $0 < \nu_j \leqslant 2$.

(3) $\lambda \equiv \alpha + i\beta$ and $\gamma \in C_\nu(\Gamma)$, $0 < \nu \leqslant 1$ where $|\lambda(z)| = 1$.

We shall seek the solution of the Problem **A** in the class of functions continuous in the closed domain $G + \Gamma$.† Theorem 3.1 implies that inside the domain G the solution to be determined (if it exists) belongs to the class $C_{\frac{p-2}{p}}(G)$. In the closed domain the solution in general does not belong to this class. It is evident that the continuity

* We make use of the terminology of N. I. Muskhelishvili's monograph [60a].

† Sometimes we shall also consider solutions in the class of discontinuous functions. These cases will always be distinguished by a special remark.

properties in the closed domain depend also on the smoothness properties of the boundary of the domain and on the functions α, β, γ entering the boundary condition. Below we shall prove that if the conditions I are satisfied the solution of Problem **A** may be sought in the class of of functions continuous in the Hölder sense in the closed domain. This result follows from the theorem, the proof of which will now be given.

THEOREM 4.1. *If the conditions I are satisfied and the Problem* **A** *has a solution $w(z)$ continuous in the closed domain $G+\Gamma$, then $w(z) \in C_\tau(G+\Gamma)$ where*

$$\tau = \sigma\nu'\nu'', \qquad 0 < \tau \leqslant \frac{p-2}{p}, \tag{1.3}$$

and

$$\sigma = \min\left(\nu, \frac{p-2}{p}\right), \qquad \nu' = \min\left(1, \frac{1}{\nu_1}, \dots, \frac{1}{\nu_k}\right),$$
$$\nu'' = \min(1, \nu_1, \dots, \nu_k). \tag{1.4}$$

PROOF. The solution of Problem **A** is representable in the form $w = w_0 + \tilde{w}$ where w_0 is to solution of the homogeneous equation $\mathfrak{C}(w) = 0$ and consequently it is given by the relation (Ch. III, §4)

$$w_0(z) = \Phi(z)e^{\omega(z)}, \qquad \omega(z) = \frac{1}{\pi}\iint_G \left(A + B\frac{\overline{w}_0}{w_0}\right)\frac{d\xi\,d\eta}{\zeta - z},$$

$\tilde{w}$ is a particular solution of the non-homogeneous equation $\mathfrak{C}(w) = F$, which may be taken in the form (Ch. III, §13.4)

$$\tilde{w}(z) = -\frac{1}{\pi}\iint_G \Omega_1(z,\zeta)F(\zeta)\,d\xi\,d\eta -$$
$$-\frac{1}{\pi}\iint_G \Omega_2(z,\zeta)\overline{F(\zeta)}\,d\xi\,d\eta. \tag{1.5}$$

The functions ω and $\tilde{w} \in C_{\frac{p-2}{p}}(G+\Gamma)$ and the function Φ is holomorphic in G, continuous in $G+\Gamma$ and satisfies the boundary condition

$$\mathrm{Re}[\overline{\lambda_0(z)}\Phi(z)] = \gamma_0(z) \qquad (\text{on } \Gamma), \tag{1.6}$$

where

$$\overline{\lambda_0(z)} = \overline{\lambda(z)}e^{\omega(z)}, \qquad \gamma_0(z) = \gamma(z) - \mathrm{Re}[\overline{\lambda(z)}\widetilde{w}].$$

Obviously, λ_0 and $\gamma_0 \in C_\sigma(\Gamma)$, $\sigma = \min\left(\nu, \frac{p-2}{p}\right)$. The theorem will be proved if we now prove that the solution $\Phi(z)$ of the Riemann–Hilbert problem (1.6) belongs to the class $C_\tau(G+\Gamma)$.

We consider first the case of the circle $|z| < 1$. Then $\lambda_0(z)$ can be represented on Γ in the form

$$\overline{\lambda_0(z)} = |\lambda_0(z)| z^{-n} e^{\chi(z)} e^{-p(z)}, \tag{1.7}$$

where $\chi(z) = p + iq$ is a function holomorphic in the circle $|z| < 1$, the imaginary part of which on the circumference $|z| = 1$ is given by $q = -\arg\lambda_0(z) + n\arg z$; the integer n is so chosen that every branch of $q(z)$ is a single-valued function on the circumference (and, consequently inside it as well). The function χ may be constructed by means of the Schwarz integral

$$\chi(z) = \frac{1}{2\pi}\int_\Gamma q(t)\frac{t+z}{t-z}\frac{dt}{t}. \tag{1.8}$$

Since $q \in C_\sigma(\Gamma)$, $\chi(z) \in C_\sigma(G+\Gamma)$ (Theorem 1.10). Moreover, it follows from (1.8) that (see Ch. I, §3)

$$C_\sigma(\chi, G+\Gamma) \leqslant M_\sigma C_\sigma(q, \Gamma), \qquad M_\sigma = \text{const}.$$

Introducing the expression (1.7) into the boundary condition (1.6) we obtain

$$\mathrm{Re}[z^{-n}e^{\chi(z)}\Phi(z)] = \gamma_1(z), \qquad \gamma_1 = \frac{\gamma_0 e^{p(z)}}{|\lambda_0(z)|}. \tag{1.9}$$

Evidently, $\gamma_1 \in C_\sigma(\Gamma)$. If $n < 0$ (1.9) implies that

$$\Phi(z) = \frac{z^n e^{-\chi(z)}}{2\pi i}\int_\Gamma \gamma_1(t)\frac{t+z}{t-z}\frac{dt}{t} + ic_0 z^n e^{-\chi(z)},$$

where c_0 is a real constant. Hence, in view of the continuity of $\Phi(z)$ we have

$$c_0 = 0\,, \qquad \int_0^{2\pi} \gamma_1(e^{i\vartheta})e^{-ki\vartheta}d\vartheta = 0 \qquad (k = 0, \ldots, -n+1)\,.$$

These relations ensure the continuity of $\Phi(z)$ at the point $z = 0$. Therefore $\Phi(z)$ has the form

$$\Phi(z) = \frac{e^{-\chi(z)}}{\pi i}\int_\Gamma \frac{\gamma_1(t)t^n\,dt}{t-z}\,.$$

It follows immediately from the above result that $\Phi(z) \in C_\sigma(G+\Gamma)$. Moreover,

$$C_\sigma(\Phi, G+\Gamma) \leqslant M'_\sigma C_\sigma(\gamma, \Gamma)\,.$$

If $n \geqslant 0$ the solution of the problem (1.9) is given by the formula

$$\Phi(z) = \frac{z^n e^{-\chi(z)}}{2\pi i}\int_\Gamma \gamma_1(t)\frac{t+z}{t-z}\frac{dt}{t} + e^{-\chi(z)}\sum_{k=0}^{2n} c_k z^k\,, \qquad (1.10)$$

where c_k are complex constants which satisfy the conditions

$$c_{2n-k} = -\bar{c}_k \qquad (k = 0, 1, \ldots, n)\,.$$

It follows from (1.10) that $\Phi(z) \in C_\sigma(G+\Gamma)$. Thus, in the case of the circle $|z| < 1$ our theorem has been proved.

If G is a simply-connected domain, by means of the holomorphic function $z = \varphi(\zeta)$ we can map G conformally onto the circle $|\zeta| < 1$; then the boundary condition (1.6) takes the form

$$\mathrm{Re}[\overline{\lambda_1(\zeta)}\Phi_1(\zeta)] = \gamma_1(\zeta) \qquad (|\zeta| = 1)\,, \qquad (1.11)$$

where $\lambda_1(\zeta) = \lambda_0[\varphi(\zeta)]$, $\Phi_1(\zeta) = \Phi[\varphi(\zeta)]$, $\gamma_1(\zeta) = \gamma_0[\varphi(\zeta)]$. Since according to the assumption $\Gamma \in C^1_{\mu,\nu_1,\ldots\nu_k}$, $0 < \mu \leqslant 1$, $0 < \nu_j \leqslant 2$, we have $\varphi \in C_{\nu''}$ in the circle $|\zeta| \leqslant 1$ where $\nu'' = \min(1, \nu_1, \ldots, \nu_k)$. It is therefore evident that $\lambda_1(\zeta)$ and $\gamma_1(\zeta) \in C_{\sigma\nu''}(\Gamma')$ where Γ' is the circumference

$|\zeta| = 1$. In view of the proposition proved above the holomorphic function $\Phi_1(\zeta)$ satisfying the boundary condition (1.11) and continuous in $|\zeta| \leqslant 1$ belongs to the class $C_{\sigma\nu''}$ in the circle $|\zeta| \leqslant 1$. Since the function $\zeta = \psi(z)$ inverse to $\varphi(\zeta)$ belongs to the class $C_{\nu'}(G+\Gamma)$ where $\nu' = \min\left(1, \frac{1}{\nu_1}, \ldots, \frac{1}{\nu_k}\right)$ (Theorem 1.9), $\Phi(z) = \Phi_1[\psi(z)]$ belongs to the class $C_{\sigma\nu_1\nu''}(G+\Gamma)$. Hence, $w(z) \in C_\tau(G+\Gamma)$ where $\tau = \sigma\nu'\nu''$. This completes the proof of the theorem in the case of a simply-connected domain.

Let us now consider the case of a multiply-connected domain. Let $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$ be curves of the class $C^1_{\mu,\nu_1,\ldots,\nu_k}$ bounding the domain G, Γ_0 being the exterior contour containing inside the curves $\Gamma_1, \ldots, \Gamma_m$. The function $\Phi(z)$ continuous in $G+\Gamma$ and holomorphic in G may be represented by the Cauchy formula, namely

$$\Phi(z) = \frac{1}{2\pi i}\int\limits_{\Gamma}\frac{\Phi(\zeta)\,d\zeta}{\zeta - z} = \Phi_0(z) + \ldots + \Phi_m(z)\,, \qquad (1.12)$$

where

$$\Phi_j(z) = \frac{1}{2\pi i}\int\limits_{\Gamma_j}\frac{\Phi(\zeta)\,d\zeta}{\zeta - z} \qquad (j = 0, 1, \ldots, m)\,,$$

$\Phi_0(z)$ is holomorphic in the simply-connected domain G_0 bounded by the curve Γ_0; $\Phi_j(\zeta), j \geqslant 1$, is holomorphic in the infinite simply-connected domain G_j bounded by the curve Γ_j, and $\Phi_j(\infty) = 0$ According to (1.12) the boundary condition (1.6) can be written in the form

$$\mathrm{Re}[\overline{\lambda_0(z)}\,\Phi_j(z)] = \lambda_j(z) \quad (\text{on } \Gamma_j)\,, \qquad (1.13)$$

where

$$\lambda_j(z) = \gamma_0(z) - \sum_{\substack{k=0\\k\neq j}}^{m} \mathrm{Re}[\overline{\lambda_0(z)}\,\Phi_k(z)]\,.$$

Since $\Phi_0(z), \ldots, \Phi_{j-1}(z), \Phi_{j+1}(z), \ldots, \Phi_m(z)$ are holomorphic along Γ_j, $\gamma_j \in C_\sigma(\Gamma_j)$. Therefore, in view of the result proved above, the solution Φ_j of the boundary value

problem (1.13) belongs to the class $C_{\sigma\nu'\nu''}(G_j+\Gamma_j)$. Consequently, $\Phi = \Phi_0+\Phi_1+\ldots+\Phi_m \in C_{\sigma\nu'\nu''}(G+\Gamma)$, i.e. the solution $w(z)$ of Problem A belongs to the class $C_\tau(G+\Gamma)$ where $\tau = \sigma\nu'\nu''$. This completes the proof of our theorem.

1.3. In the following considerations we shall assume that the conditions I are satisfied. Therefore, according to Theorem 4.1, we shall seek the solutions of Problem A in the class of functions $C_\alpha(\bar{G})$, $0<\alpha<1$. Besides, we shall now prove that Problem A can be reduced to the case of a canonical domain bounded by the circles $\Gamma_0, \Gamma_j, \ldots, \Gamma_m$ where Γ_0 is the unit circle whose centre lies at the point $z = 0$, belonging to the domain G, and $\Gamma_1, \ldots, \Gamma_m$ lie inside Γ_0. This reduction can be performed by means of the conformal mapping $z = \varphi(\zeta)$ (Ch. I, §2) which results in the following transformation of the equation (1.1) and the boundary condition (1.2):

$$\partial_{\bar{\zeta}}w_1+A_1(\zeta)w_1+B_1(\zeta)\overline{w}_1 = F_1(\zeta)\,,$$
$$\operatorname{Re}[\lambda_1(\zeta)w_1(\zeta)] = \gamma_1(\zeta) \qquad (\text{on } \Gamma)\,,$$

where

$$A_1(\zeta) = \overline{\varphi'(\zeta)}A(\varphi)\,, \qquad B_1(\zeta) = \overline{\varphi'(\zeta)}B(\varphi)\,,$$
$$F_1(\zeta) = \overline{\varphi'(\zeta)}F(\varphi)\,, \qquad \lambda_1(\zeta) = \lambda[\varphi(\zeta)]\,, \qquad \gamma_1(\zeta) = \gamma[\varphi(\zeta)]\,.$$

Since $\varphi(\zeta)$ is Hölder continuous in the closed domain (Theorem 1.8 and Theorem 1.9) $\lambda_1(\zeta)$ and $\gamma_1(\zeta)$ are also Hölder continuous on the boundary of the new (canonical) domain. It was proved in Ch. III, §1 that the functions A_1, B_1, F_1 belong to a $L_p(\bar{G})$, $p \geqslant p_1 > 2$.

Thus, we have obtained a new problem which is completely equivalent to the original problem but has the merit of having the boundary consisting of circles, the remaining conditions I being valid in the previous form.

We shall later observe that this circumstance considerably simplifies the investigation of Problem A. Therefore, hereafter—whenever it turns out to be expedient—we shall take for G the canonical domain of the form indicated above, without stating this fact every time.

The investigation of Problem **A** may always be reduced to the case in which $F \equiv 0$, i.e. we may confine ourselves to the determination of the solution of the homogeneous equation

$$\mathfrak{C}(w) \equiv \partial_{\bar{z}} w + Aw + B\bar{w} = 0 , \tag{1.14}$$

satisfying the boundary condition of the form (1.2). To this end it is sufficient to represent the solution of the problem in form of the sum

$$w = w_0 + \tilde{w} , \tag{1.15}$$

where $\tilde{w}$ is a particular solution of the equation (1.1). Such a particular solution can be obtained for instance by means of the formula (1.5). Then we have for w_0 the boundary value problem for the homogeneous equation (1.14) with the boundary condition

$$\operatorname{Re}[\overline{\lambda(z)}\, w_0(z)] = \gamma(z) - \operatorname{Re}[\overline{\lambda(z)}\, \tilde{w}(z)] \quad (\text{on } \Gamma) .$$

The investigation of Problem **A** may also be reduced to the problem of the determination of the solution of the non-homogeneous equation (1.1), satisfying the homogeneous boundary condition ($\gamma \equiv 0$). This reduction can be obtained again by means of the substitution of the form (1.15) if for $\tilde{w}$ we take a continuously differentiable function satisfying the boundary condition (1.2).

§2. The adjoint boundary value Problem Å′. Necessary and sufficient conditions of solubility of Problem A

2.1. It is important for future considerations to introduce the so-called adjoint boundary value problem A′. For this purpose let us examine the formula (9.3) of Ch. I. If in this formula we take for w' a solution of the homogeneous equation adjoint to (1.14)

$$\mathfrak{C}'(w') \equiv \partial_{\bar{z}} w' - Aw' - \bar{B}\bar{w}' = 0 , \tag{2.1}$$

we obtain

$$\operatorname{Re}\left[\frac{1}{2i}\int\limits_{\Gamma} ww'\,dz - \iint\limits_{G} w'\mathfrak{C}(w)\,dx\,dy\right] = 0 . \tag{2.2}$$

Let us moreover assume that w' satisfies the boundary condition

$$\operatorname{Re}[\lambda(z)z'(s)w'(z)] = 0\ , \qquad z'(s) = \frac{dz(s)}{ds} \qquad (\text{on } \Gamma)\ , \tag{2.3}$$

i.e.

$$w'(z) = i\overline{\lambda(z)}\,\overline{z'(s)}\,\chi(z) \qquad (\text{on } \Gamma)\ , \tag{2.4}$$

where $\chi(z)$ is a real function of the point z of the boundary Γ. In view of Theorem 4.1 this function satisfies the Hölder condition.

Substituting (2.4) in (2.2) we have

$$\operatorname{Re}[\tfrac{1}{2}\int_\Gamma \overline{\lambda(t)}\chi(t)\overline{t'(s)}w(t)\,dt] \equiv \tfrac{1}{2}\int_\Gamma \chi(t)\operatorname{Re}[\overline{\lambda(t)}w(t)]\,ds$$
$$= \operatorname{Re}\iint_G w'\mathfrak{C}(w)\,dx\,dy\ .$$

The last relation implies that if w is the solution of Problem **A**, i.e. it satisfies the equation (1.1) and the boundary condition (1.2), then the following relation holds:

$$\tfrac{1}{2}\int_\Gamma \chi(t)\gamma(t)\,ds - \operatorname{Re}\iint_G w'(z)F(z)\,dx\,dy = 0$$

or, according to (2.4)

$$\frac{1}{2i}\int_\Gamma \lambda(t)w'(t)\gamma(t)\,dt - \operatorname{Re}\iint_G w'F(z)\,dx\,dy = 0\ . \tag{2.5}$$

Thus, the fulfilment of the last relation is a necessary condition of the solubility of Problem **A**. Here w' is an arbitrary solution of the equation (2.1) satisfying the boundary condition (2.3).

Below we shall prove the sufficiency of this condition. Therefore the problem of the determination of the solution w' of the equation (2.1) satisfying the boundary condition (2.3) will hereafter be called *the homogeneous boundary value problem adjoint to Problem* **A**, or briefly *Problem* **Å′**.

In order to simplify the treatment we shall assume in this section that $F \equiv 0$. We have already indicated that this fact does not affect the generality of the considerations. Under this assumption the relation (2.5) takes the form

$$\int_\Gamma \lambda(t) w'(t) \gamma(t) dt = 0 . \tag{2.6}$$

Besides, we shall assume in this paragraph that $\Gamma \in C^1_\mu$, $0 < \mu \leqslant 1$. This assumption also does not influence the generality of the results (see §1.3).

2.2. According to the formula (10.8) of Ch. III, the generalized Cauchy formula for the adjoint equation (2.1) has the form

$$w'(z) = -\frac{1}{2\pi i}\int_\Gamma \Omega_1(\zeta, z, G) w'(\zeta) d\zeta - \Omega_2(\zeta, z, G) \overline{w'(\zeta)}\, \overline{d\zeta} ,$$

If w' is a solution of Problem $\mathring{\mathrm{A}}'$ we have in view of (2.4)

$$w'(z) = -\frac{1}{2\pi}\int_\Gamma \left(\Omega_1(t, z, G)\bar{\lambda} + \overline{\Omega_2(t, z, G)}\lambda\right)\chi ds . \tag{2.7}$$

Passing to the limit in the last relation (the point $z \in G$ tends to a boundary point ζ) and making use of the formula (14.2) of Ch. III, we obtain

$$w'(\zeta) = \frac{i}{2}\overline{\lambda(\zeta)}\,\overline{\zeta'(\sigma)}\chi(\zeta) + \int_\Gamma K_0(t, \zeta)\chi(t) ds , \quad \zeta \in \Gamma , \tag{2.8}$$

where σ is the length of the arc corresponding to the point ζ of the contour Γ, and

$$K_0(t, \zeta) = -\frac{1}{2\pi}[\Omega_1(t, \zeta, G)\overline{\lambda(t)} + \overline{\Omega_2(t, \zeta, G)}\lambda(t)] . \tag{2.9}$$

Since χ is a real function, inserting the expression (2.8) into the boundary condition (2.3) we have

$$\begin{gathered}\int_\Gamma K_1(t, \zeta)\chi(t) ds = 0 , \\ K_1(t, \zeta) = \mathrm{Re}[\lambda(\zeta)\zeta'(\sigma) K_0(t, \zeta)] .\end{gathered} \tag{2.10}$$

Thus, the solution of the homogeneous Problem Å′ is representable in the form (2.7) where $\chi(t)$ is a real function of the point t of the boundary Γ, satisfying the homogeneous integral equation (2.10).

2.3. The equation (2.10) contains an integral in the sense of the Cauchy principal value. Therefore it belongs to the class of so-called singular integral equations. At the present time the theory of these equations is well developed. Instead of the Fredholm theorems the so-called Noether theorems generalizing the former ones, are valid for this class of equations.

We shall now quote without proof some results of the theory of singular integral equations ([60a], Ch. II).

Equation (2.10) is a particular case of a more general equation of the form

$$K\varphi \equiv a(\zeta)\varphi(\zeta) + \frac{1}{\pi i}\int_{\Gamma} \frac{K(\zeta, t)\varphi(t)\,dt}{t-\zeta} = f(\zeta)\,, \quad \zeta \in \Gamma\,, \tag{2.11}$$

where $a(\zeta)$, $K(\zeta, t)$, $f(\zeta) \in C_\nu(\Gamma)$, $0 < \nu \leqslant 1$, and the functions $a(\zeta)+K(\zeta,\zeta)$ and $a(\zeta)-K(\zeta,\zeta)$ vanish nowhere on Γ. In the theory of singular integral equations an important role is played by the concept of the index.

By *the index* of the equation (2.11) we understand an integer $\varkappa$ equal to the increment of the function

$$\frac{1}{2\pi}\arg\frac{a(\zeta)-K(\zeta,\zeta)}{a(\zeta)+K(\zeta,\zeta)}\,,$$

as the point ζ describes once the boundary Γ of the domain G in the direction leaving the domain on the left, i.e. we have

$$\varkappa = \frac{1}{2\pi}\Delta_\Gamma \arg\frac{a(\zeta)-K(\zeta,\zeta)}{a(\zeta)+K(\zeta,\zeta)}\,. \tag{2.12}$$

For the equation (2.10) $\varkappa = 0$, since in this case $a(\zeta) \equiv 0$ $K(\zeta,\zeta) \neq 0$.

We now consider the adjoint homogeneous equation

$$K'\psi \equiv a(\zeta)\psi(\zeta) - \frac{1}{\pi i}\int_{\Gamma} \frac{K(t,\zeta)\psi(t)\,dt}{t-\zeta} = 0\,. \tag{2.13}$$

Let k and k' be the numbers of the linearly independent solutions of the homogeneous equations $K\varphi = 0$ and $K'\psi = 0$, respectively.

These numbers are finite and satisfy the relation

$$k - k' = \varkappa\,. \tag{2.14}$$

The equation (2.11) *is soluble if and only if the following relations are satisfied*

$$\int_\Gamma f(\zeta)\psi_j(\zeta)\,d\zeta = 0 \qquad (j = 1, \ldots, k')\,, \tag{2.15}$$

where $\psi_1, \ldots, \psi_{k'}$ *is the complete system of linearly independent solutions of the adjoint homogeneous equation* (2.13).

If $\Gamma \in C^1_\mu$, $0 < \mu \leqslant 1$, $a(\zeta)$, $K(\zeta, t) \in C_\nu(\Gamma)$, $0 < \nu \leqslant 1$, *a number* σ, $0 < \sigma < 1$, *can be found such that all solutions of the homogeneous equations* $K\varphi = 0$ *and* $K'\psi = 0$ *belong to the class* $C_\sigma(\Gamma)$; σ *depends only on* μ *and* ν.

If $f \in C_\tau(\Gamma)$, $0 < \tau \leqslant 1$, *the solutions of the equation* $K\varphi = f\,(K'\psi = f)$ *if they exist, belong to a class* $C_\varrho(\Gamma)$, $0 < \varrho < 1$, *where* ϱ *depends on* μ, ν, τ *and is independent of the form of the functions* $a(\zeta)$, $K(\zeta, t)$ *and* $f(\zeta)$. *Moreover, if the equation* $K\varphi = f$ *has solutions, a solution can be indicated which has the form* $\varphi = Hf$ *where* H *is a linear operator acting from* $C_\tau(\Gamma)$ *into* $C_\varrho(\Gamma)$.

In particular, the equation (2.11) *is soluble for an arbitrary right-hand side if and only if the adjoint homogenous equation* (2.13) *has no non-trivial solutions. In this case the equation* (2.11) *always has a solution which satisfies the condition of the form*

$$C_{\nu'}(\varphi, \Gamma) \leqslant M_0 C_\nu(f, \Gamma)\,,\;\; 0 < \nu' \leqslant \nu < 1\,,\;\; M_0 = \text{const.} \tag{2.16}$$

To prove the last assertion it is sufficient to observe that $K\varphi$ is a linear operator in a space $C_{\nu'}(\Gamma)$ ([60a], Ch. II, §49) and consequently the inverse operator K^{-1} is also linear, if it exists.

Let us observe that $K\varphi$ is a linear operator also in $L_p(\Gamma)$, $p > 1$. Regarding (2.11) as a linear equation in L_p we can prove the validity of the preceding results also for $f \in L_p(\Gamma)$, $p > 1$, preserving the remaining assumptions in the former form (see the paper of Khvedelidse, [91]). The inequality (2.16) has in this case the form

$$L_p(\varphi, \Gamma) \leqslant M_p L_p(f, \Gamma), \quad p > 1 . \tag{2.17}$$

2.4. Let $\chi_1, \ldots, \chi_k$ be the complete system of linearly independent solutions of the equation (2.10). Substituting these functions into the right-hand side of the equation (2.7) we obtain solutions of Problem Å′. However, some of these solutions may turn out to be trivial solutions. We have seen above (Ch. III, §14) that this fact takes place when the function $\overline{\lambda(t)}\chi(t)\overline{t'(s)}$ takes on every boundary contour $\Gamma_j (j = 0, 1, \ldots, m)$ the values of a function $\Phi_j(z)$ holomorphic in G_j, and $\Phi_0(\infty) = 0$. Let $\chi_1, \ldots, \chi_{l'}$ be the solutions of the equation (2.10) to which the linearly independent solutions $w_1', \ldots, w_{l'}$ of Problem Å′ correspond. In this case the remaining solutions of the equation (2.10) satisfy the boundary condition of the form

$$\chi(t) = i\lambda(t)t'(s)\Phi^-(t) \quad (\text{on } \Gamma) , \tag{2.18}$$

where $\Phi(z)$ is a function holomorphic outside $\bar{G}$, and $\Phi_0(\infty) = 0$. It follows from (2.18) that Φ^- satisfies the boundary condition

$$\mathrm{Re}[\lambda(t)t'(s)\Phi^-(t)] = 0 \quad (\text{on } \Gamma) . \tag{2.19}$$

This problem will be termed *the concomitant problem* to Problem Å′, or briefly *Problem* Å$'_*$. Let l'_* be the number of linearly independent solutions of this problem. Obviously,

$$l' + l'_* = k , \tag{2.20}$$

where k is the number of solutions of the homogeneous equation (2.10).

2.5. Let us now return to the investigation of Problem **A**. If w be its solution the boundary values have the form

$$w(t) = \lambda(t)\gamma(t) + i\lambda(t)\mu(t)\,, \qquad t \in \Gamma\,, \tag{2.21}$$

where $\mu(t)$ is a real function of the point t of the contour Γ, for the time being unknown. In view of Theorem 4.1 it is continuous in the Hölder sense. Making use of the generalized Cauchy formula (10.6), Ch. III, we have

$$w(z) = w_1(z) + w_2(z)\,, \tag{2.22}$$

where

$$w_1(z) = \frac{1}{2\pi i}\int_\Gamma \Omega_1(z, t, G)\,\lambda(t)\gamma(t)\,dt - \\ - \frac{1}{2\pi i}\int_\Gamma \Omega_2(z, t, G)\,\overline{\lambda(t)}\gamma(t)\,\overline{dt}\,, \tag{2.23}$$

$$w_2(z) = \frac{1}{2\pi}\int_\Gamma \Omega_1(z, t, G)\,\lambda(t)\mu(t)\,dt + \\ + \frac{1}{2\pi}\int_\Gamma \Omega_2(z, t, G)\,\overline{\lambda(t)}\mu(t)\,\overline{dt}\,. \tag{2.24}$$

If we pass to the limit in the relation (2.22) (the point z tends to a boundary point ζ, from the domain G) we obtain

$$w^+(\zeta) = \tfrac{1}{2}\lambda(\zeta)[\gamma(\zeta) + i\mu(\zeta)] + w_1(\zeta) + w_2(\zeta)\,.$$

Inserting the last result into the boundary condition (1.2) and taking into account that γ and μ are real functions we obtain

$$\int_\Gamma K_1(\zeta, t)\mu(t)\,ds = \gamma_0(\zeta)\,, \tag{2.25}$$

where

$$\gamma_0(\zeta) = \gamma(\zeta) - \operatorname{Re}[\overline{\lambda(\zeta)}\,w_1^+(\zeta)] \equiv -\operatorname{Re}[\overline{\lambda(\zeta)}\,w_1^-(\zeta)]\,. \tag{2.26}$$

Thus, all solutions of Problem **A** are represented by the formula (2.22) where μ satisfies the equation (2.25).

In particular, the solution of the homogeneous Problem Å is given by the formula (2.24) where μ is a solution of the homogeneous integral equation

$$\int_\Gamma K_1(\zeta, t)\mu(t)ds = 0 , \tag{2.27}$$

which is adjoint to the equation (2.10). Since the indices of the equations (2.10) and (2.7) are equal to zero, in view of the formula (2.14) the numbers of the linearly independent solutions of these equations are

$$k = k' . \tag{2.28}$$

2.6. For Problem Å_* concomitant to Problem Å we have the boundary condition

$$\mathrm{Re}[\overline{\lambda(t)}\Phi^-(t)] = 0 , \tag{2.29}$$

where $\Phi^-(t)$ is the boundary value of a function holomorphic outside $G+\Gamma$, and $\Phi(\infty) = 0$. Denoting the numbers of the linearly independent solutions of the homogeneous problems Å and Å_* by l and l_*, respectively, and taking into account the relations (2.20) and (2.28) we have

$$l+l_* = k . \tag{2.30}$$

2.7. Let us now return to the non-homogeneous equation (2.25). According to (2.15) the necessary and sufficient condition of the solubility of (2.25) is given by the relations

$$\int_\Gamma \gamma_0(t)\chi_j(t)ds = 0 \quad (j = 1, \ldots, k) , \tag{2.30a}$$

where $\chi_1, \ldots, \chi_k$ is the complete system of solutions of the adjoint homogeneous singular integral equation which in this case is identical with the equation (2.10). We have already seen that these solutions constitute two groups, namely $\chi_1, \ldots, \chi_{l'}$ and $\chi_{l'+1}, \ldots, \chi_k$; according to (2.4) and (2.18) we have

$$\chi_j(t) = i\lambda(t)t'(s)w_j'(t) \qquad (j = 1, \ldots, l', t \in \Gamma) , \tag{2.31}$$

$$\chi_j(t) = i\lambda(t)t'(s)\Phi_j(t) \qquad (j = l'+1, \ldots, k, t \in \Gamma) , \tag{2.32}$$

where w_j' and Φ_j are the solutions of Problems $\mathring{\mathbf{A}}'$ and $\mathring{\mathbf{A}}_*'$, respectively.

Now, in accordance with (2.26) and (2.31) for $j \leqslant l'$

$$\int_\Gamma \gamma_0(t)\chi_j(t)\,ds = i\int_\Gamma \gamma(t)\lambda(t)w_j'(t)\,dt - \operatorname{Re}\left[i\int_\Gamma w_1^+(t)\,w_j'(t)\,dt\right].$$

Hence, by virtue of (2.2) and (2.6)

$$\int_\Gamma \gamma_0(t)\chi_j(t)\,ds = 0 \qquad (j = 1, \ldots, l')\,.$$

Further, for $j > l'$, according to (2.32) and (2.26)

$$\int_\Gamma \gamma_0(t)\chi_j(t)\,ds = \operatorname{Re}\left[i\int_\Gamma w_1^-(t)\,\Phi_j^-(t)\,dt\right] = 0\,,$$

since w_1 and Φ_j are holomorphic outside $\bar{G}$, and $w_1(\infty) = \Phi_j(\infty) = 0$.

Thus, all relations (2.30a) are satisfied if the relations (2.6) hold. Consequently, we have proved that the relation (2.6) is sufficient for the solubility of Problem **A**. If we consider Problem **A** for the non-homogeneous equation (1.1), it is evident that the sufficient condition of its solubility is given by the relation (2.5). We have therefore proved *

THEOREM 4.2. *The non-homogeneous boundary value Problem* **A** *is soluble if and only if the condition* (2.5) *is satisfied, w' being an arbitrary solution of the adjoint homogeneous boundary value Problem* $\mathring{\mathbf{A}}'$.

This theorem immediately implies

THEOREM 4.3. *The non-homogeneous boundary value Problem* **A** *is soluble for an arbitrary right-hand side if and only if the adjoint homogeneous Problem* $\mathring{\mathbf{A}}'$ *has no solution.*

* This theorem was proved in the author's paper [14a] by a somewhat different method. The formula (14.12) of Ch. III was there employed; with the help of this formula Problem **A** is reduced to a singular integral equation distinct from (2.25).

2.8. If the non-homogeneous Problem A is soluble, i.e. if the conditions (2.5) are satisfied, the solution is given by the formula

$$w(z) = w_0(z) + \sum_{k=1}^{l} c_k w_k(z) , \qquad (2.33)$$

where $w_1, \ldots, w_l$ is the complete system of linearly independent solutions of the homogeneous Problem Å, $c_1, \ldots, c_l$ are arbitrary real constants and w_0 is a particular solution of the non-homogeneous Problem A. This particular solution may always be represented in the form

$$w_0(z) = \int_\Gamma \gamma(t)\, M_0(z, t)\, ds , \qquad (2.34)$$

where $M_0(z, t)$ is a given function which can be constructed by means of the equation (1.1) and the boundary condition (1.2), but is independent of γ. Let us observe that if $\gamma \in C_\nu(\Gamma)$ then according to Theorem 4.1 $w_0 \in C_\sigma(\bar{G})$ and it can be proved that

$$C_\sigma(w_0, \bar{G}) \leqslant M C_\nu(\gamma, \Gamma), \quad \sigma = \min\left(\nu, \frac{p-2}{p}\right). \qquad (2.35)$$

2.9. The preceding results can be generalized to the case of the general linear elliptic system of partial differential equations of the first order. It was found in §7 of Ch. II, that such a system can always be reduced to the form

$$\begin{gathered} -v_y + a_{11}u_x + a_{12}u_y + a_1 u + b_1 v = f_1 , \qquad (2.36) \\ v_x + a_{21}u_x + a_{22}u_y + a_2 u + b_2 v = f_2 , \\ a_{11} > 0, \ \Delta = a_{11}a_{22} - \tfrac{1}{4}(a_{12} - a_{21})^2 \geqslant \Delta_0 > 0, \ \Delta_0 = \text{const.} \end{gathered}$$

Let us consider in connection with this system the boundary condition

$$\alpha u + \beta v = \gamma \qquad (2.37)$$

It was proved in §7 of Ch. II, that by means of a suitable transformation of the independent variables and by a substitution of the form

$$U = \sqrt{\Delta}u\,, \qquad V = v - \frac{a_{12} - a_{21}}{2}u \tag{2.38}$$

the system of equations (2.36) is reduced to the canonical form and the boundary condition (2.37) takes the form

$$\alpha_* U + \beta V = \gamma_* \qquad (\text{on } \Gamma_*)\,, \tag{2.39}$$

where Γ_* is the image of the contour Γ and

$$\alpha_* = \frac{1}{\sqrt{\Delta}}\left(\alpha + \frac{a_{12} - a_{21}}{2}\beta\right) \tag{2.40}$$

If $\alpha^2 + \beta^2 > 0$ on Γ, then $\alpha_*^2 + \beta_*^2 > 0$ on Γ_*.

In respect of the coefficients and the free terms of the system (2.36) it is sufficient to make the following restrictions:

1) $a_{ik} \in D_{1,p}(\bar{G})$, $\quad p > 2$; $\quad$ 2) $a_i, b_i f_i \in L_p(\bar{G})$, $\quad p > 2$.

Then the coefficients and the free terms of the transformed canonical system belong to the class $L_p(\bar{G}_*)$, $p > 2$. If furthermore, the domain belongs to C^1_μ, $0 < \mu < 1$, $\alpha, \beta, \gamma \in C_\nu(\Gamma)$, then it is readily seen that the theorems proved for Problem **A** remain valid also for the problem (2.36)–(2.37) (see also §3.2 and §9.4).

§3. Index of Problem A. Reduction of the boundary condition of Problem A to the canonical form

3.1. We shall find below that in the investigation of Problem **A** an important item is the introduction of the so-called *index* which we shall now define.

Let us denote by $\Delta_\Gamma f(t)$ the increment of the function $f(t)$ as the point t describes once the curve Γ in the direction leaving the domain G on the left. Let us introduce the integer

$$n = n_0 + n_1 + \ldots + n_m \equiv \frac{1}{2\pi}\Delta_\Gamma \arg \lambda(t)\,, \tag{3.1}$$

where

$$n_j = \frac{1}{2\pi} \Delta_{\Gamma_j} \arg \lambda(t) \qquad (j = 0, \ldots, m) . \tag{3.2}$$

Let us represent the complex function $\lambda(t)$ by the unit vector $\lambda(t)$ with components $\alpha(t)$ and $\beta(t)$ assuming that the origin of the vector is fixed. As the point t describes once the boundary Γ of the domain G in the direction leaving this domain on the left the vector λ returns to its original position having performed about its origin n^+ complete revolutions clockwise and n^- complete revolutions in the opposite direction.

It is readily observed that the number n given by the relation (3.1) is equal to the difference $n^+ - n^-$ which will hereafter be called *the index* of the function $\lambda(t)$ with respect to the boundary Γ of the domain G or else *the index of the boundary value problem* **A.** We observe that the index of Problem **A** is invariant with respect to comformal mappings of the domain and substitutions of the form

$$w = w_0 e^{\omega} ,$$

where ω is a function belonging to the class $D_{1,p}(\bar{G})$, $p > 2$.

We can now by making use of the formula (3.2) calculate the index of the adjoint problem **A′**. We have

$$n' = \frac{1}{2\pi} \Delta_\Gamma \arg \left(\overline{\lambda(t)}\, \overline{t'(s)}\right) = -\frac{1}{2\pi} \Delta_\Gamma \arg \lambda(t) - \frac{1}{2\pi} \Delta_\Gamma \arg t'(s) .$$

Since on describing the outer boundary contour Γ_0, $\arg t'(s)$ acquires an increment 2π and on describing the inner boundary contour Γ_j an increment -2π, we have

$$n' = -n + m - 1 . \tag{3.3}$$

3.2. We also note that the transformation reducing the system of equations of general form (2.36) to the canonical form leaves the index of the boundary condition (2.37) invariant. This result follows form the fact that the

imaginary parts of $\lambda = \alpha + i\beta$ and $\lambda_* = \alpha_* + i\beta_*$ are identical and therefore the absolute value of the difference of their arguments does not exceed π. Hence, in view of the continuity of $\lambda(z)$ and $\lambda_*(z)$, it can easily be found that the increments of $\arg\lambda(z)$ and $\arg\lambda_*(z)$ along Γ and Γ_*, respectively, are identical.

3.3. Let $z_1, \ldots, z_m$ be some fixed points situated outside G but inside the contours $\Gamma_1, \ldots, \Gamma_m$, respectively. Moreover, let us specify inside the domain G entirely arbitrary points $a_1, \ldots, a_k$, where $k = n$ as $n \geqslant 0$ and $k = -n$ as $n < 0$. Let us introduce the notation

$$\Omega_n(z) = \begin{cases} \prod_{i=1}^{n} (z - a_i), & n > 0, \\ 1, & n = 0, \\ \prod_{i=1}^{-n} (z - a_i)^{-1}, & n < 0. \end{cases} \tag{3.4}$$

In particular, if $a_1 = \ldots = a_k = 0$ (we assume that the point $z = 0$ belongs to the domain G), then it is evident that

$$\Omega_n(z) = z^n. \tag{3.5}$$

Considering now the function

$$\lambda_0(z) = \lambda(z)\overline{\Omega_n(z)} \prod_{k=1}^{m} (\bar{z} - \bar{z}_k)^{-n_k} e^{-\overline{\mathrm{T}A}}, \tag{3.6}$$

$$\mathrm{T}A = -\frac{1}{\pi} \iint_G \frac{A(\zeta)}{\zeta - z}\, d\xi\, d\eta,$$

we easily find that

$$|\lambda_0(z)| > 0, \quad z \in \Gamma, \quad \Delta_{\Gamma_j} \arg\lambda_0(z) = 0 \quad (j = 0, \ldots, m).$$

Hence

$$\lambda_0(z) = |\lambda_0(z)| e^{i\sigma(z)}, \quad z \in \Gamma, \tag{3.7}$$

where

$$\sigma(z) = \arg\lambda(z) - \arg\Omega_n(z) + \sum_{k=1}^{m} n_k \arg(z - z_k) + \mathrm{Im}(\mathrm{T}A); \tag{3.8}$$

Since $\lambda \in C_\nu(\Gamma)$ and $\mathrm{T}A \in C_\alpha(\Gamma)$, $\alpha = \dfrac{p-2}{p}$, $\sigma(z)$ is a single-valued real function on every contour Γ_j and belongs to $C_\tau(\Gamma)$, $0 < \tau = \min(\nu, \alpha)$.

It will be shown below (§5.3) that the function $\sigma(t)$ can be represented in the form

$$\sigma(t) = p(t) - i\sigma_*(t) + \pi a(t) ,$$

where $p(z)$ is a function holomorphic in G and continuous in the Hölder sense in $G+\Gamma$, σ_* is the imaginary part of $p(t)$ and $a(t)$—a piecewise constant continuous function on Γ; $a = 0$ on Γ_0 and $a = a_j = \text{const.}$ on $\Gamma_j (j = 1, \dots, m)$. Under these conditions p and a are uniquely expressible by $\sigma(t)$. Such being the case, according to (3.7) and (3.8) the boundary condition (1.2) can be written in the form

$$\operatorname{Re}[\overline{\Omega_n(t)}\, e^{-\pi i a(t)} w_0(t)] = \gamma_0(t) \quad (\text{on } \Gamma) , \tag{3.9}$$

where

$$w_0(z) = w(z)\, e^{-ip(z)+\mathrm{T}A} \prod_{k=1}^{m} (z-z_k)^{n_k}, \tag{3.10}$$

$$\gamma_0(t) = \frac{\gamma(t)}{|\lambda(t)|}\, e^{\sigma_*(t)+\operatorname{Re}\mathrm{T}A} |\Omega_n(t)| \prod_{k=1}^{m} |t-z_k|^{n_k}. \tag{3.11}$$

It is readily seen that the function w_0 satisfies the equation

$$\partial_{\bar z} w_0 + B_0 \overline{w}_0 = 0 , \tag{3.12}$$

where

$$B_0 = B \exp\left\{-2i\left(\operatorname{Re} p(z) - \operatorname{Im}\mathrm{T}A - \sum_{k=1}^{m} n_k \vartheta_k\right)\right\}. \tag{3.13}$$

Here $\vartheta_j = \arg(z - z_j) (j = 1, \dots, m)$. Evidently, $B_0 \in L_p(\overline{G})$, $p > 2$, since $|B_0| = |B|$.

Thus, Problem **A** can always be reduced to an equivalent problem for an equation of the form (3.12) with the boundary condition of a simpler form (3.9). This form of Problem **A** will be called *canonical* (see also [39a]).

The homogeneous problem Å is, obviously, equivalent to the homogeneous problem for the equation (3.12) with the boundary condition

$$\mathrm{Re}[\overline{\Omega_n(t)}\, e^{-\pi i a(t)} w] = 0 \quad (\text{on } \Gamma), \tag{3.14}$$

and the adjoint homogeneous boundary value problem Å′ is reduced to the equivalent problem for the adjoint equation

$$\partial_{\bar z} w' - \bar{B}_0 \bar{w}' = 0 \tag{3.15}$$

with the boundary condition of the form

$$\mathrm{Re}[\Omega_n(z)\, z'\, e^{\pi i a(z)} w'(z)] = 0 , \quad z' = \frac{dz(s)}{ds} \quad (\text{on } \Gamma). \tag{3.16}$$

§4. Properties of the zeros of the solutions of the homogeneous Problem Å. Criteria of solubility of the Problems Å and A

In this paragraph we shall investigate some properties of the zeros of generalized analytic functions satisfying the boundary condition

$$\mathrm{Re}[\overline{\lambda(z)}\, w(z)] = 0 \quad (\text{on } \Gamma). \tag{4.1}$$

A relation will be derived which connects the numbers of the interior and boundary zeros with the index of the problem. We shall obtain a fundamental relation between the numbers of solutions of the homogeneous Problems Å and Å′. Moreover, criteria of solubility of the non-homogeneous Problem **A** will be established.

In this section, if not otherwise stated, we shall always assume that $\Gamma \in C^1_\mu$, $0 < \mu \leqslant 1$, and $\lambda(z) \in C_\nu(\Gamma)$, $0 < \nu < 1$.

4.1. We know already that a function $w(z)$ of the class $\mathfrak{A}_p(G)$, $p > 2$, continuous in the domain G can be represented in the form (Ch. III, §4.2)

$$w(z) = \Phi(z)\, e^{\omega(z)} , \tag{4.2}$$

where $\Phi(z)$ is holomorphic in G and $\omega(z)$ is continuous on the entire plane in the Hölder sense, $\omega \in C_{\frac{p-2}{p}}(E)$; besides, $\omega(z)$ is holomorphic outside $G+\Gamma$ and vanishes at infinity. It follows from the last formula that if $w(z)$ does not identically vanish its zeros in the domain G are isolated and every such zero has a multiplicity which is an integer, i.e. if at the point $z_0 \in G$ the function $w(z)$ vanishes, then an integer k exists such that in the vicinity of z_0 we have the relation

$$w(z) = (z-z_0)^k w_0(z) , \tag{4.3}$$

where $w_0(z)$ is a continuous function and $w_0(z_0) \neq 0$. It is evident now that on every closed set belonging to G there may be only a finite number of the zeros of the function $w(z)$. In respect of the zeros of the function $w(z)$ which are situated on the boundary of the domain G these propositions are not valid even in the case when $w(z)$ is continuous in the closed domain $G+\Gamma$. Nevertheless, it turns out that these properties are preserved for the solutions of the homogeneous problem $\mathring{A}$ in the whole closed domain $G+\Gamma$ (obviously under the assumptions already made with respect to the data of the problem).

THEOREM 4.4. *Let $\Gamma \in C^1_\mu$, $0 < \mu \leqslant 1$. If $w(z)$ is a non-trivial solution of Problem $\mathring{A}$ we have*

$$w(z) = P(z)\hat{w}(z) , \tag{4.4}$$

where $P(z)$ is a polynomial all roots of which belong to $G+\Gamma$ and $\hat{w}(z)$ is continuous and vanishes nowhere in $G+\Gamma$.

PROOF. By the conformal mapping $z = \varphi(\zeta)$ the domain G can be transformed into a domain G' bounded by circles the union of which will be denoted by Γ'. It is evident that the function $w_1(\zeta) = w[\varphi(\zeta)]$ is continuous in $G'+\Gamma'$, belongs to the class $\mathfrak{A}_p(G')$, $p > 2$, and satisfies the boundary condition

$$\operatorname{Re}[\overline{\lambda_1(\zeta)}\, w_1(\zeta)] = 0 \qquad (\text{on } \Gamma'), \qquad \lambda_1 = \lambda[\varphi(\zeta)] . \tag{4.5}$$

It should be taken into account here that, by Theorem 1.8, $\varphi(\zeta)$ satisfies the Lipschitz condition in $G'+\Gamma'$. Therefore, obviously, $\lambda_1(\zeta) \in C_\nu(\Gamma')$. By means of a transformation of the form (3.10) the boundary condition (4.5) is reduced to the form

$$\operatorname{Re}[\zeta^{-n} e^{-\pi i \alpha(\zeta)} w_0(\zeta)] = 0 \quad (\text{on } \Gamma'), \tag{4.6}$$

where

$$w_0(\zeta) = w_1(\zeta) e^{\omega(\xi,\eta)} \prod_{k=1}^{n} (\zeta - \zeta_k)^{n_k}, \tag{4.7}$$

$$\omega = -ip(\zeta) + \mathrm{T}A .$$

Outside an arbitrarily small neighbourhood of the point $\zeta = 0$ the function $w_*(\zeta) = \zeta^{-n} w_0(\zeta)$ belongs to the class $\mathfrak{A}_p$, $p > 2$, and is continuous up to Γ'. Hence, according to the generalized principle of symmetry it is continuously continuable outside G' the class being preserved in a wider domain G'' which contains inside the closed domain $G'+\Gamma'$. Consequently, the function $w_0(\zeta)$ equal to $\zeta^n w_*(\zeta)$ may have in $G'+\Gamma'$ only a finite number of zeros. In view of (4.7) this assertion holds also with respect to the function $w_1(\zeta)$. Therefore there exists a polynomial $P_1(\zeta)$ such that

$$w_1(\zeta) = P_1(\zeta) w_2(\zeta) , \tag{4.7a}$$

where $w_2(\zeta)$ is continuous and vanishes nowhere in $G'+\Gamma'$. Returning to the function $w(z)$ we have

$$w(z) = P_1[\psi(z)] w_2[\psi(z)] , \tag{4.7b}$$

where $\psi(z)$ is the function inverse to $\varphi(\zeta)$. It follows form this relation that $w(z)$ can have in $G+\Gamma$ only a finite number of zeros. Denoting them by $z_1, \ldots, z_k$ we have $P_1[\psi(z_j)] = 0 \, (j = 1, \ldots, k)$. We can therefore set $w(z) = P(z)\hat{w}(z)$ where

$$P(z) = \prod_{j=1}^{k} (z - z_j)^{n_j} ,$$

$$\hat{w}(z) = w_2[\psi(z)] \prod_{j=1}^{k} \left(\frac{\psi(z) - \psi(z_j)}{z - z_j} \right)^{n_j} . \tag{4.8}$$

Since $\Gamma \in C^1_\mu$, $\psi(z)$ has the derivative $\psi'(z)$ which is continuous and does not vanish in $G+\Gamma$ (Ch. I, §2). This fact and (4.8) imply immediately that $\hat{w}(z)$ is continuous and does not vanish in $G+\Gamma$.

REMARK. If $\Gamma \in C^1_{\mu,\nu_1,\dots\nu_k}$ and $w(z_0)=0$ at a boundary point z_0 which is not a corner point, it again follows from (4.7b) that near z_0 the function $w(z)=(z-z_0)^{n_0}w_0(z)$ where n_0 is a positive integer, w_0 is continuous in the vicinity of z_0, and $w_0(z_0)\neq 0$. If z_0 is a corner point it follows from (4.7b) in view of the formula (2.3), Ch. I, that

$$w(z)=(z-z_0)^{\frac{n_0}{\nu_0}}w_0(z) \qquad (\text{near } z_0)\,, \tag{4.8a}$$

where n_0 is a positive integer, $\nu_0\pi$ is equal to the interior angle at z_0, w_0 is continuous in the vicinity of the point z_0 and $w_0(z_0)\neq 0$.

4.2. If $w(z)$ is a non-trivial solution of Problem Å, then according to the last theorem we can represent $w(z)$ in the form (4.4) where $P(z)$ is a polynomial, $\hat{w}(z)$ is continuous and does not vanish in $G+\Gamma$.

Assume that

$$\vartheta(z)=\arg\lambda(z), \quad \theta(z)=\arg P(z), \quad \hat{\theta}(z)=\arg\hat{w}(z)\,. \tag{4.9}$$

Let $a_1, \dots, a_l$ be the roots of the polynomial $P(z)$ which are situated inside the domain G and $a_{j1}, \dots, a_{jl}$ the roots of $P(z)$ which belong to the boundary curve $\Gamma_j (j=0,1,\dots,m)$. Let us denote by k_i and k_{ji} the multiplicities of the roots a_i and a_{ji}, respectively.

In view of (4.4) and (4.9) the boundary condition (4.1) may be written in the form

$$e^{i(\theta+\hat{\theta}-\vartheta)}=i(-1)^k \qquad (k\text{—an integer}). \tag{4.10}$$

Specifying the values of θ, $\hat{\theta}$ and ϑ at a point z_j of the curve Γ_j, which is not a root of the polynomial $P(z)$, and describing once this curve in the positive direction we have in view of (4.10)

$$2\pi n_j \equiv \Delta_{\Gamma_j}\vartheta=\Delta_{\Gamma_j}\theta+\Delta_{\Gamma_j}\hat{\theta} \qquad (j=0,1,\dots,m)\,. \tag{4.11}$$

It is readily observed that for the exterior contour Γ_0 we have

$$\begin{aligned}\Delta_{\Gamma_0}\theta &= \pi\sum_{i=0}^{l_0} k_{0i} + 2\pi\sum_{i=1}^{l} k_i + 2\pi\sum_{j=1}^{m}\sum_{i=1}^{l_j} k_{ji}\\ &= \pi N_{\Gamma_0} + 2\pi(N_{\Gamma_1} + \dots + N_{\Gamma_m}) + 2\pi N_G\,, \qquad (4.12)\end{aligned}$$

For the interior contours

$$\Delta_{\Gamma_j}\theta = -\pi\sum_{i=1}^{l_j} k_{ji} = -\pi N_{\Gamma_j} \qquad (j = 1, \dots, m)\,, \qquad (4.13)$$

where

$$N_G = k_1 + \dots + k_l, \qquad N_{\Gamma_j} = k_{j1} + \dots + k_{jl_j} \qquad (j = 0, \dots, m)$$

denote the numbers of zeros of the function $w(z)$ which lie in the domain G and on the contour Γ_j, respectively. It should be taken into account that a zero is counted the number of times equal to its multiplicity.

In view of (4.12) and (4.13) the relation (4.11) may be written thus:

$$\begin{aligned}2n_0 &= N_{\Gamma_0} + 2(N_{\Gamma_1} + \dots + N_{\Gamma_m}) + 2N_G + 2\hat{n}_0,\\ 2n_j &= -N_{\Gamma_j} + 2\hat{n}_j \qquad (j = 1, \dots, m)\,,\end{aligned} \qquad (4.14)$$

where $\hat{n}_j = \frac{1}{2\pi}\Delta_{\Gamma_j}\hat{\theta}$. Adding these relations and taking into account (3.2) we obtain

$$2n = N_{\Gamma_0} + N_{\Gamma_1} + \dots + N_{\Gamma_m} + 2N_G + 2\hat{n}\,, \qquad (4.15)$$

$$\hat{n} = \hat{n}_0 + \hat{n}_1 + \dots + \hat{n}_n\,. \qquad (4.16)$$

But, according to the principle of the argument (Ch. III, §4.10) $\hat{n} = \frac{1}{2\pi}\Delta_\Gamma \arg\hat{w} = 0$ since $\hat{w}$ is a generalized analytic function continuous in $G+\Gamma$ and vanishing nowhere. Thus, we have the following relationship:

$$\begin{aligned}2n &= 2N_G + N_\Gamma\,, \qquad (4.17)\\ N_\Gamma &= N_{\Gamma_0} + N_{\Gamma_1} + \dots + N_{\Gamma_m}\,.\end{aligned}$$

The last results are easily generalized to the case in which the solution of Problem **A** is sought for in a more general class of single-valued functions having a finite number of singular points which are situated in the domain or on its boundary. In what follows we shall confine ourselves to the consideration of the case when in the vicinity of every singular point z_0 the solution $w(z)$ of Problem **Å** has the form

$$w(z) = O(|z-z_0|^{-\nu_0}), \tag{4.17a}$$

where ν_0 is a positive number. It follows immediately from the formula (4.2) that the number ν_0 is an integer if z_0 lies inside the domain G. Under our conditions ν_0 is an integer also when the singular point z_0 belongs to the boundary of the domain.

Let Γ' be a boundary arc containing one singular point z_0 of the solution w of Problem **Å**. Without affecting the generality of the results we may assume that $z_0 = 0$ and Γ' is the interval $[-\varrho', \varrho']$ of the real axis. The general case is reduced to the present one by means of a conformal mapping. Representing w in the form (4.2) we can achieve the fact that $\omega(z)$ takes on Γ' values equal to $\arg\lambda(z)$. Such being the case, the function $\Phi(z)$ is holomorphic inside a semi-circle $K_\varrho - |z| < \varrho < \varrho'$, $\mathrm{Im}(z) > 0$, continuous on its closure $\bar{K}_\varrho$ with the exception of the point $z = 0$ near which it has the form $\Phi = 0(|z|^{-\nu_0})$. Moreover, this function satisfies the boundary condition

$$\mathrm{Re}[\Phi(z)] = 0, \quad \text{if} \quad z \in [-\varrho, \varrho], \quad z \neq 0 .$$

Continuing the function Φ according to the principle of symmetry into the lower semi-circle $K'_\varrho - |z| < \varrho$, $\mathrm{Im}(z) < 0$—we easily find that Φ has a pole at the point $z = 0$, i.e. ν_0 is an integer, thus completing the proof (see also [60a], Ch. I, §15).

Thus, we have established that every solution $w(z)$ of the homogeneous problem **Å** having inside the domain G

and on its boundary Γ a finite set of singular points of the type (4.17a), is representable in the form

$$w(z) = R(z)\hat{w}(z) , \tag{4.17b}$$

where $R(z)$ is a rational function the zeros and poles of which are situated in $G+\Gamma$ and $\hat{w}$ is a function of the class $\mathfrak{A}_p(G), p > 2$, continuous in the Hölder sense and vanishing nowhere in $G+\Gamma$.

It follows immediately from the formula (4.17b) that the following relation is valid

$$2N_G + N_\Gamma - 2P_G - P_\Gamma = 2n , \tag{4.17c}$$

where N_G, P_G, N_Γ and P_Γ are the numbers of the zeros and poles of the function $w(z)$ inside G and on Γ, respectively.

The formula (4.17b) implies the following theorem.

If for $n < 0$ the numbers of the interior and boundary poles of a solution $w(z)$ of the homogeneous problem $\mathring{\mathbf{A}}$ *satisfy the inequality*

$$2P_G + P_\Gamma < -2n , \tag{4.17d}$$

then $w(z) \equiv 0$.

4.3. Let us now expose a number of theorems which follow easily from the formulae (4.4), (4.14) and (4.17).

THEOREM 4.5. *If the index n of Problem* $\mathbf{A}$ *is a negative number the homogeneous Problem* $\mathring{\mathbf{A}}$ *has no non-trivial solutions.*

A different proof of this theorem will be given below (§5.4).

THEOREM 4.6. *If for $n = 0$ Problem* $\mathring{\mathbf{A}}$ *has non-trivial solutions they are given by the formula*

$$w(z) = c_0 w_0(z) , \tag{4.18}$$

where c_0 is an arbitrary real constant and $w_0(z)$ is a particular solution of Problem $\mathring{\mathbf{A}}$ *which vanishes nowhere in $G+\Gamma$.*

PROOF. For $n = 0$ it follows from (4.17) that $N_\Gamma = N_G = 0$. Therefore the polynomial $P(z)$ in the formula

(4.4) reduces to a constant c_0. Thus, every non-trivial solution of Problem Å has the form (4.18) and, consequently, it vanishes nowhere in $G+\Gamma$.

We shall now prove that for $n=0$ two solutions of Problem Å are always linearly independent. In fact, if w_1 and w_2 are non-trivial solutions of Problem Å their linear combination $w=c_1w_1+c_2w_2$ with real coefficients is also a solution of this problem. But in view of the boundary condition $w_1=i\lambda\chi_1$ and $w_2=i\lambda\chi_2$ on Γ, where χ_1 and χ_2 are real functions which vanish nowhere on Γ. Therefore the constants c_1 and c_2 can always be so chosen that at a fixed point z_0 belonging to Γ, $w(z_0)=c_1w_1(z_0)+c_2w_2(z_0)=0$. But this is possible only if $c_1w_1(z)+c_2w_2(z)\equiv 0$. Hence, all non-trivial solutions of Problem Å have the form (4.18) where c_0 is an arbitrary real constant. This completes the proof.

THEOREM 4.7. *If for $n>0$ the homogeneous Problem Å has a non-trivial solution $w(z)$ then this solution has zeros in the closed domain $G+\Gamma$; the number N_G of the zeros inside G and the number N_Γ of the zeros on Γ are connected by the relation*

$$N_\Gamma+2N_G=2n\,. \tag{4.19}$$

Moreover, on every boundary contour Γ_j there may exist only an even number of the zeros (their multiplicity being taken into account).

PROOF. The theorem follows from the relations (4.14); they also imply that N_{Γ_j} are even numbers.

It follows from (4.19) that the number of the interior zeros of a non-trivial solution of the homogeneous problem Å does not exceed n, and the number of the boundary zeros $\leqslant 2n$. The extreme cases are also possible:

$$N_\Gamma=0\,,\quad N_G=n\,,\quad N_\Gamma=2n\,,\quad N_G=0\,. \tag{4.19a}$$

The degree of the polynomial $P(z)$ appearing in the relation (4.4) is

$$p=N_\Gamma+N_G=2n-N_G \tag{4.20}$$

Consequently,

$$n \leqslant p \leqslant 2n\,. \tag{4.21}$$

We also have the following

THEOREM 4.8. *If $n \geqslant 0$ the homogeneous Problem $\mathring{A}$ cannot have more than $2n+1$ linearly independent solutions.*

PROOF. If w_0 is a solution of Problem $\mathring{A}$ the function $w = z^{-n}w_0$ satisfies the equation $\partial_{\bar{z}}w + Aw + Be^{-2in\varphi}\bar{w} = 0$ ($\varphi = \arg z$) everywhere in G except for the point $z = 0$ at which it may have a pole of the order $\leqslant n$. Therefore it is representable in the form (Ch. III, §15.4)

$$w(z) = w_*(z) + c_1\tilde{w}_{-1}(z) + \ldots + c_{2n}\tilde{w}_{-2n}(z) \tag{4.22}$$

(c_k is a real constant)

where $w_*(z)$ satisfies the equation $\partial_{\bar{z}}w + Aw + \mathrm{B}e^{-2in\varphi}\bar{w} = 0$, it is continuous in $G+\Gamma$ and satisfies the boundary condition

$$\mathrm{Re}[w_*(z)] = -\sum_{k=1}^{2n} c_k \mathrm{Re}[\tilde{w}_{-k}(z)] \quad (\text{on } \Gamma)\,. \tag{4.23}$$

Since the corresponding homogeneous problem—$\mathrm{Re}[w_*(z)] = 0$—according to Theorem 4.6 cannot have more than one (linearly independent) solution, the boundary value problem (4.23) does not have more than $2n+1$ linearly independent solutions. Returning to the relation (4.22) we find that the original Problem $\mathring{A}$ has at most $2n+1$ solutions.

4.4. THEOREM 4.9. *The difference between the numbers of solutions of the homogeneous Problems $\mathring{A}$ and $\mathring{A}'$ is equal to the difference between the corresponding indices,* i.e.

$$l - l' = n - n' = 2n + 1 - m\,.^* \tag{4.24}$$

PROOF. It follows from (2.20) and (2.30) that

$$l - l' = l'_* - l_*\,, \tag{4.25}$$

* This formula was proved by the author in the paper [14a] in a somewhat different way (see also §6.4.).

where l_* and l'_* are the numbers of solutions of the concomitant homogeneous boundary value Problems $\mathring{A}_*$ and $\mathring{A}'_*$ with the boundary conditions (2.29) and (2.19). These numbers can easily be calculated; to this end we may assume that the boundary value problems have been reduced to the canonical form. Moreover, without affecting the generality we may assume that the exterior boundary contour Γ_0 is the circle $|z| = 1$; the origin of the coordinate system lies as before in the domain G under consideration.

Since the boundary conditions of the problems $\mathring{A}$ and $\mathring{A}'$ may be taken in the form $\operatorname{Re}[z^{-n}w(z)] = 0$ and $\operatorname{Re}[z^n z' w'(z)] = 0$ the boundary conditions of the concomitant problems $\mathring{A}_*$ and $\mathring{A}'_*$ have the form

$$\operatorname{Re}[t^{-n}\Phi^-(t)] = 0 , \qquad t \in \Gamma, \tag{4.26}$$

$$\operatorname{Re}[t^n t'(s)\Psi^-(t)] = 0 , \qquad t \in \Gamma, \tag{4.27}$$

where Φ and Ψ are the unknown functions which are holomorphic in the domains $G_0, G_1, \ldots, G_m$. We have also $\Phi(\infty) = \Psi(\infty) = 0$.

From the boundary condition (4.26) on the interior boundary curves we obtain at once

$$\Phi(z) = ic_j z^n , \qquad \text{if} \qquad z \in G_j \qquad (j = 1, \ldots, m) , \tag{4.28}$$

where c_j are arbitrary real constants. The conditions (4.27) may also be written in the form

$$\frac{d}{ds}\operatorname{Re}\int^z t^{-n}\Psi(t)\,dt = 0 , \qquad z \in \Gamma_j , \qquad (j = 1, \ldots, m) .$$

Since the integral appearing in this relation is a function holomorphic in the domain G_j we have

$$\Psi(z) = 0 , \qquad \text{if} \qquad z \in G_j \qquad (j = 1, \ldots, m) . \tag{4.29}$$

Let us now proceed to the consideration of the boundary conditions on the exterior contour Γ_0. We shall begin with the case $n \geqslant 0$. Here, under the condition $\Phi(\infty) = 0$

it follows immediately from (4.26) that $\Phi(z)=0$ for $z \in G_0$. Consequently, in view of (4.28) the number of solutions of the concomitant homogeneous problem $\mathring{A}_*$ is equal to m, i.e.

$$l_* = m\,. \tag{4.30}$$

Since according to the assumption Γ_0 is the circle $|t|=1$, $t'(s)=it$ and the condition (4.27) takes the form

$$\operatorname{Re}[t^{n+1}\Psi^-(t)] = 0\,.$$

Taking into account that $\Psi^-(\infty)=0$ we easily find that this problem has $2n+1$ linearly independent solutions

$$\begin{gathered} iz^{-n-1}\,, \quad i(z^{-n-2}+z^{-n})\,, \quad z^{-n-2}-z^{-n}\,, \;\ldots, \\ i(z^{-1}+z^{-2n-1})\,, \quad z^{-1}-z^{-2n-1}\,. \end{gathered} \tag{4.31}$$

Therefore, according to (4.29) the number of solutions of the concomitant homogeneous problem $\mathring{A}'_*$ is equal to $2n+1$, i.e.

$$l'_* = 2n+1\,. \tag{4.32}$$

From (4.30), (4.32) and according to (4.25) the formula (4.24) follows at once.

Assume now that $n<0$. Then the boundary value problem (4.26) for the exterior domain G_0 has $-(2n+1)$ solutions (the condition $\Phi(\infty)=0$ being taken into account)

$$\begin{gathered} iz^{n}\,, \quad i(z^{n-1}+z^{n+1})\,, \quad z^{n-1}-z^{n+1}\,, \;\ldots \\ \ldots,\; i(z^{-1}+z^{2n+1})\,, \quad z^{-1}-z^{2n+1}\,. \end{gathered} \tag{4.33}$$

Thus, according to the formula (4.28) we have in the case under consideration

$$l_* = -2n+m-1\,. \tag{4.34}$$

It follows from (4.27) that $\Psi(z)=0$ if $z \in G_0$, i.e.

$$l'_* = 0\,. \tag{4.35}$$

According to (4.25) from (4.34) and (4.35) we again obtain the formula (4.24). This completes the proof of the theorem.

Theorem 4.8 establishes the upper bound for the number of solutions of Problem $\mathring{A}$. Theorem 4.9 enables us to find also the lower bound of this number. Since $l' \geqslant 0$, it is evident that $l \geqslant 2n+1-m$. In general, the following inequality is valid

$$\max(0, 2n+1-m) \leqslant l \leqslant 2n+1 . \tag{4.36}$$

But for $n < 0$ and $n > m-1$ even more precise results may be obtained.

Theorem 4.5 and the formula (4.9) imply immediately

THEOREM 4.10. *If* $n < 0$,

$$l = 0 , \quad l' = m-2n-1 . \tag{4.37}$$

If $n > m-1$, *then*

$$l = 2n+1-m , \quad l' = 0 . \tag{4.38}$$

In particular, in the case of a simply-connected domain ($m = 0$) we have

THEOREM 4.11. *If* $m = 0$, *then*: (1) *for* $n < 0$ *the homogeneous Problem* $\mathring{A}$ *has no non-trivial solutions* ($l = 0$) *and the adjoint homogeneous Problem* $\mathring{A}'$ *has exactly* $l' = -2n-1$ *solutions*; (2) *for* $n \geqslant 0$ *the homogeneous Problem* $\mathring{A}$ *has exactly* $l = 2n+1$ *solutions and the adjoint Problem* $\mathring{A}'$ *has no non-trivial solutions.*

In particular, for $n = 0$ *Problem* $\mathring{A}$ *has one (linearly independent) solution*; *this solution does not vanish anywhere in* $G+\Gamma$.

A different proof of the last theorem will be given in §7.

Theorem 4.3 and Theorem 4.10 imply at once the following

THEOREM 4.12. *If* $n > m-1$ *the non-homogeneous Problem* A *is always soluble and its general solution is given by the formula*

$$w(z) = w_0(z) + \sum_{j=1}^{2n+1-m} c_j w_j(z) \tag{4.39}$$

$$(c_j\text{—}\textit{real constant}) ,$$

where $w_1, \ldots, w_{2n+1-m}$ is the complete system of solutions of the homogeneous Problem $\mathring{\mathbf{A}}$ *and w_0 is a particular solution of the non-homogeneous Problem* **A**. *If $n < 0$ the non-homogeneous Problem* **A** *has a solution (a unique solution) if and only if the following relations are satisfied*

$$\int_\Gamma \gamma(t)\, w_j'(t)\, \lambda(t)\, dt = 0 \qquad (j = 1, \ldots, m-2n-1)\,, \tag{4.40}$$

where $w_1', \ldots, w_{m-2n-1}'$ is the complete system of solutions of the adjoint homogeneous Problem $\mathring{\mathbf{A}}'$.

4.5. The theorems proved above are important also because they make possible the identification of various properties of Problem **A** by means of fairly simple criteria, without constructing the solutions, which in general is very difficult to perform. From this point of view we have obtained the most complete result in the case of the index satisfying the conditions

$$n < 0\,, \qquad n > m-1\,. \tag{4.41}$$

Under these conditions Theorem 4.10 and Theorem 4.12 make it possible to establish facts concerning the solubility and insolubility of Problem **A** and to determine the numbers of solutions of the homogeneous Problems $\mathring{\mathbf{A}}$ and $\mathring{\mathbf{A}}'$. We have not yet such a complete result when the index of the problem satisfies the condition

$$0 \leqslant n \leqslant m-1\,. \tag{4.42}$$

In these special cases the theorems given above do not enable us to determine the numbers l and l' separately. Obviously, it is sufficient to find one of them, then the other is given by the formula (4.24). Therefore we may confine ourselves to the case of the index of the problem satisfying the condition

$$0 \leqslant n < \tfrac{1}{2}m\,, \tag{4.43}$$

since for $n \geqslant \frac{1}{2}m$ this condition is satisfied for the index of the adjoint Problem $\mathring{\mathbf{A}}'$, which is equal to $m-n-1$.

In order to calculate the number of solutions of the

homogeneous Problem Å it is evidently sufficient to take the boundary condition in the form (see §3.3)

$$\mathrm{Re}[\overline{\Omega_n(z)}\,w(z)] = 0\,, \quad \Omega_n(z) = \prod_{k=1}^{n}(z-a_k)\,, \quad a_k \in G\,. \quad (4.44)$$

Then the boundary condition of the adjoint problem is given by

$$\mathrm{Re}[\Omega_n(z)z'(s)w'(z)] = 0 \quad (\text{on } \Gamma) \qquad (4.45)$$

or, introducing a new function

$$\hat{w}'(z) = \Omega_n(z)\hat{w}'(z)\,, \qquad (4.46)$$

we have

$$\mathrm{Re}[z'(s)\hat{w}'(z)] = 0 \quad (\text{on } \Gamma)\,. \qquad (4.47)$$

Evidently, $\hat{w}'$ is a generalized analytic function, continuous in $G+\Gamma$. In order to calculate the number of solutions of the problem (4.47) let us consider the adjoint problem

$$\mathrm{Re}[\hat{w}(z)] = 0 \quad (\text{on } \Gamma)\,. \qquad (4.48)$$

Since the index of the latter problem is equal to zero, according to Theorem 4.8 the number of its solutions $\hat{l}$ does not exceed the unity, i.e.

$$\hat{l} = 0 \quad \text{or} \quad \hat{l} = 1\,. \qquad (4.49)$$

We shall later give examples which show that both these cases can occur (§5.1). The index of the boundary value problem (4.47) is equal to $m-1$ and hence the number of its solutions, according to the formulae (4.24) is given by the relation

$$\hat{l}' = 2(m-1)+1+\hat{l}-m = m-1+\hat{l}\,.$$

Consequently, by (4.49),

$$\hat{l}' = m-1 \quad \text{or} \quad \hat{l}' = m\,. \qquad (4.50)$$

Thus, the general solution of the boundary value problem (4.47) has the form

$$\hat{w}'(z) = c_1\hat{w}_1'(z) + \ldots + c_{\hat{l}'}\hat{w}_{\hat{l}'}'(z) \tag{4.51}$$

(c_k is a real constant) .

The solution of the problem (4.45) follows from the formula (4.46) if $\hat{w}'(z)$ be subjected to the following conditions

$$\hat{w}'(a_1) = 0, \ldots, \hat{w}'(a_n) = 0 . \tag{4.52}$$

These relations yield $2n$ linear equations for the determination od $\hat{l}'$ real constants $c_1, \ldots, c_{\hat{l}'}$. By virtue of (4.50) and (4.43) $\hat{l}' \geqslant 2n$. Therefore the rank of the system of equations (4.52) satisfies the condition

$$0 \leqslant r \leqslant 2n . \tag{4.53}$$

It is now easily observed that the number of solutions of the problem (4.45) is given by the formula

$$l' = \hat{l}' - r , \tag{4.54}$$

i.e. according to (4.50)

$$l' = m - r - 1 \qquad \text{or} \qquad l' = m - r . \tag{4.55}$$

The number of solutions of Problem Å in view of (4.24) and (4.54) is determined by the relation

$$l = 2n + 1 + \hat{l}' - r - m , \tag{4.56}$$

i.e.

$$l = 2n + 1 - r \qquad \text{or} \qquad l = 2n - r . \tag{4.57}$$

The following question naturally arises. Can the number r take all the values satisfying the condition (4.53) or are some of them excluded?

It is evident that Problem Å has the minimum number of solutions ($l = 0$ or $l = 1$) when $r = 2n$. We shall prove below that exactly these cases are most frequently encountered. It will also be established that if the conditions

(4.43) are satisfied the following inequality holds (§6.4; Appendix to Ch. IV)

$$0 \leqslant l \leqslant n+1\,. \tag{4.58}$$

The condition $l = n+1$ can take place. It occurs, however, for a special form of boundary value problems (§ 5.8).

§5. Investigation of special classes of boundary value problems of the type A in the case $0 \leqslant n \leqslant m-1$

5.1. To begin with we consider two simple but typical problems corresponding to the case $n = 0$ and $m \geqslant 1$.

For $n = 0$ according to (4.53) and (4.56) $l = 1 + \hat{l} - m$. Consequently, in view of (4.50)

$$l = 0 \quad \text{or} \quad l = 1\,. \tag{5.1}$$

We prove that both these cases can occur. Let us consider for instance the following problem.

It is required to find a function $\Phi(z)$ holomorphic in G, continuous in $G+\Gamma$, which satisfies the boundary condition

$$\operatorname{Re}[e^{\pi i a(z)}\Phi(z)] = 0 \quad (\text{on } \Gamma)\,, \tag{5.2}$$

where $a(z)$ is a real piecewise constant function,

$$a = 0 \quad \text{on} \quad \Gamma_0\,, \quad a = a_k = \text{const} \quad \text{on} \quad \Gamma_k \tag{5.3}$$
$$(k = 1, 2, \ldots, m)\,.$$

It is evident that the index of this problem is equal to zero.

If all $a_k = 0 (\operatorname{mod} 1)$ the problem (5.2) has the solution $\Phi = ic$ (c is a real constant) and consequently the case $l = 1$ occurs.

Let us now assume that at least one constant a_k can be found, which satisfies the condition

$$a_k \neq 0 (\operatorname{mod} 1)\,. \tag{5.4}$$

It will be proved below that in this case the problem (5.2) has no non-trivial solutions. Therefore this case corresponds to the case $l = 0$.

By means of the principle of symmetry we find that the solution of the problem is analytically continuable through the circles $\Gamma_0, \Gamma_1, \dots, \Gamma_m$. We may therefore differentiate both sides of the relation (5.2) with respect to the arc of the appropriate circle. Thus, we obtain

$$\operatorname{Re}[e^{\pi i a(z)}\Phi'(z)z'(s)] = 0 \quad (\text{on } \Gamma). \tag{5.5}$$

But in view of (5.2) the function $ie^{-\pi i a}\overline{\Phi(z)}$ takes real values on Γ. Multiplying both sides of the relation (5.5) by this function we have

$$\operatorname{Re}[\overline{i\Phi(z)}\Phi'(z)z'(s)] = 0 \quad (\text{on } \Gamma).$$

Integrating the above relation over Γ and applying Green's identity we obtain

$$\operatorname{Re}\Big[i\int_\Gamma \overline{\Phi(z)}\Phi'(z)\,dz\Big] \equiv -2\iint_G |\Phi'(z)|^2\,dx\,dy = 0\,,$$

i.e. $\Phi'(z) \equiv 0$. Consequently, $\Phi = c = \text{const}$. Now, the boundary condition (5.2) and the inequality (5.4) imply that $c = 0$.

The problem (5.2) was investigated by a somewhat different method by Kveselava, [39a].

5.2. It is of interest for many reasons to investigate the following special boundary value problems

$$\operatorname{Re}[z'(s)\psi(z)] = 0 \quad (\text{on } \Gamma), \tag{5.6}$$

$$\operatorname{Re}[e^{\pi i a(z)}z'(s)\psi(z)] = 0 \quad (\text{on } \Gamma), \tag{5.7}$$

where $\psi(z)$ is the unknown function which is holomorphic in G, and $a(z)$ is as before a piecewise constant function of the form (5.3) satisfying the condition (5.4). The indices of these problems are equal to $m-1$ but the numbers of their solutions are distinct. They are given respectively by

$$l' = m\,, \quad l' = m-1\,. \tag{5.8}$$

The above relations follow at once from the formula (4.24) if it is taken into account that the adjoint boundary conditions have the form

$$\mathrm{Re}[\Phi(z)] = 0 \quad \text{and} \quad \mathrm{Re}[e^{-\pi i \alpha}\Phi(z)] = 0 . \tag{5.9}$$

We have seen in the previous subsection that in the first case $l = 1$ and in the second $l = 0$ (of course if the condition (5.4) is satisfied). We shall now prove that there is no difficulty in constructing all solutions of the problem (5.6).

Let $u_k(x, y)$ be a function harmonic in G and satisfying the boundary condition

$$\begin{gathered} u_k = 0 \text{ on } L_0, \; u_k = \delta_{ki} \text{ on } L_i, \; i = 1, \dots, m \\ (k = 1, \dots, m; \; \delta_{kk} = 1 , \; \delta_{ki} = 0 \text{ for } i \neq k) . \end{gathered} \tag{5.10}$$

The function u_k is the harmonic measure of the contour Γ_k with respect to the domain G. Obviously, these functions depend only on the domain. They can be constructed by means of the formulae

$$u_k(x, y) = \int\limits_{\Gamma_k} \frac{\partial g(t, z)}{\partial n_t} ds_t \quad (k = 1, \dots, m) , \tag{5.11}$$

where $g(z, \zeta)$ is the Green function for the domain G of the Dirichlet problem. It is known that this function has the form

$$g(z, \zeta) = \frac{1}{2\pi} \ln|z-\zeta| + g_0(z, \zeta) , \tag{5.12}$$

where $g_0(z, \zeta)$ is a function harmonic in G with respect to z, and it satisfies the boundary condition (ζ is a fixed point)

$$g_0 = -\frac{1}{2\pi} \ln|z-\zeta| \quad \text{for} \quad z \in \Gamma . \tag{5.13}$$

Let us consider the function $g_*(z, \zeta)$ conjugate to $g(z, \zeta)$ with respect to the point z; it is given by the curvilinear integral

$$g_*(z, \zeta) = \int\limits_{z_0}^{z} \frac{\partial g(t, \zeta)}{\partial n_t} ds_t + \text{const} . \tag{5.14}$$

This function is, obviously, multi-valued in G. If the point z describes a closed curve homological to Γ_k which does not contain the point ζ, $g_*(z, \zeta)$ acquires an increment equal to $u_k(\zeta)$. If this curve bounds a domain belonging to G and containing the point ζ, g_* acquires an increment equal to 1.

The relations (5.10) imply at once that

$$\frac{du_k}{ds} \equiv \frac{\partial u_k}{\partial z}\frac{dz}{ds} + \frac{\partial u_k}{\partial \bar{z}}\frac{d\bar{z}}{ds} = 0 \quad (\text{on } \Gamma), \qquad (5.15)$$

i.e.

$$\operatorname{Re}\left[z'(s)\frac{\partial u_k}{\partial z}\right] = 0 \quad (\text{on } \Gamma).$$

These relations are valid along every line $u_k = \text{const}$.

Hence, the problem (5.6) has m solutions

$$\Phi_k(z) = 2\frac{\partial u_k}{\partial z} \quad (k = 1, \ldots, m). \qquad (5.16)$$

These functions depend only on the domain. They are linearly independent. This fact follows at once from the linear independence of the harmonic functions $u_1, \ldots, u_m$. Since the problem (5.6), as we have seen before, has m linearly independent solutions its general solution is given by the formula

$$\Phi(z) = c_1\Phi_1(z) + \ldots + c_m\Phi_m(z), \qquad (5.17)$$

where $c_1, \ldots, c_m$ are arbitrary real constants.

According to (5.15) we have

$$\frac{\partial u_k}{\partial n} = \frac{1}{i}\frac{\partial u_k}{\partial z}\frac{dz}{ds} - \frac{1}{i}\frac{\partial u_k}{\partial \bar{z}}\frac{d\bar{z}}{ds} = \frac{2}{i}\frac{\partial u_k}{\partial z}\frac{dz}{ds} = \frac{1}{i}\Phi_k(z)z'(s),$$

i.e.

$$\Phi_k(z) = \overline{iz'(s)}\frac{\partial u_k}{\partial n} \quad (\text{on } \Gamma) \quad (k = 1, \ldots, m), \quad (5.18)$$

where n is the outer normal of Γ. Let us prove that the function $\Phi_k(z)$ does not vanish anywhere on Γ. It follows

from the boundary condition (5.10) that inside G the following inequalities hold

$$0 < u_k(x, y) < 1 \quad (\text{in } G) \quad (k = 1, \ldots, m), \quad (5.19)$$

Hence, we have

$$\frac{\partial u_k}{\partial n} \leqslant 0 \quad \text{on} \quad \Gamma_j \quad \text{for} \quad j \neq k, \quad \frac{\partial u_k}{\partial n} \geqslant 0 \quad \text{on} \quad \Gamma_k .$$

We shall now prove that the equality sign does not occur in the last inequalities. If it did occur the function $\Phi_k(z)$ would have a zero on Γ and the corresponding point would be a critical point for the harmonic function u_k. In other words, through this point there passes besides Γ at least one more line $u_k = \text{const.} = \delta_{kj}$ which evidently is partly situated also inside the domain G. The last result, however, contradicts the inequalities (5.19). It has therefore been established that

$$\frac{\partial u_k}{\partial n} < 0 \text{ on } \Gamma_j \text{ for } j \neq k, \quad \frac{\partial u_k}{\partial n} > 0 \text{ on } \Gamma_k \quad (5.20)$$
$$(j, k = 1, \ldots, m) .$$

Hence, *the functions $\Phi_k(z)$ do not vanish anywhere on the boundary Γ of the domain G.*

If we now turn to the relation (4.17) we find that every function Φ_k has exactly $m-1$ zeros (their multiplicity being taken into account) inside the domain G. In fact, since the index of the problem (5.6) is equal to $m-1$ we have according to the formula (4.17)

$$N_\Gamma + 2N_G = 2m-2 , \quad (5.21)$$

where N_Γ and N_G are the numbers of the boundary and interior zeros of the solution of the problem (5.6). Since it was proved that for Φ_k $N_\Gamma = 0$ it follows from (5.21) that $N_G = m-1$.

5.3. We now consider the following particular case of Problem **A**.

Problem **D**. *It is required to find a function $\Phi(z)$ holomorphic in G, continuous in $\bar{G}$ and satisfying the condition*

$$\mathrm{Re}[\Phi(z)] = \gamma(z) \quad (\text{on } \Gamma) . \quad (5.22)$$

Since the adjoint homogeneous problem (5.6) has m linearly independent solutions (5.16), according to the general Theorem 4.2, for the solubility of the boundary value Problem **D** it is necessary and sufficient that the following conditions be satisfied:

$$\frac{1}{i}\int_{\Gamma} \gamma(z)\Phi_k(z)\,dz \equiv \int_{\Gamma} \gamma(z)\frac{\partial u_k}{\partial n}\,ds = 0 \qquad (5.23)$$
$$(k = 1, \dots, m)\,.$$

Let us now take for γ a function of the form

$$\gamma(z) = \gamma_0(z) + c_j \quad \text{on} \quad \Gamma_j \ (j = 0, 1, \dots, m)\,, \qquad (5.24)$$

where c_j are real constants, $c_0 = 0$ and γ_0 is a function continuous on Γ. We shall prove that it is always possible to choose (in a unique way) such values for the constants c_j that all the conditions (5.23) are satisfied.* To this end it is necessary and sufficient that the constants c_j satisfy the relations

$$\sum_{j=1}^{m} A_{kj}c_j = -\int_{\Gamma} \gamma_0 \frac{\partial u_k}{\partial n}\,ds \qquad (k = 1, \dots, m) \qquad (5.25)$$

where

$$A_{kj} = \int_{\Gamma_j} \frac{\partial u_k}{\partial n}\,ds \qquad (j, k = 1, \dots, m)\,. \qquad (5.26)$$

Making use of the conditions (5.10) and applying Green's identity we obtain

$$A_{kj} = \int_{\Gamma} u_j \frac{\partial u_k}{\partial n}\,ds = \frac{1}{i}\int_{\Gamma} u_j\Phi_k(z)\,dz$$
$$= \iint_{G} \overline{\Phi}_j\Phi_k\,dx\,dy\,. \qquad (5.27)$$

* In this modified formulation the Dirichlet problem was investigated by Muskhelishvili, [60a] (Ch. III, §60). Below, a somewhat different method of solution of this problem will be indicated.

In other words the determinant of the system (5.25) is the Gramm determinant for the linearly independent system of functions $\Phi_1, \ldots, \Phi_m$. Hence, the system (5.25) has a unique solution which can be written in the form

$$c_j = -\int_{\Gamma} \gamma_0 \frac{\partial u_j^*}{\partial n} ds \qquad (j = 1, \ldots, m), \tag{5.28}$$

where

$$u_j^* = a_{j1}u_1 + \ldots + a_{jm}u_m . \tag{5.29}$$

Let us call the functions u_j^* depending only on the domain, *the canonical harmonic functions of the domain.*

It follow at once from (5.27) that

$$A_{kj} = A_{jk}, \qquad \text{i.e.} \qquad \int_{\Gamma_j} \frac{\partial u_k}{\partial n} ds = \int_{\Gamma_k} \frac{\partial u_j}{\partial n} ds . \tag{5.30}$$

Therefore $a_{jk} = a_{kj}$ and, moreover,

$$\sum_{i=1}^{m} A_{ki}a_{ij} = \sum_{i=1}^{m} A_{ik}a\delta_{ijkj} . \tag{5.31}$$

Hence, taking into account the relations (5.29) and (5.26) we obtain

$$\int_{\Gamma_k} \frac{\partial u_j^*}{\partial n} ds = \delta_{kj} . \tag{5.32}$$

The constants a_{jk} depend only on the domain under consideration. The harmonic functions u_j^* also depend only on the domain, since they are uniquely determined by their boundary values

$$u_j^* = 0 \text{ on } \Gamma_0, \; u_j^* = a_{jk} \text{ on } \Gamma_k \qquad (j, k = 1, \ldots, m) . \tag{5.33}$$

Since u_k^* are linearly independent, for the system of linearly independent solutions of the problem (5.6) also the functions

$$\Phi_k^*(z) = 2\frac{\partial u_k^*}{\partial z} = i\bar{z}'(s)\frac{\partial u_k^*}{\partial n} \qquad (k = 1, \ldots, m) . \tag{5.34}$$

may be taken. Here the derivative $z'(s)$ is taken along the arc of the line $u_k^* = \text{const.}$, and the derivative of this function is taken along the normal to this arc.

Making use of the relations (5.32), (5.10) and Green's identity we obtain

$$\delta_{kj} = \int_{\Gamma_j} \frac{\partial u_k^*}{\partial n} ds = \int_{\Gamma} u_j \frac{\partial u_k^*}{\partial n} ds = \frac{1}{i} \int_{\Gamma} u_j \Phi_k^* dz$$
$$= \iint_G \overline{\Phi}_j \Phi_k^* dx\, dy\,. \quad (5.35)$$

Thus, *the systems of functions Φ_k and Φ_j^* are biorthogonal with respect to the domain G*. It follows from (5.35) that the following relations are valid

$$a_{ik} = \sum_{j=1}^{m} a_{ij}\delta_{jk} = \iint_G \Phi_k^* \sum_{j=1}^{m} a_{ij}\Phi_j\, dx\, dy$$
$$= \iint_G \Phi_k^* \overline{\Phi_j^*}\, dx\, dy\,. \quad (5.36)$$

Now, the relations (5.28) may be written in the form

$$c_k = i \int_{\Gamma} \gamma_0 \Phi_k^*(z)\, dz \qquad (k = 1, \ldots, m)\,. \quad (5.37)$$

Hence, Problem **D** for the right-hand side of the form (5.24) has always a solution—moreover, a unique solution—if the constants c_k are determined according to the formulae (5.28) or, which is equivalent, by (5.37) $c_0 = 0$).

We now write the boundary condition (5.22) in the form

$$\mathrm{Re}[\Phi(z)] = \gamma_0(z) + c(z) \qquad (\text{on } \Gamma)\,, \quad (5.38)$$

where

$$\gamma_0(z) = \gamma(z) - c(z)\,, \quad (5.39)$$

$$c(z) = 0 \text{ on } \Gamma_0,\ c(z) = c_k = \frac{1}{i}\int_{\Gamma} \gamma \Phi_k^*(z)\, dz \ (\text{on } \Gamma_k) \quad (5.40)$$
$$(k = 1, \ldots, m)\,.$$

The function γ_0 satisfies the conditions (5.23) and consequently there exists a function $\psi_0(z)$ holomorphic in G which satisfies the boundary condition

$$\mathrm{Re}[\psi_0(z)] = \gamma_0 \quad (\text{on } \Gamma) . \tag{5.41}$$

This relation implies that

$$\gamma(z) = \psi_0(z) - i\gamma_*(z) + c(z) \quad (\text{on } \Gamma) , \tag{5.42}$$

where γ_* is the imaginary part of the function ψ_0.

Thus, *every function* γ *of the class* $C_\nu(\Gamma)$, $0 < \nu < 1$, *is representable in the form* (5.42) *whwre* $\psi_0(z)$ *is a function holomorphic in* G *and satisfying the boundary condition* (5.41), *and* $c(z)$ *is a piecewise constant function given by the formula* (5.40).

5.4. On the basis of the last result we can now give a new proof of Theorem 4.5. We have seen above (§3.3) that the boundary condition of the homogeneous Problem Å can be written in the form

$$\mathrm{Re}[z^{-n}e^{i\sigma(z)}w(z)] = 0 \quad (\text{on } \Gamma) .$$

Making now use of the representation of w in the form (4.2) the last boundary condition may be written thus:

$$\mathrm{Re}[z^{-n}e^{i\sigma_0(z)}\Phi(z)] = 0 \quad (\text{on } \Gamma) , \tag{5.43}$$

where Φ is a function holomorphic in G and $\sigma_0(z)$ is a real function continuous in the Hölder sense on Γ. Let us now represent σ_0 according to the formula (5.42), and assume that $n < 0$. Then the boundary condition (5.43) takes the form

$$\mathrm{Re}[e^{\pi i a(z)}\Phi_0(z)] = 0 , \qquad \Phi_0 = z^{-n}e^{i\psi_0(z)}\Phi(z) , \tag{5.44}$$

where a is a piecewise constant function, $a = 0$ on Γ_0, $a = a_k =$ const. on $\Gamma_k (k = 1, \ldots, m)$, and Φ_0 is a function holomorphic in G and continuous in $G + \Gamma$. Besides, $\Phi_0(0) = 0$, since according to the assumption $n < 0$. But we have seen above (§5.1) that under these conditions the boundary value problem (5.44) has no non-trivial solutions; this completes the proof of Theorem 4.5.

5.5. It is useful to introduce the following multi-valued analytic functions

$$w_k^*(z) = u_k^*(z) + iv_k^*(z)\,, \qquad k = 1, \ldots, m\,, \tag{5.45}$$

where $v_k^*(z)$ are harmonic functions conjugate to $u_k^*(z)$. They are determined by the formulae

$$v_k^*(z) = \int_{z_0}^{z} \frac{\partial u_k^*}{\partial n} ds + c_k\,, \tag{5.46}$$

where c_k are real constants and n is the normal to the integration line connecting a fixed point z_0 with the variable point z of the domain G. These functions may be called *the conjugate canonical harmonic functions of the domain.*

If, in the domain G, the point z describes a closed curve homological to the boundary contour Γ_j ($j = 1, \ldots, m$), according to (5.46) and (5.32) w_k^* acquires an increment $i\delta_{kj}$. We shall find that these functions enable us to construct a function which conformally maps the domain under consideration onto one of the canonical (multiply-connected) domains.

Let us consider the following bilinear form

$$h_0(z, \zeta) = u_1(\zeta) w_1^*(z) + \ldots + u_m(\zeta) w_m^*(z)\,. \tag{5.46a}$$

This function is analytic in z and when z describes, in the domain G, a closed curve homological to Γ_j ($j = 1, \ldots, m$) it acquires an increment equal to $iu_j(\zeta)$. Let us now consider the analytic function

$$\frac{1}{2\pi} h(z, \zeta) = g(z, \zeta) + ig_*(z, \zeta) - h_0(z, \zeta)\,, \tag{5.47}$$

where $g_*(z, \zeta)$ is a harmonic function conjugate to $g(z, \zeta)$ with respect to the point z. If z describes, in the domain G, a closed curve homological to $\Gamma_k (k = 1, \ldots, m)$ then g_*, as it was indicated above, acquires an increment equal to $u_k(\zeta)$. Therefore the function

$$h_*(z, \zeta) = \dot{h}(z, \zeta) - \ln(z - \zeta)$$

is a single-valued analytic function of z in the domain G. Hence, the function

$$\varphi(z,\zeta) = e^{h(z,\zeta)} \equiv (z-\zeta)\,e^{h_*(z,\zeta)} \tag{5.48}$$

is single-valued and analytic in G. It vanishes for $z = \zeta$ and satisfies the boundary conditions

$$\begin{aligned} \varphi &= e^{ip(z,\zeta)} \quad \text{for} \quad z \in \Gamma_0\,, \\ \varphi &= e^{ip(z,\zeta)} e^{-2\pi u_k^*(z)} \quad \text{for} \quad z \in \Gamma_k \quad (k = 1, \dots, m)\,, \end{aligned} \tag{5.49}$$

where

$$p(z,\zeta) = 2\pi[g_*(z,\zeta) - u_1(\zeta)v_1^*(z) - \dots - u_m(\zeta)v_m^*(z)]\,. \tag{5.50}$$

By means of these relations it is readily observed that the function $w = \varphi(z,\zeta)$ (ζ is a fixed point of the domain G) establishes a conformal mapping of the domain G onto the canonical domain G_0 which is bounded by the unit circle Γ_0' with centre at the origin, and by concentric arcs of circles Γ_k' the radii of which are equal respectively to $e^{-2\pi u_k^*(\zeta)}$ ($k = 1, \dots, m$). It is also evident that the origin of coordinates $w = 0$ corresponds to the point $z = \zeta$, [96].

It is also possible to obtain from the formula (5.49) a number of other conformal mappings of the domain G onto canonical domains of various types, [96].

5.6. It is of interest to consider especially the following non-homogeneous boundary value problem:

Problem $\boldsymbol{D'}$. *It is requisite to find a function $\Phi(z)$ holomorphic in G, continuous in $\bar{G}$ and satisfying the boundary condition*

$$\operatorname{Re}[e^{-\pi i\alpha(z)}\Phi(z)] = \gamma(z) \qquad (\text{on } \Gamma)\,, \tag{5.51}$$

where $\alpha(z)$ is a piecewise constant function, $\alpha = 0$ on Γ_0, $\alpha = \alpha_k = \text{const.}$ on $\Gamma_k (k = 1, \dots, m)$, and we assume that the condition (5.4) is satisfied. For instance, the following problem leads to Problem $\boldsymbol{D'}$:

It is required to find in G a solution w, $w \in C(\bar{G})$, of the equation

$$\partial_{\bar{z}} w + A(z)w = F(z) \qquad (A, F \in L_p(\bar{G})\,,\ p > 2)\,, \tag{5.52}$$

satisfying the boundary condition

$$\operatorname{Re}[w(z)] = \gamma(z) \quad (\text{on } \Gamma)\,. \tag{5.53}$$

In fact, the general solution of the equation (5.52) can be represented in the form (see Ch. III, §4.4)

$$w(z) = w_0(z) + \Phi(z)\, e^{\omega(z)}\,, \tag{5.54}$$

where

$$\omega(z) = \frac{1}{\pi} \iint\limits_G \frac{A(\zeta)}{\zeta - z}\, d\xi\, d\eta\,, \tag{5.55}$$

$$w_0(z) = -\frac{e^{\omega(z)}}{\pi} \iint\limits_G \frac{e^{-\omega(\zeta)} F(\zeta)}{\zeta - z}\, d\xi\, d\eta\,. \tag{5.56}$$

Let ω_0 and ω_1 be the real and imaginary parts of $\omega(t)$. Inserting the expression (5.54) into (5.53) we obtain

$$\operatorname{Re}[\Phi(z)\, e^{i\omega_1(z)}] = \gamma_1, \qquad \gamma_1 = \gamma e^{-\omega_0} - e^{-\omega_0} \operatorname{Re}(w_0)\,. \tag{5.57}$$

Since $\omega_1 \in C_\nu(E)$, $\nu = \dfrac{p-2}{p}$, this function may be represented according to the formula (5.42) in the form

$$\omega_1(z) = p(z) - i\omega_*(z) - \pi a(z) \quad (\text{on } \Gamma)\,, \tag{5.58}$$

where $p(z)$ is a function holomorphic in G, ω_* is its imaginary part and $a(z)$ is a piecewise constant function on Γ which, according to the formula (5.40), is uniquely expressible by ω_1, namely

$$a = 0 \text{ on } \Gamma_0, \qquad a = a_k = \frac{-1}{\pi i} \int\limits_\Gamma \omega_1 \Phi_k^*(z)\, dz \text{ on } \Gamma_k \tag{5.59}$$

$$(k = 1, \ldots, m)\,.$$

Bearing in mind that $\omega_1 = \operatorname{Im}[\omega(z)]$ and $\operatorname{Re}[\Phi_k^*(z) z'] = 0$ (on Γ) we have

$$a_k = \frac{1}{\pi} \operatorname{Im}\left\{\frac{-1}{i} \int\limits_\Gamma \omega(z)\, \Phi_k^*(z)\, dz\right\} = \frac{-2}{\pi} \operatorname{Im} \iint\limits_G \partial_{\bar{z}} \omega \Phi_k^*(z)\, dx\, dy$$

or, taking into account that $\partial_{\bar{z}}\omega = -A(z)$ we obtain

$$a_k = \frac{2}{\pi}\operatorname{Im}\iint_G A(\zeta)\Phi_k^*(\zeta)\,d\xi\,d\eta \qquad (k = 0, \ldots, m)\,. \quad (5.60)$$

Substituting now (5.58) into (5.57) we have

$$\operatorname{Re}[e^{\pi i\alpha(z)}\Phi_*(z)] = \gamma_*\,, \quad (5.61)$$

where

$$\Phi_*(z) = e^{ip(z)}\Phi(z)\,, \qquad \gamma_* = e^{-\omega_*}\gamma_1\,. \quad (5.62)$$

If $\alpha(z) = N_k$ on Γ_k where N_k are integers, then the boundary condition (5.61) reduces to the boundary condition (5.22) which was already considered in §5.3. It is of interest to notice that this case occurs in problems of infinitesimal bending of surfaces of the second order, of positive curvature. We shall see in Ch. V (§3.5, §4, §6.6) that in this case the fundamental equations of the problem are reduced to the form (5.52) where

$$A(z) = -\partial_{\bar{z}}\ln\sqrt{a\sqrt{\mathrm{K}}}\,, \qquad F = 0\,. \quad (5.63)$$

Here K is the principal curvature of the surface and a is the determinant of the first fundamental quadratic form with respect to the isometric—conjugate coordinate system (Ch. II, §6.3). Hence, in view of the formulae (5.60), (5.34)

$$\begin{aligned} -a_k &= \frac{2}{\pi}\operatorname{Im}\iint_G \frac{\partial\ln\sqrt{a\sqrt{\mathrm{K}}}}{\partial\bar{z}}\Phi_k^*(z)\,dx\,dy \\ &= \frac{2}{\pi}\operatorname{Im}\frac{1}{2i}\int_\Gamma \ln\sqrt{a\sqrt{\mathrm{K}}}\,\Phi_k^*(z)\,dz \\ &= \frac{1}{\pi}\operatorname{Im}\int_\Gamma \ln\sqrt{a\sqrt{\mathrm{K}}}\frac{\partial u_k^*}{\partial n}\,ds = 0\,. \end{aligned}$$

We now return to the general case, when the condition (5.4) is satisfied. Then, according to Theorem 4.2, the problem (5.51) has a solution if and only if the following relations hold:

$$\int_L \gamma\psi_k(z)\,dz = 0 \qquad (k = 1, \ldots, m-1)\,. \quad (5.64)$$

where $\psi_1, \ldots, \psi_{m-1}$ is the complete system of solutions of the adjoint homogeneous problem (5.7). We have already seen that this problem has $m-1$ solutions. It is important to note that Problem **D′** has a unique solution, since we found in §5.1 that the corresponding homogeneous problem has no solution.

In particular, for $m = 1$ the conditions (5.64) are absent Hence, Problem **D′** in the case of a double-connected domain is always soluble. This is a significant result which does not occur in the case of Problem **D**.

Thus, *in the case of a double-connected domain the problem of the determination of a solution of the equation* (5.52), *according to the boundary condition of the form* (5.53), *has always a solution and the solution is unique if the coefficient A satisfies the condition*

$$\operatorname{Im} \iint_G A(\zeta)\Phi_1^*(\zeta)\,d\xi\,d\eta \neq \frac{\pi}{2}N \qquad (N\text{—an integer}). \tag{5.65}$$

We observe that the equation (5.52) is the complex form of the elliptic system

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + au + bv = f, \qquad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - bu + av = g.$$

5.7. We now consider the general homogeneous boundary value problem of Riemann–Hilbert, in the class of analytic functions

$$\operatorname{Re}[\overline{\lambda(z)}\Phi(z)] = 0 \qquad (\text{on } \Gamma). \tag{5.66}$$

In view of the formulae (3.6) and (3.7) this boundary condition may be written in the form

$$\operatorname{Re}[\overline{\Omega_n(z)}e^{-i\sigma(z)}\Phi_0(z)] = 0 \qquad (\text{on } \Gamma), \tag{5.67}$$

where

$$\left.\begin{aligned} \sigma(z) &= \arg\lambda(z) - \arg\Omega_n(z) + \sum_{k=1}^{n} n_k \arg(z-z_k), \\ \Phi_0(z) &= \Phi(z)\prod_{k=1}^{m}(z-z_k)^{n_k}. \end{aligned}\right\} \tag{5.68}$$

Representing now the function $\sigma(z)$ according to the relation (5.42) in the form

$$\sigma(z) = q(z) - i\sigma_*(z) + \pi a(z) ,$$

where $q(z)$ is a function holomorphic in G and $a(z)$ a piecewise constant function of the form

$$a = 0 \text{ on } \Gamma_0, \qquad a = a_k = \frac{1}{\pi i}\int_\Gamma \sigma(z)\Phi_k^*(z)\,dz \text{ on } \Gamma_k, \tag{5.69}$$
$$(k = 1, \ldots, m)$$

the boundary condition (5.66) takes the form

$$\operatorname{Re}[\overline{\Omega_n(z)}\,e^{-\pi i a(z)}\Phi_*(z)] = 0 \qquad (\text{on } \Gamma) , \tag{5.70}$$

where

$$\Phi_*(z) = e^{-iq(z)}\Phi_0(z) .$$

The piecewise constant function $a(z)$ depends not only on the function $\lambda(z)$ but also on the choice of the points $a_1, \ldots, a_n$ of the domain G. Considering these points for the time being as variable we have (in view of (5.69) and (5.68))

$$\frac{\partial a_k}{\partial a_j} = \frac{1}{\pi i}\int_\Gamma \frac{\partial\sigma}{\partial a_j}\Phi_k^*(z)\,dz = \frac{-1}{2\pi}\int_\Gamma \frac{\Phi_k^*(z)}{z-a_j}\,dz = \frac{1}{i}\Phi_k^*(a_j) ,$$

or according to (5.34)

$$\frac{\partial u_k^*}{\partial a_j} - \frac{i}{2}\frac{\partial a_k}{\partial a_j} = 0 \qquad (j = 1, \ldots, n) . \tag{5.71}$$

The last result indicates that u_k^* and $\frac{1}{2}a_k$ are conjugate harmonic functions with respect to every point a_j. Consequently,

$$a_k = a_k^0 + 2\sum_{j=1}^{n} v_k^*(a_j) \qquad (k = 1, \ldots, m) , \tag{5.72}$$

where v_k^* is the harmonic function conjugate to u_k^* and a_k^0 are fixed real constants. Therefore when the points a_j describe a closed curve situated in the domain G, in

view of the relations (5.72) α_k acquires an increment equal to an even number or to zero. Hence $e^{\pi i\alpha(z)}$ is a single-valued function of the points $a_1, \dots, a_n$.

By a similar device it can be proved that the constants α_k are independent of the choice of the points $z_1, \dots, z_m$ lying outside G and appearing in the expression (5.68).

If $a_1 = \dots = a_n = a$ the relations (5.72) take the form

$$\alpha_k(a) = \alpha_k^0 + 2n\overset{*}{v}_k(a) \qquad (k = i, \dots, m) . \tag{5.73}$$

We can now indicate conditions under which the boundary value problem (5.66) is reduced to the simpler form

$$\operatorname{Re}[\overline{\Omega_n(z)}\Phi_k(z)] = 0 . \tag{5.74}$$

It is evident that this fact takes place if and only if $\alpha(z)$ is an integer on every contour $\Gamma_k (k = 1, \dots, m)$. In order that this result holds it is necessary and sufficient that there exist such points $a_1, \dots, a_n$ in the domain G for which the following relations are valid:

$$\alpha_k^0 + 2\sum_{j=1}^{n} \overset{*}{v}_k(a_j) = N_k \qquad (k = 1, \dots, m) , \tag{5.75}$$

where N_k are integers. Since the numbers α_k^0 depend on the arbitrarily chosen function λ, for the given points a_j a function $\lambda(z)$ can always be indicated such that the relations (5.75) hold.

In particular, if there exists a fixed point a of the domain G such that

$$\alpha_k^0 + 2n\overset{*}{v}_k(a) = N_k \qquad (k = 1, \dots, m) , \tag{5.76}$$

then the boundary condition (5.66) reduces to the equivalent form

$$\operatorname{Re}[(z-a)^{-n}\Phi(z)] = 0 \qquad (\text{on } \Gamma) . \tag{5.77}$$

5.8. We shall now give one more simple example showing that Problem **A** can have $n+1$ linearly independent

solutions if $0 \leqslant n \leqslant m-1$, [141]. Below, in §6 we shall find that this number is the maximum number. Let us assume that there exists a function $\zeta = \psi(z)$ which maps conformally the domain G onto the ζ-plane with cuts $a_0b_0, a_1b_1, \ldots, a_mb_m$ along the real axis, and mapping the point $z = a$ onto the point $\zeta = \infty$. In other words, near the point a the function $\psi(z)$ has the expansion

$$\psi(z) = \frac{1}{z-a}\left(b_0 + b_1(z-a) + \ldots\right), \qquad b_0 \neq 0 \,. \tag{5.78}$$

Besides, on the contours Γ_j the function $\psi(z)$ takes real values, since according to the assumption these contours are mapped onto the segments of the real axis. Therefore $\psi(z)$ satisfies the boundary condition

$$\mathrm{Re}[i\psi(z)] = 0 \qquad (\text{on } \Gamma) \,. \tag{5.79}$$

It should also be borne in mind that $\psi(z)$ is continuous in the closed domain G.

It is readily seen that under these assumptions the functions

$$\begin{gathered} i(z-a)^n, \qquad i(z-a)^n\psi(z), \\ i(z-a)^n\psi^2(z), \ldots, i(z-a)^n\psi^n(z) \end{gathered} \tag{5.80}$$

are linearly independent solutions of the problem (5.77). Let us denote this problem by $\mathbf{A}_n(a)$ and the number of its solutions by l_n. Evidently, $l_n \geqslant n+1$. Let us now observe that if Φ_n is a solution of the problem $\mathbf{A}_n(a)$, then $(z-a)\Phi_n$ is a solution of the problem $\mathbf{A}_{n+1}(a)$. But the problem $\mathbf{A}_{n+1}(a)$ has also a solution given by the function $i^n(z-a)^{n+1}\psi^{n+1}(z)$ which does not vanish at the point $z = a$, and consequently cannot be represented in the form $(z-a)\Phi_n$. Hence, $l_n < l_{n+1}$. Since $l_0 = 1$ and $l_m = m+1$ (the last result follows from Theorem 4.10) we have

$$1 \leqslant n+1 \leqslant l_n < l_{n+1} \leqslant m+1 \qquad (n = 0, 1, \ldots, m-1) \,.$$

This at once implies that

$$l_n = n+1, \quad 0 \leqslant n \leqslant m-1 \,. \tag{5.81}$$

Accordingly, the number of solutions of the adjoint homogeneous problem $\mathring{A}'_n(a)$ the boundary condition of which has the form

$$\operatorname{Re}\left[(z-a)^n \frac{dz}{ds}\Phi(z)\right] = 0 \quad (\text{on } \Gamma), \tag{5.82}$$

is equal to

$$l'_n = m-n. \tag{5.83}$$

5.9. We now consider the boundary value problem of the form (5.77), for the case of an arbitrary (multiply-connected) domain G. The solution of this problem may be sought in the form

$$\Phi(z) = -c_n - c_{n-1}(z-a) - \ldots - c_1(z-a)^{n-1} + \\ + (z-a)^n\Phi_0(z), \tag{5.84}$$

where $c_1, \ldots, c_n$ are unknown complex constants and Φ_0 is a new unknown holomorphic function. Then for Φ_0 we have the boundary condition

$$\operatorname{Re}[\Phi_0(z)] = \operatorname{Re}\sum_{k=1}^{n} c_k(z-a)^{-k}. \tag{5.85}$$

According to the relations (5.23) the condition of solubility of this problem has the form

$$\operatorname{Re}\sum_{k=1}^{n} c_k \frac{1}{2\pi i}\int_{\Gamma} \frac{\Phi^j(z)}{(z-a)^k}dz \equiv \operatorname{Re}\sum_{k=1}^{n}\frac{c_k}{(k-1)!}\Phi_j^{(k-1)}(a) = 0$$
$$(j = 1, \ldots, m).$$

Thus, we have the following system of equations:

$$c'_1\Phi_j(a) + \bar{c}'_1\overline{\Phi_j(a)} + \ldots + c'_n\Phi_j^{(n-1)}(a) + \bar{c}'_n\overline{\Phi_j^{(n-1)}(a)} = 0$$
$$(j = 1, \ldots, m), \tag{5.86}$$

where

$$c'_k = \frac{c^k}{(k-1)!} \quad (k = 1, \ldots, n).$$

We have seen before (§5.3) that the functions $\Phi_k(z)$ depend only on the domain. Therefore, the matrices of the coefficients of the system (5.86) depend only on the domain G and on the choice of the point a. Consequently, the rank r of this matrix depends on the domain G and on the point a, i.e. $r = r(a, G)$.

Since we assume that $2n < m$ we have evidently

$$0 < r \leqslant 2n. \tag{5.87}$$

The boundary value problem (5.85) has $2n-r+1$ linearly independent solutions. Consequently, in view of the formula (5.84), the number of solutions of the boundary value problem (5.77) is given by the formula

$$l = 2n-r+1\,. \tag{5.87a}$$

Let ν be a number satisfying the inequality (5.87). We shall denote by $M_\nu(a, G)$ the set of points $a \in G$ for which the rank of the matrix of the coefficients of the system (5.86) is equal to ν.

We found in the previous subsection that there exist domains G for which $M_n(a, G)$ is non-empty. Now we shall prove that for every domain $GM_{2n}(a, G)(2_n < m)$ is an open set everywhere dense in G.

Let us first prove the following lemma:

LEMMA 1. *Let $f_1(z), \ldots, f_n(z)$ be functions holomorphic in G. Let $\Delta_n(z)$ be a determinant of the n-th degree the k-th row of which has the elements*

$$f_k, \bar{f}_k, f'_k, \bar{f}'_k, \ldots, f_k^{\left(\frac{n-2}{2}\right)}, \overline{f_k^{\left(\frac{n-2}{2}\right)}} \tag{5.88}$$

if n is an even number, and

$$f_k, \bar{f}_k, \ldots, f_k^{\left(\frac{n-3}{2}\right)}, \overline{f_k^{\left(\frac{n-3}{2}\right)}}, f_k^{\left(\frac{n-1}{2}\right)} \tag{5.89}$$

if n is odd. In order that the system of functions $f_1, \ldots, f_n$ holomorphic in the domain G be linearly independent over the field of real numbers it is necessary and sufficient

that there exists at least one point z_0 of the domain G, such that

$$\Delta_n(z_0) \neq 0 . \tag{5.90}$$

PROOF. The sufficiency can be proved very simply. If $f_1, \ldots, f_n$ are linearly independent, then

$$c_1 f_1(z) + \ldots + c_n f_n(z) \equiv 0 , \tag{5.91}$$

where $c_1, \ldots, c_n$ are some real constants, and $c_1^2 + \ldots + c_n^2 > 0$. Differentiating successively $\left[\frac{n-2}{2}\right]$ times the relation (5.91) and taking into consideration the complex conjugate relations we obtain a linear homogeneous system of equations for $c_1, \ldots, c_n$, its determinant being $\Delta_n(z)$, and if $\Delta_n(z) \neq 0$ at at least one point, then obviously all c_j vanish. Thus the sufficiency of the condition (5.90) is proved.

Proceding to the proof of the necessity of this condition we first examine the case $n = 2$. If

$$\Delta_2(z) \equiv f_1(z)\overline{f_2(z)} - \overline{f_1(z)} f_2(z) \equiv 0 \quad (\text{in } G) ,$$

the system of equations

$$c_1 f_1(z) + c_2 f_2(z) = 0 , \quad c_1 \bar{f}_1(z) + c_2 \bar{f}_2(z) = 0$$

has a non-trivial real solution c_1, c_2 and in the vicinity of points at which $f_1(z) \neq 0$ c_2 obviously cannot vanish identically. Therefore, we have

$$a f_1(z) + f_2(z) \equiv 0 , \qquad a = \frac{c_1}{c_2} . \tag{5.92}$$

Differentiating both sides of the above relation with respect to $\bar{z}$ we obtain

$$a_{\bar{z}} f_1(z) \equiv 0 , \quad \text{i.e.} \quad a_{\bar{z}} = 0 .$$

Since a is a real quantity it follows that $a = \text{const}$. Consequently, in view of (5.92) the functions f_1 and f_2 are linearly independent, which contradicts our assumption.

In order to prove the lemma for an arbitrary n it is sufficient to show that if the lemma holds for k functions

$f_1, \ldots, f_k, k < n$, then it also holds for $k+1$ functions $f_1, \ldots, f_k, f_{k+1}$. Let us assume that this is not the case, i.e. there exists a neighbourhod G_0 of the point z_0 in which

$$\Delta_k(z) \neq 0, \quad a\Delta_{k+1}(z) \equiv 0 \quad (\text{in } G_0).$$

Let us first assume that k is even. Then, according to the last relation the system of equations

$$\sum_{i=1}^{k+1} c_i f_i^{(j)}(z) = 0, \quad \sum_{i=1}^{k+1} c_i \overline{f_i^{(j)}(z)} = 0 \tag{5.93}$$

$$\left(j = 1, \ldots, \frac{k-2}{2}\right),$$

$$\sum_{i=1}^{k+1} c_i f_i^{(k/2)}(z) = 0 \tag{5.94}$$

has a non-trivial solution $c_1, \ldots, c_k, c_{k+1}$, and c_{k+1} cannot vanish anywhere in G_0. The last fact follows immediately from the condition $\Delta_k(z) \neq 0$ in G_0.

Dividing all relations (5.93) and (5.94) by c_{k+1} we obtain

$$\sum_{i=1}^{k} a_i f_i^{(j)}(z) = -f_{k+1}^{(j)}(z), \quad \sum_{k=1}^{k} a_i \overline{f_i^{(j)}(z)} = -\overline{f_{k+1}^{(j)}(z)} \tag{5.95}$$

$$\left(j = 0, 1, \ldots, \frac{k-2}{2}\right)$$

$$\sum_{i=1}^{k} a_i f_i^{(k/2)}(z) + f_{k+1}^{(k/2)}(z) = 0, \quad a_i = \frac{c_i}{c_{k+1}}. \tag{5.96}$$

Since $\Delta_k(z) \neq 0$ the system (5.95) has a unique non-trivial real solution $a_1, \ldots, a_k$ satisfying the relation (5.96). Differentaiting the relations (5.95) with respect to $\bar{z}$ and taking into account the same relations and also (5.96), we obtain

$$\sum_{i=1}^{k} \frac{\partial a_i}{\partial z} f_i^{(j)}(z) = 0, \quad \sum_{i=1}^{k} \frac{\partial a_i}{\partial \bar{z}} \overline{f_i^{(j)}(z)} = 0$$

$$\left(j = 1, \ldots, \frac{k-2}{2}\right).$$

But the determinant of this system is equal to $\Delta_k(z)$ which according to the asumption does not vanish in G_0. Hence,

$$\frac{\partial a_i}{\partial \bar{z}} \equiv 0, \quad \text{i.e.} \quad a_i = \text{const.} \qquad (i = 1, \ldots, k).$$

This means in view of the relation (5.95) that the functions $f_1, f_2, \ldots, f_k, f_{k+1}$ are linearly independent, which contradicts the assumption of the lemma. It is easy to carry out a similar proof for an odd k. Thus, the lemma will be proved in the most general case.

Let us now return to the boundary value Problem $\mathring{\mathbf{A}}_n(a)$:

$$\operatorname{Re}[(z-a)^{-n}\Phi(z)] = 0 \quad (\text{on } \Gamma) \tag{5.77}$$

and let us assume that the point a is so chosen that the determinant $\Delta_{2n}(a)$ of the system of functions $\Phi_1(z), \ldots, \Phi_{2n}(z)$ does not vanish. Such being the case, the system (5.86) has evidently only the trivial solution $c_1 = 0, \ldots, c_n = 0\,(2n < m)$. Consequently, the problem (5.85) has a unique solution $\Phi_0 = ic$ (c is a real constant) and according to the formula (5.84) the boundary value problem (5.77) has the unique solution

$$\Phi = ic(z-a)^n.$$

Thus, in the case under consideration the problem $\mathbf{A}_n(a)$ has the minimum number of solutions, i.e. $l = 1$ and (5.87a) implies that $r = 2n$. The set of points a for which this takes place is open and everywhere dense in G. Hence, the Problem $\mathring{\mathbf{A}}_n(a)$ may have more than one solution only in exceptional cases—when the point a and perhaps the domain G are chosen in a special way. These cases are possible; it follows from the example of the previous subsection.

5.10. We now consider the boundary condition (5.74) which can be written in the following form (Problem $\mathbf{A}(a_1, \ldots, a_n)$):

$$\operatorname{Re}\left[\frac{\Phi(z)}{\Omega_n(z)}\right] = 0, \qquad \Omega_n(z) = \prod_{k=1}^{n} (z-a_k), \tag{5.97}$$

where Φ is the unknown function, holomorphic in G. This function is representable in the form (we assume that $a_i \neq a_k$ for $i \neq k$)

$$\frac{\Phi(z)}{\Omega_n(z)} = -\sum_{k=1}^{n} \frac{c_k}{z-a_k} + \Phi_0(z)\,. \tag{5.98}$$

Then we have the following boundary condition for Φ_0:

$$\mathrm{Re}[\Phi_0(z)] = \mathrm{Re}\sum_{k=1}^{n} \frac{c_k}{z-a_k}\,. \tag{5.99}$$

According to the relations (5.23) the condition of solubility of this problems has the form

$$\mathrm{Re}\sum_{k=1}^{n} c_k\Phi_j(a_k) = 0 \qquad (j = 1, \dots, m)\,. \tag{5.100}$$

Let r be the rank of the matrix of the coefficients of this system. It is evident that this rank depends only on the choice of the points $a_1, \dots, a_n$ and the domain G, i.e. $r = r(a_1, \dots, a_n, G)$, and

$$0 \leqslant r(a_1, \dots, a_n, G) \leqslant 2n\,. \tag{5.101}$$

The number of solutions of the homogeneous problem (5.97) is evidently equal to $2n+1-r$. Let us denote by $M_r(a_1, \dots, a_n, G)$ the set of points $a_1, \dots, a_n$ of the domain G for which the condition (5.101) is satisfied. It is now important to find out whether there exists for every fixed r, $0 \leqslant r \leqslant 2n$, a domain G for which the set $M_r(a_1, \dots, a_n, G)$ is non-empty. We shall later find that $M_{2n}(a_1, \dots, a_n, G)$ is non-empty for any domain G. Moreover, this set is open and everywhere dense in G.

We shall now prove the following lemma:

LEMMA 2. *Let $f_1(z), \dots, f_n(z)$ be functions holomorphic in G. Let $\Delta_n(z_1, \dots, z_{n'})$, $n' = \left[\frac{n+1}{2}\right]$, be a determinant*

whose kth row has the elements

$$f_k(z)_1, \quad \overline{f_k(z_1)}, \dots, f_k(z_{n'}), \quad f_k z_{n'}) ,$$

if n is an even number and

$$f_k(z_1), \quad \overline{f_k(z_1)}, \dots, f_k(z_{n'-1}), \quad \overline{f_k(z_{n'-1})}, \quad f_k(z_{n'}) ,$$

if n is odd. The functions $f_1, \dots, f_n$ are linearly independent over the field of real numbers if and only if there exists at least one system of points $z_1, \dots, z_{n'}$, of the domain G for which

$$\Delta_n(z_1, \dots, z_{n'}) \neq 0 . \tag{5.102}$$

PROOF. The sufficiency of this condition can be proved exactly as for the condition (5.90), we shall therefore omit the reasoning. The necessity of the condition (5.102) for $n = 2$ is obvious, since in this case the lemmas 1 and 2 are identical. Consequently, in order to prove the lemma for an arbitrary $n > 2$ it is sufficient to show that if the lemma holds for the functions $f_1, \dots, f_k$ it is also valid for the functions $f_1, \dots, f_k, f_{k+1} (k \leqslant n-1)$. Let us assume that this is not the case, i.e. that there exists a system of points $z_1, \dots, z_k$, for which

$$\Delta_k(z_1, \dots, z_{k'}) \neq 0, \quad k' = \left[\frac{k+1}{2}\right], \tag{5.103}$$

while for any point z we have the relation

$$\Delta_{k+1}(z_1, \dots, z_{k'}, z) = 0 .$$

In view of the last relation the system of equations

$$\begin{gathered}\sum_{i=1}^{k+1} c_i f_i(z_j) = 0, \quad \sum_{i=1}^{k+1} c_i \overline{f_i(z_j)} = 0 \quad (j = 1, \dots, k') , \\ \sum_{i=1}^{k+1} c_i f_i(z_j) = 0\end{gathered} \tag{5.104}$$

has a non-trivial solution $c_1, \ldots, c_k, c_{k+1}$, and it is readily observed that c_{k+1} vanishes nowhere in the domain G. Hence, dividing all relations (5.104) by c_{k+1} we obtain

$$\sum_{i=1}^{k} a_i f_i(z_j) = -f_{k+1}(z_j), \qquad \sum_{i=1}^{k} a_i \overline{f_i(z_j)} = -\overline{f_{k+1}(z)} \quad (5.105)$$
$$(j = 1, \ldots, k'),$$

$$\sum_{i=1}^{k} a_i f_i(z) + f_{k+1}(z) = 0, \qquad a_i = \frac{c_i}{c_{k+1}}. \qquad (5.106)$$

In view of (5.103) the system (5.105) has a unique solution $a_1, \ldots, a_k$, a_j being real and depending only on $z_1, \ldots, z_{k'}$; they are independent of z and satisfy the relation (5.106) for an arbitrary value of z. Thus, $f_1, \ldots, f_k, f_{k+1}$ are linearly independent, which contradicts our assumption.

Obviously, the set of points for which the condition (5.102) is satisfied is everywhere dense and open in the set of all possible systems $(z_1, \ldots, z_{n'})$, $z_i \in G$, i.e. in the product $\underbrace{G \times G \times \ldots \times G}_{n'}$. In other words, there does not exist an open subset $G \times G \times \ldots \times G$ all points of which satisfy the condition

$$\Delta_n(z_1, \ldots, z_{n'}) \equiv 0 .$$

Let us now return to the problem (5.97) and let us assume that the points $a_1, \ldots, a_n$ are so chosen that the determinant $\Delta_n(a_1, \ldots, a_n)$ for the functions $\Phi_1, \ldots, \Phi_{2n-1}$ does not vanish. Since according to the assumption, $2n-1 < m$ the functions $\Phi_1, \ldots, \Phi_{2n-1}$ are linearly independent; this result is possible in view of Lemma 2. Such being the case, the system of equations (5.100) has only the trivial solution $c_1 = \ldots = c_n = 0$ and, consequently, the problem (5.97) has only solutions of the form

$$\Phi = ic\Omega_n(z) \qquad (c\text{—real constant}).$$

Thus, in the case under consideration Problem $\mathbf{A}(a_1, \ldots, a_n)$ has the minimum number of solutions, i.e. $l = 1$ and, consequently, $r = 2n$. Moreover, it is evident that the set $M_{2n}(a_1, \ldots, a_n, G)$ for which this result holds is open and everywhere dense in $G \times G \times \ldots \times G$. This fact proves that the homogeneous boundary value problem (10.97) can have more than one solution only in special cases, namely when the system of points $a_1, \ldots, a_n$ and perhaps the domain G are chosen in a special way. It is evident, however, that such cases are possible. *

§6. On the conditions of correctness of Problem A

6.1. It is usually said that *a boundary value problem is correct with respect to a group of parameters entering the statement of the problem if its solution always exists, is unique and varies continuously as the parameters are continuously varied.* It should be observed that the term "parameter" is used in a very wide sense. A parameter can be a quantity of any nature on which the solution of the problem depends (e.g. constant parameters, coefficients and right-hand sides of the equation and the boundary condition, the boundary of the domain, etc.). Moreover, the continuous change and continuous dependence can be understood in many ways. It can be considered in the sense of the ordinary Euclidean metric or the metric of the space C; in other cases—in the sense of metrics of various Banach spaces or in the sense of convergence in various topological spaces.

In particular, for Problem **A** it is sensible to investigate problems of correctness both with respect to the coefficients and the right-hand sides of the equation and the boundary condition, and with respect to some continuous (small) deformations of the boundary Γ of the domain G.

* During the preparation of this paragraph B. Bojarski became interested in these problems and obtained a number of further results on the topic, [11g]; they are presented in an Appendix to the present Chapter.

In this section we shall consider some criteria of correctness of Problem **A** with respect to the right-hand sides of the equation and the boundary condition. We shall return to the problem of correctness of Problem **A** with respect to other data in §7.4, Ch. V, §10.5, and in the Appendix to Ch. IV.

In the following subsection of this section we shall indicate conditions the addition of which to the boundary value problem

$$\mathfrak{C}(w) \equiv \partial_{\bar{z}} w + Aw + B\overline{w} = F(z) \quad (\text{in } G)\,, \tag{6.1}$$

$$\operatorname{Re}[\overline{\lambda(z)}\, w(z)] = \gamma(z) \quad (\text{on } \Gamma) \tag{6.2}$$

ensures the existence and uniqueness of the solution of the modified problem, and the continuous dependence on the data of the problem.

6.2. It is evident that Problem **A** can be correct with respect to the functions F and γ only if the homogeneous Problem $\mathring{\mathbf{A}}$ has no solution and the non-homogeneous Problem **A** is always soluble. The last fact occurs only if the adjoint Problem $\mathring{\mathbf{A}}'$ has no non-trivial solutions). In this case, however, according to Theorem 4.3 Problem **A** has the unique solution w which is representable by the formula

$$w(z) = \int_{\Gamma} \mathbf{M}(z, \zeta)\gamma(\zeta)\,ds + \iint_{G} \big(\mathbf{M}_1(z, \zeta)F(\zeta) + \mathbf{M}_2(z, \zeta)\overline{F(\zeta)}\big)\,d\xi\,d\eta\,, \tag{6.3}$$

where $\mathbf{M}, \mathbf{M}_1$ and $\mathbf{M}_2$ are functions uniquely determined by means of the coefficients of the equation (6.1) and the boundary condition (6.2), but are entirely independent of the right-hand sides F and γ. The right-hand side of the relation (6.3) is a linear operator associating with every pair of functions F and γ ($F \in L_p(\bar{G})$, $p > 2$, $\gamma \in C_\nu(\Gamma)$, $0 < \nu < 1$) a solution of Problem **A** belonging to the class $C_\sigma(G+\Gamma)$, $\sigma = \min\left(\nu, \dfrac{p-2}{p}\right)$. Consequently, if the condition $l = l' = 0$ is satisfied Problem **A** is correct with

respect to the functions F and γ in the classes $L_p(\bar{G})$, $p > 2$, and $C_\nu(\Gamma)$, $0 < \nu < 1$, respectively.

It follows from (4.24) that in this case we have

$$m = 2n + 1 , \tag{6.4}$$

i.e. *Problem* **A** *may be correct with respect to* γ *and* F *only for domains of an even connectedness* ($m = 1, 3, \ldots$). The last condition was first indicated to me by Bojarski, [11a, a′].

Theorems 4.10 and 4.12 imply that Problem **A** is certainly incorrect if $n < 0$ or $n > m-1$. In the first case ($n < 0$) *Problem* **A** *is not always soluble but when a solution exists, it is unique.* In the second case ($n > m-1$) *Problem* **A** *is always soluble but the solution is not unique.* Thus, in general, Problem **A** is not correctly formulated. It is therefore important to examine the necessary modifications of the statement of Problem **A** which would ensure the existence and uniqueness of the solution and in a certain sense the continuous dependence of the solution on the data of the problem. These problems are of a great interest, since many problems of geometry and mechanics reduce to Problem **A** (see Ch. V and VI).

The modifications of the formulation of Problem **A** in the sense indicated above can be carried out in various ways. For instance, for $n > m-1$, by adding to the boundary condition (6.2) the relations

$$l_j(w) = c_j \quad (j = 1, \ldots, k) , \tag{6.5}$$

where l_j are some homogeneous additive functionals in w and c_j are constants, it may be shown that the modified problem is soluble and the solution is unique. It is, for instance, sufficient to take functionals which satisfy the conditions

$$l_j(w_i) = \delta_{ji} \quad (\delta_{ii} = 1, \delta_{ji} = 0, j \neq i) , \tag{6.6}$$

where $w_i (i = 1, 2, \ldots, 2n+1-m)$ is the complete system of solutions of the homogeneous Problem **Å**. Such functionals, however, are inconvenient in parctice because

their structure depends on the solutions of the homogeneous problem. Of a greater interest from the practical point of view are functionals l_j which are entirely independent of the data of the problem and hence of the solutions of the corresponding homogeneous problem. Such functionals in the case of simply-connected domains were indicated by Joachim Nitsche [64a] and in the case of multiply-connected domains by Schmidt, [97a]. The conditions given by Nitsche, however, are of a very special nature and are hardly subject to a geometric or mechanical interpretation.

Below we shall indicate some other ways of modifying the formulation of Problem **A**, which suit the geometric and mechanical nature of the problem better.

6.3. We shall prove in this subsection that we may take for the additional conditions (6.5) relations specifying the values of the unknown solution on a certain finite set of points of the domain and its boundary.

Let $z_1, \ldots, z_k$ and $z'_1, \ldots, z'_{k'}$ be arbitrarily fixed points of the domain G and its boundary Γ, respectively, while the following conditions are satisfied. (1) the numbers k and k' satisfy the relation

$$2k+k' = 2n+1-m \quad (n > m-1); \tag{6.7}$$

(2) there are m curves, e.g. $\Gamma\,\Gamma_1, \ldots, \Gamma_m$, among $(m+1)$ boundary curves $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$, on every of which an *odd number* of fixed points z'_j are situated.

The system of fixed points of the domain G and its boundary Γ satisfying these conditions will be called *the normally distributed* $(k, k', G+\Gamma)$ *set.* In the choice of k and k' the following extreme cases are possible: (1) $k = 0$, $k' = 2n+1-m$ and (2) $k = n-m$, $k' = m+1$. Consequently, $0 \leqslant k \leqslant n-m$, $m+1 \leqslant k' \leqslant 2n+1-m$.

Let us now specify on this set the values of the unknown solution of Problem **A**:

$$\begin{aligned} w(z_j) &= a_j+ib_j \qquad (j = 1, \ldots, k)\,, \\ w(z'_j) &= \lambda_j(\gamma_j+ic_j) \quad (j = 1, \ldots, k')\,, \end{aligned} \tag{6.8}$$

where a_j, b_j and c_j are arbitrarily specified real constants, $\lambda_j = \lambda(z'_j)$ and $\gamma_j = \gamma(z'_j)$. The relations (6.8) concerning the boundary points z'_j are of course in accordance with the boundary condition of Problem **A**:

$$\mathrm{Re}[\bar{\lambda}_j w(z_j)] = \gamma_j \qquad (j = 1, \ldots, k') .$$

We shall now prove that *Problem* **A** *with the additional conditions* (6.8) *has always a unique solution.*

According to Theorem 4.12 the general solution of the problem (when $m < n+1$) has the form

$$w(z) = w_0(z) + \sum_{j=1}^{2n+1-m} d_i w_j(z) , \tag{6.9}$$

where w_0 is a particular solution of the non-homogeneous Problem **A**, d_j are real arbitrary constants, w_j is the complete system of solutions of the homogeneous Problem **Å**. If this solution is subjected to the additional conditions (6.8) we obtain the following linear system of equations for the determination of the constants d_j

$$\begin{aligned} &\sum_{i=1}^{2n+1-m} d_i w_i(z_j) = a_j + ib_j - w_0(z_j) \qquad (j = 1, \ldots, k) , \\ &\sum_{i=1}^{2n+1-m} d_i w_i(z'_j) = \lambda_j(\gamma_j + ic_j) - w_0(z'_j) \quad (j = 1, \ldots, k') . \end{aligned} \tag{6.10}$$

We prove that this system has a solution for an arbitrary right-hand side. Let d_j^0 be a non-trivial solution of the corresponding homogeneous system. Then the function

$$w_*(z) = \sum_{i=1}^{2n+1-m} d_i^0 w_i(z) \tag{6.11}$$

is a solution of the homogeneous Problem **Å** and, moreover, it satisfies the homogeneous point conditions

$$\begin{aligned} w_*(z_j) &= 0 \qquad (j = 1, \ldots, k) , \\ w_*(z'_j) &= 0 \qquad (j = 1, \ldots, k') . \end{aligned} \tag{6.12}$$

This means that the interior points $z_1, \ldots, z_k$ and the boundary points $z'_1, \ldots, z'_k$ are zeros of $w_*(z)$. According to the assumption, there are on every of m boundary contours $\Gamma_1, \ldots, \Gamma_m$ odd numbers of points of the set $z'_1, \ldots, z'_k$. Now, according to Theorem 4.7 on every boundary contour there may be situated only an even number of zeros of the solution of the homogeneous Problem Å. Therefore, on every one of these contours there will be at least one more zero of the function $w_*(z)$. Consequently, $w_*(z)$ has at least $N_\Gamma = k' + m$ boundary and $N_G = k$ interior zeros. But these numbers in view of (6.7) satisfy the relation

$$2N_G + N_\Gamma = 2n+1\ ,$$

which contradicts Theorem 4.7. It follows that $w_*(z) \equiv 0$ and, consequently, $d_j^0 = 0$ $(j = 1, \ldots, 2n+1-m)$. Thus, the system (6.10) has a solution for an arbitrary right-hand side. We have therefore proved

THEOREM 4.13. *If on a normally distributed $(k, k', G + + \Gamma)$ point set the conditions of the form* (6.8) *are given, then for $n > m-1$ Problem* Å *has always a solution satisfying these conditions, this solution is uniquely determined and depends continously on F and γ. It also depends continuously on the points z_j and z'_j and is a linear function of the constants $a_j, b_j (j = 1, \ldots, k)$ and $c_j (j = 1, \ldots, k')$.*

In subsequent chapters the geometric and mechanical nature of the conditions (6.8) will be elucidated.

6.4. We now consider the case $n \leqslant m-1$. Then Problem **A** in general has not always a solution in the class of continuous functions. Therefore, the following natural question arises: in these cases can the existence and uniqueness of the solution of Problem **A** be ensured by an extension of the class of the unknown solution? We shall prove below that this can be done if the solution of the problem is sought in the class of functions possessing a number of prescribed poles inside the domain G. In Chapters V and VI the mechanical and geometrical

interpretation of such a formulation of the problem will be shown.

Let $z_1, \ldots, z_k$ be some fixed points inside the domain G, where $k = m - n$. We shall seek the solution of Problem **A** in the class of functions of the form

$$w(z) = Q(z)\tilde{w}(z), \qquad Q(z) = \prod_{j=1}^{k} (z - z_j)^{-1}, \tag{6.13}$$

where $\tilde{w}(z)$ is a function continuous in $G + \Gamma$, which is a solution of the following boundary value problem

$$\begin{aligned} &\partial_{\bar{z}}\tilde{w} + A\tilde{w} + \tilde{B}\overline{\tilde{w}} = \tilde{F} \qquad (\text{in } G), \\ &\mathrm{Re}[\overline{\tilde{\lambda}(z)}\tilde{w}(z)] = \gamma \qquad (\text{on } \Gamma), \end{aligned} \tag{6.14}$$

where

$$\tilde{B} = B\frac{\overline{Q(z)}}{Q(z)}, \qquad \tilde{F} = \frac{F(z)}{Q(z)}, \qquad \tilde{\lambda} = \lambda(z)\overline{Q(z)}.$$

The index of this boundary value problem is given by

$$\tilde{n} = n + k = m.$$

Completing the problem (6.14) by the point conditions

$$\bar{\lambda}_j w(z_j') \equiv \overline{\tilde{\lambda}}_j \tilde{w}(z_j') = \gamma_j + ic_j \qquad (j = 0, 1, \ldots, m),$$

where z_j' is an arbitrarily fixed point on $\Gamma_j (j = 0, 1, \ldots, m)$ and c_j are arbitrary real constants, we arrive at a problem which in view of Theorem 4.13 is always soluble and has a unique solution representable by the formula

$$\tilde{w}(z) = \tilde{w}_*(z) + \sum_{i=0}^{m} c_i \tilde{w}_i(z),$$

where $\tilde{w}_*$ is the solution of the above indicated problem corresponding to the case $c_i = 0 (i = 0, 1, \ldots, m)$ and $\tilde{w}_i$ are solutions of the homogeneous problem ($\tilde{F} \equiv 0, \gamma \equiv 0$) satisfying the relations

$$\tilde{w}_i(z_j') = \delta_{ij} \qquad (i, j = 0, 1, \ldots, m). \tag{6.15}$$

Hence, in view of the formula (6.13) we infer that the following theorem is valid:

THEOREM 4.14. *If* $n < m$ *Problem* $\mathring{\mathbf{A}}$ *has a unique solution of the form* (6.13) *satisfying the following additional conditions*

$$\overline{\lambda}_j w(z'_j) = \gamma_j + ic_j \quad (j = 0, 1, \ldots, m), \tag{6.16}$$

where z'_j *is an arbitrarily fixed point of the contour* Γ_j *and* $c_0, c_1, \ldots, c_m$ *are arbitrarily fixed real constants.*

We can now formulate the necessary and sufficient conditions of the solubility of Problem **A** in the class of regular functions. It is evident that these conditions consist in satisfying the relations

$$\sum_{i=0}^{m} c_i \widetilde{w}_i(z_j) = -\widetilde{w}_*(z_j) \quad (j = 1, \ldots, m-n). \tag{6.17}$$

Thus, we have $2(m-n)$ real equations for the determination of $m+1$ real constants.

Let r be the rank of the matrix of the system (6.17), $r \leqslant \min(m+1, 2m-2n)$. Obviously, Problem $\mathring{\mathbf{A}}$ has $l = m+1-r$ solutions and the non-homogeneous Problem **A** will be soluble if and only if the following relations are satisfied:

$$\sum_{j=1}^{m-n} X_{ij} \widetilde{w}_*(z_j) = 0 \quad (i = 1, \ldots, l' = 2m-2n-r),$$

where X_{ij} are the minors of the matrix of the system (6.17). Evidently, these relations should be identical with the necessary and sufficient conditions (2.5) of the solubility of Problem **A** (§2.1).

Since $l = m+1-r$, $l' = 2m-2n-r$, we again obtain the formula

$$l - l' = 2n+1-m,$$

yielding the difference between the numbers of solutions of the adjoint homogeneous Problems $\mathring{\mathbf{A}}$ and $\mathring{\mathbf{A}}'$ (§4.4).

If $n < 0$, in view of the inequality (4.17d) we obtain at once that the homogeneous Problem $\mathring{\mathbf{A}}$ has no solution possessing poles inside G if the number of the latter is smaller that $-n$. Accordingly, for $n < 0$ $l = m+1-r = 0$, i.e. the rank of the matrix of the system (6.17) is $m+1$. It is therefore easy to prove

THEOREM 4.15. *If $n < 0$, Problem* **A** *is always soluble in the class of functions having inside G arbitrarily specified poles the number of which is given by the relation*

$$p = \tfrac{1}{2}(m-2n-1) \qquad \textit{if } m \textit{ is odd}\,,$$

or

$$p = \tfrac{1}{2}(m-2n) \qquad \textit{if } m \textit{ even}\,.$$

Moreover, if m is an odd number the solution of the problem is uniquely determined, and if m is an even number the set of solutions of the problem contains linearly one arbitrary real constant.

In the case of a simply-connected domain this theorem was obtained in a somewhat different way by Schmidt, [97b].

The last results reveal to a certain extent the degree of incorrectness of Problem **A**. Adding to the conditions of Problem **A** some new conditions restricting in a certain way the class of the unknown solutions of the equation, or, conversely, extending this class by taking into account solutions having poles, or, finally, according to a definite rule restricting and at the same time extending the class of solutions, we arrive at a modified statement of the problem, which is already correct. In particular, for $n > m-1$ we have to narrow in a certain way (e.g. by adding point conditions of the form (6.8)) the class of continuous solutions, in order to ensure the correctness of Problem **A**. Therefore, in this case $(n > m-1)$ Problem **A** will be called *the quasi-correct problem.*

In other cases, when—on the contrary—we have to extend the class of solutions by taking into account

functions possessing poles we shall say that Problem **A** is *pseudo-correct.*

We finally observe that in the case of pseudo-correct problems we can obtain a correct formulation in the class of continuous solutions by an incomplete specification of the right-hand sides of the equation and the boundary condition. In the class of analytic functions, such a modified formulation of the Dirichlet problem was investigated by Muskhelishvili, [60a] (Ch. III, §1). For Problem **A** a similar investigation was carried out by Nitsche, [64a], in the case of a simply-connected domain.

6.5. We now consider separately the case $0 \leqslant n \leqslant m-1$. If $r < m+1$ Problem **Å** has a continuous solution which has at least one zero on each of $m-r$ distinct boundary contours. In fact, with no loss of generality we may assume that the principal minor of the order r of the matrix of the system (6.17) does not vanish. Then the solution of the corresponding homogeneous system has the form

$$c_i = A_{ir}c_r + \ldots + A_{im}c_m \qquad (i = 0, \ldots, r-1)\,.$$

Accordingly, the homogeneous Problem **Å** has continuous solutions of the form

$$w(z) = Q(z) \sum_{j=r}^{m} c_j \tilde{w}'_j(z)\,,$$

where

$$\tilde{w}'_j(z) = \tilde{w}_j(z) + \sum_{i=0}^{r-1} A_{ij}\tilde{w}_i(z) \qquad (j = r, r+1, \ldots, m)\,.$$

But in view of (6.15) $\tilde{w}'_j(z'_i) = 0$ if $i \geqslant r$ and $i \neq j$. Thus, the homogeneous Problem **Å** has a continuous solution $w = Q\tilde{w}'_r(z)$ which has $m-r$ zeros $z'_{r+1}, \ldots, z'_m$ situated, according to the assumption, on distinct contours constituting the boundary. Consequently, according to Theorem 4.7 the solution has at least $2(m-r)$ zeros on Γ, and by virtue of (4.19) we have the inequality

$$r \geqslant m-n\,,$$

yielding the lower bound for r. We therefore have the inequality

$$l = m+1-r \leqslant n+1\,,$$

yielding the upper bound for the number of solutions of the homogeneous Problem Å; we have seen already (§5.8) that this upper bound is really achieved.

It has thus been established that for $0 \leqslant n \leqslant m-1$ *the number of solutions of the homogeneous Problem* Å *satisfies the inequality*

$$0 \leqslant l \leqslant n+1\,, \tag{6.18}$$

which was already indicated above (§4.5). This inequality is rigorous in the sense that the extreme values of l, $l = 0$ and $l = n+1$, are possible. But the considerations of the previous section proved that the cases $l = 0$ take place most often if $n \leqslant \frac{1}{2}m$. The remaining cases are encountered only if the boundary condition is chosen in a special way (i.e. the functions λ), as well as the coefficients of the equation $\mathfrak{C}(w) = 0$, and the domain.

We have indicated in §5 examples corresponding to the cases $l = 0, 1$ and $n+1$. It is of interest to indicate examples corresponding to the other cases—$1 < l < n+1$.

In order to become acquainted with other results concerning the special cases $0 \leqslant n \leqslant m-1$ of Problem Å the reader is referred to the Appendix to Chapter 4, where incidentally the proof of the inequality (6.18) is given, first announced in the paper [11f].

§7. Solution of Problem A by means of two-dimensional integral equations. Application of the generalized principle of symmetry. Generalized Schwarz integral

7.1. In this section we shall derive new integral equations of Fredholm type which will enable us to solve Problem A without reference to the formulae giving general representations of the solutions by means of contour integrals. As distinct from the previous consi-

derations, we shall deal here with equations containing integrals over the domain ([14a], §8.9).

We shall restrict ourselves to the case of a simply-connected domain, the remaining assumptions concerning the data of the problem being preserved. Then, with no loss of generality we may consider the domain G to be the circle $|z| < 1$. Besides, we shall assume that the boundary condition of Problem **A** is reduced to the canonical form (§3)

$$\begin{aligned} \partial_{\bar{z}} w + B\overline{w} &= F \qquad (\text{in } G)\,, \\ \operatorname{Re}[z^{-n} w(z)] &= \gamma(z) \qquad (\text{or } \Gamma)\,, \end{aligned} \tag{7.1}$$

where n is an integer. It is obvious that the unknown solution of Problem **A** satisfies the integral equation

$$w(z) - \frac{1}{\pi}\iint_G \frac{B(\zeta)\overline{w(\zeta)}}{\zeta - z}\, d\xi\, d\eta = \Phi(z) + \mathrm{T}_G F\,, \tag{7.2}$$

where $\Phi(z)$ is a function analytic in G and

$$\mathrm{T}_G F = -\frac{1}{\pi}\iint_G \frac{F(\zeta)\, d\xi\, d\eta}{\zeta - z}\,. \tag{7.3}$$

The integral equation (7.2) will be equivalent to Problem **A** if we succeed in finding such a representation of the analytic function Φ that the solution of the equation (7.2) satisfies the boundary condition (7.1).

7.2. We first consider the case $n \geqslant 0$ and represent the function Φ in the form

$$\begin{aligned} \Phi(z) = \Phi_0(z) + \frac{z^{2n+1}}{\pi}\iint_G \frac{\overline{B(\zeta)}\, w(\zeta)}{1 - \bar{\zeta} z}\, d\xi\, d\eta\, - \\ - \frac{z^{2n+1}}{\pi}\iint_G \frac{\overline{F(\zeta)}}{1 - \bar{\zeta} z}\, d\xi\, d\eta\,, \end{aligned} \tag{7.4}$$

where Φ_0 is a new unknown function, analytic in G. Substituting the expression (7.4) in the right-hand side of

the relation (7.2) we obtain the following integral equation of Fredholm type for the unknown function w:

$$w(z)+P_n(B\overline{w}) = \Phi_0(z)+P_nF\,, \tag{7.5}$$

where

$$P_nf = -\frac{1}{\pi}\iint_G\left(\frac{f(\zeta)}{\zeta-z}+\frac{z^{2n+1}\overline{f(\zeta)}}{1-\bar{\zeta}z}\right)d\xi\,d\eta\,. \tag{7.6}$$

Denoting by G' the exterior of the circle $|z|\leqslant 1$ we can represent P_nf in the form

$$P_nf = \mathrm{T}_Gf+z^{2n+1}\mathrm{T}_{G'}f_1\,, \tag{7.7}$$

where

$$\mathrm{T}_{G'}f_1 \equiv -\frac{1}{\pi}\iint_{G'}\frac{f_1(\zeta)\,d\xi\,d\eta}{\zeta-z}, \qquad f_1(\zeta) = \frac{\overline{f\left(\frac{1}{\bar{\zeta}}\right)}}{\zeta\bar{\zeta}^2}. \tag{7.8}$$

In view of Theorems 1.24, 1.25 and 1.29 it follows from (7.7) that the operator

$$Q_nf \equiv P_n(B\bar{f}) \tag{7.9}$$

is linear (over the field of real numbers), completely continuous in the spaces $C(\overline{G})$ and $L_q(\overline{G})$, $q\geqslant\dfrac{p}{p-1}$, it maps $C(\overline{G})$ onto $C_{\frac{p-2}{p}}(\overline{G})$ and $L_q(\overline{G})$ $\left(\text{when } q\geqslant\dfrac{2p}{p-2}\right)$ onto $C_\nu(\overline{G})$, where $\nu = 1-2\left(\dfrac{1}{p}+\dfrac{1}{q}\right)$. If $\dfrac{p}{p-1}\leqslant q\leqslant\dfrac{2p}{p-2}$, then Q_nf belongs to the class $L^{\alpha}_{p_1}(\overline{G})$ where

$$p_1 = \frac{1}{\dfrac{1}{p}+\dfrac{1}{q}-\dfrac{1}{2}+\alpha}.$$

Here α is a sufficiently small fixed positive number. Moreover, there exists an integer m such that Q_n^mf belongs to a $C_\beta(\overline{G})$, $0<\beta<1$, if $f\in L_q(G)$, $q\geqslant\dfrac{p}{p-1}$.

It is readily observed that for an arbitrary function f of the class $L_p(\bar{G}), p > 2$, the following condition is satisfied

$$\operatorname{Re}[z^{-n}P_n f] = 0 \quad (\text{on } \Gamma) . \tag{7.10}$$

Therefore, if the analytic function Φ_0 satisfies the boundary condition

$$\operatorname{Re}[z^{-n}\Phi_0(z)] = \gamma \quad (\text{on } \Gamma) , \tag{7.11}$$

then the solution of the integral equation (7.5) will be the solution of the Problem **A** under consideration. Conversely, if w is the solution of Problem **A**, then a function Φ_0 analytic in G can be found, such that it satisfies the boundary condition (7.11) and w satisfies the integral equation (7.5).

We have already seen above (§1) that the general solution of the problem (7.11) is given by the formula

$$\Phi_0(z) = \frac{z^n}{2\pi i}\int_\Gamma \gamma(t)\frac{t+z}{t-z}\frac{dt}{t} + \sum_{k=0}^{2n} c_k z^k , \tag{7.12}$$

where c_k are complex constants satisfying the relations

$$c_{2n-k} = -\bar{c}_k \quad (k = 0, 1, \ldots, n) , \tag{7.13}$$

i.e.

$$\sum_{k=0}^{2n} c_k z^k = \sum_{k=0}^{n-1} a_k(z^k - z^{2n-k}) + i\beta_k(z^k + z^{2n-k}) + ic_0 z^n ,$$

c_0, a_k and β_k being arbitrary real constants.

Thus, for $n \geqslant 0$ Problem **A** is reduced to the following equivalent integral equation of Fredholm type:

$$w + Q_n w = P_n F + \frac{z^n}{2\pi i}\int_\Gamma \gamma(t)\frac{t+z}{t-z}\frac{dt}{t} + \sum_{k=0}^{2n} c_k z^k . \tag{7.14}$$

No matter what are the complex constants c_k satisfying the conditions (7.13), the solution of the equation (7.14) is the solution of Problem **A**.

We shall now prove that the equation (7.14) has a solution for an arbitrary right-hand side belonging to $L_q(G+\Gamma)$, $q \geqslant \frac{p}{p-1}$. It was proved above that Q_n is a completely continuous operator in any L_q for $q \geqslant \frac{p}{p-1}$. Therefore, our assertion will be proved if we establish that the homogeneous equation $w+Q_n w = 0$ has no non-trivial solution of the class L_q, $q \geqslant \frac{p}{p-1}$. But by means of Theorem 1.29 it is readily found that solutions of the class L_q $\left(q \geqslant \frac{p}{p-1}\right)$ of this equation are continuous in $G+\Gamma$. Therefore, the problem is reduced to the proof of the fact that the equation $w+Q_n w = 0$ has no non-trivial continuous solution. This equation may be written in the form

$$w(z) - \frac{1}{\pi}\iint_G \frac{B(\zeta)\overline{w(\zeta)}}{\zeta - z}\,d\xi\,d\eta = \frac{z^{2n+1}}{\pi}\iint_G \frac{\overline{B(\zeta)w(\zeta)}}{1-\bar{\zeta}z}\,d\xi\,d\eta\,. \qquad (7.15)$$

Its right-hand side is holomorphic in G and continuous in $G+\Gamma$, and the integral appearing in the left-hand side is a function continuous on the entire plane, holomorphic outside $G+\Gamma$ and vanishing at infinity. Assume that $z \in \Gamma$. Let us multiply both sides of the relation (7.15) by

$$\frac{1}{2\pi i}\,\frac{dz}{z-t}\,, \qquad t \in G\,,$$

and let us integrate the result over Γ. Then, making use of the Cauchy theorem and Cauchy formula, we obtain

$$\frac{1}{2\pi i}\int_\Gamma \frac{w(z)\,dz}{z-t} = \frac{t^{2n+1}}{\pi}\iint_G \frac{\overline{B(\zeta)w(\zeta)}}{1-t\bar{\zeta}}\,d\xi\,d\eta\,.$$

If the left- and right-hand sides of the last relation be expanded into the powers of t, $|t| < 1$, we obtain

$$\int_{\Gamma} w(z) e^{-ik\theta} d\theta = 0 \qquad (k = 0, 1, \ldots, 2n, z = e^{i\theta}) . \tag{7.16}$$

Besides, the solution of the equation (7.15) satisfies the homogeneous equation $\mathfrak{C}(w) = 0$ and the homogeneous boundary condition

$$\mathrm{Re}[z^{-n} w(z)] = 0 \qquad (\text{on } \Gamma) . \tag{7.17}$$

Therefore, w may be represented in the form (Ch. III, §4)

$$w(z) = \Phi(z) e^{p(z)} , \tag{7.18}$$

where Φ is a function analytic in G with respect to z, and

$$p(z) = -\frac{1}{\pi} \iint_G \left\{ \frac{f(\zeta)}{\zeta - z} - \frac{z\overline{f(\zeta)}}{1 - \bar{\zeta} z} \right\} d\xi \, d\eta , \qquad f = B \frac{\overline{w}}{w} .$$

Since $\mathrm{Re}[ip(z)] = 0$ on Γ, the boundary condition (7.17) according to (7.18) has now the form

$$\mathrm{Re}[z^{-n} \Phi(z)] = 0 . \tag{7.19}$$

Now, according to the formula (7.12) the general solution of this problem has the form

$$\Phi(z) = \sum_{k=0}^{2n} c_k z^k ,$$

where c_k are complex constants satisfying the condition (7.13). Therefore, in view of (7.18) the solution of the equation (7.15) should have the form

$$w(z) = \left(\sum_{k=0}^{2n} c_k z^k \right) e^{p(z)} .$$

Inserting the last result into the relations (7.16) we obtain

$$\sum_{k=0}^{2n} c_k \int_{\Gamma} z^k z^{-l} e^{p(z)} dz = 0 \qquad (l = 0, 1, \ldots, 2n) . \tag{7.20}$$

This implies that $c_k = 0$ $(k = 0, 1, \dots, 2n)$, since the determinant of the system (7.20) does not vanish; this fact follows from the identity of this determinant with the Gramm determinant for the system of linearly independent functions

$$z^k e^{\frac{1}{2}p(z)} \qquad (k = 0, \dots, 2n;\ p(z) = \overline{p(z)},\ z \in \Gamma).$$

Therefore, the homogeneous equation (7.15) has no non-trivial solution. Consequently, the non-homogeneous equation (7.14) has a unique solution for an arbitrary right-hand side belonging to $L_q(G+\Gamma)$, for $q \geqslant \dfrac{p}{p-1}$.

Thus, it has been proved that for $n \geqslant 0$ the non-homogeneous Problem **A** has always a solution and the homogeneous Problem $\mathring{\mathbf{A}}$ $(F \equiv 0, \gamma \equiv 0)$ has exactly $2n+1$ linearly independent solutions, for it is equivalent to the integral equation

$$w + Q_n w = i c_0 z^n + \sum_{k=0}^{n-1} \alpha_k (z^k - z^{2n-k}) + i\beta_k (z^k + z^{2n-k}), \tag{7.21}$$

the right-hand side of which is a linear combination with arbitrary real coefficients of linearly independent functions

$$iz^n, \quad z^k - z^{2n-k}, \quad i(z^k + z^{2n-k}) \tag{7.22}$$
$$(k = 0, 1, \dots, n-1).$$

7.3. We now proceed to the case $n < 0$. Now, the integral equation (7.14) used above is, evidently, not valid since it contains terms with the powers z^{2n+1} possessing therefore discontinuities of a high order at the point $z = 0$. We shall introduce, therefore, the function $w_0 = z^k w$ where $k = -n$. Since $k > 0$, w_0 is continuous in G and satisfies the equation

$$\partial_{\bar{z}} w_0 + B_0 \overline{w}_0 = z^k F(z), \qquad B_0 = B \frac{z^k}{\bar{z}^k} \tag{7.23}$$

and the boundary condition

$$\operatorname{Re}[z^k w) \equiv \operatorname{Re}(w_0) = \gamma \quad (\text{on } \Gamma)\,. \tag{7.24}$$

But this problem corresponds to the problem examined in the previous subsection for the case $n = 0$. Consequently, the function $w_0 = z^k w$ satisfies the integral equation

$$z^k w(z) + P_0[z^k(B_0\overline{w})] = P^0(z^k F) + \frac{1}{2\pi i}\int_\Gamma \gamma(t)\frac{t+z}{t-z}\frac{dt}{t} + ic_0\,. \tag{7.25}$$

It is easy to derive the identities

$$\frac{1}{2\pi i}\int_\Gamma \gamma(t)\frac{t+z}{t-z}\frac{dt}{t} = \frac{1}{2\pi i}\int_\Gamma \frac{\gamma(t)\,dt}{t} + \sum_{j=1}^{k-1}\frac{z^j}{\pi i}\int_\Gamma \frac{\gamma(t)\,dt}{t^{j+1}} + \frac{z^k}{\pi i}\int_\Gamma \frac{\gamma(t)\,dt}{t^k(t-z)},$$

$$\begin{aligned} P_0(z^k f) = &-\frac{1}{\pi}\iint_G \zeta^{k-1} f(\zeta)\,d\xi\,d\eta - \\ &-\frac{z}{\pi}\iint_G \left(\zeta^{k-2} f(\zeta) + \bar\zeta^k \overline{f(\zeta)}\right)d\xi\,d\eta + \\ &+ \ldots - \frac{z^{k-1}}{\pi}\iint_G \left(f(\zeta) + \bar\zeta^{2k-2}\overline{f(\zeta)}\right)d\xi\,d\eta + z^k P_k^*(f)\,, \end{aligned}$$

where

$$P_k^* f = -\frac{1}{\pi}\iint_G \left(\frac{f(\zeta)}{\zeta - z} + \frac{\bar\zeta^{2k-1}\overline{f(\zeta)}}{1 - z\bar\zeta}\right)d\xi\,d\eta\,. \tag{7.26}$$

The operator $P_k^* f$ can also be represented thus:

$$P_k^* f = T_E f_* = -\frac{1}{\pi}\iint_E \frac{f_*(\zeta)\,d\xi\,d\eta}{\zeta - z}, \tag{7.27}$$

where

$$f_*(\zeta) = \begin{cases} f(\zeta) & \text{for} \quad |\zeta| \leqslant 1\,, \\ \dfrac{\overline{f\left(\frac{1}{\bar\zeta}\right)}}{\zeta^{2k}\bar\zeta^2} & \text{for} \quad |\zeta| > 1\,. \end{cases} \tag{7.28}$$

Therefore, the general properties of the operator $Q_n f$ indicated on page 294 are also literally valid for the operator

$$Q_k^* f \equiv P_k^*(B\bar{f}) , \qquad (7.29)$$

i.e. this operator is completely continuous in any $L_q(G+\Gamma)$, for $q \geqslant \frac{p}{p-1}$.

The equation (7.25) can now be written in the form

$$w(z) + Q_k^* w = P_k^* F + \frac{1}{\pi i} \int_{\Gamma_1} \frac{\gamma(t)\, dt}{t^k (t-z)} + \sum_{j=1}^{k} a_j z^{-j} ,$$

where

$$\begin{aligned} a_j &\equiv a_j(w) \\ &= \frac{1}{\pi} \iint_G [(B\bar{w} - F)\zeta^{j-1} + \bar{\zeta}^{2k-j-1}(\bar{B}w - \bar{F})]\, d\xi\, d\eta + \\ &\qquad + \frac{1}{\pi i} \int_\Gamma \frac{\gamma(t)\, dt}{t^{k-j+1}}, \qquad j = 1, 2, \ldots, k-1 , \qquad (7.30) \\ a_k &\equiv a_k(w) \\ &= i c_0 + \frac{1}{2\pi i} \int_\Gamma \frac{\gamma(t)\, dt}{t} - \frac{1}{\pi} \iint_{G_*} (B\bar{w} + \bar{F}) \zeta^{k-1}\, d\xi\, d\eta . \end{aligned}$$

The function $w(z)$ should be continuous inside G. To this end it is necessary and sufficient that the following relations hold:

$$a_1(w) = 0 , \; a_2(w) = 0 , \; \ldots , \; a_k(w) = 0 . \qquad (7.31)$$

Thus, for $n < 0$ Problem **A** is reduced to the integral equation

$$w(z) + Q_k^* w = P_k^* F + \frac{1}{\pi i} \int_\Gamma \frac{\gamma(t)\, dt}{t^k (t-z)} , \qquad (7.32)$$

and the solution of the latter equation will satisfy the boundary condition (7.24) if and only if the relations (7.31) are satisfied.

We shall now prove that the homogeneous equation

$$w + P_k^*(B\bar{w}) = 0 \qquad (7.33)$$

has no non-trivial solution. It can easily be verified that for two arbitrary functions f and g, continuous in $G+\Gamma$, the relation

$$\operatorname{Im}\left\{\iint_G [g(f+P_k^* f)-f(g-P_k g)]\,dx\,dy\right\} \equiv 0 .$$

holds. Let now f be a non-trivial solution of the equation (7.33). Then we have the relation

$$\operatorname{Im}\iint_G f(g-P_k g)\,dx\,dy = 0 ,$$

which should be valid for any continuous function g. We have, however, proved in the preceding section that the equation $g-P_k g = h$ has a solution for an arbitrary function h of the class $L_q(\bar{G})$, $q \geqslant \frac{p}{p-1}$. Equating h to $i\bar{f}$ we arrive at the relation

$$\iint_G |f|^2\,dx\,dy = 0 ,$$

which contradicts the assumption $f \not\equiv 0$.

Thus, we have established that the non-homogeneous equation (7.32) has a unique solution which, obviously, can be expressed uniquely by F and γ in the form

$$w(z) = \int_\Gamma \mathrm{X}_k^*(z,\zeta)\gamma(\zeta)\,ds + \\ + \iint_G \Omega_1^*(z,\zeta)F(\zeta)\,d\xi\,d\eta + \iint_G \Omega_2^*(z,\zeta)\overline{F(\zeta)}\,d\xi\,d\eta . \quad (7.34)$$

The determined solution of the equation (7.32) should be subjected to the conditions (7.31) in order to obtain the required solution of Problem **A**. But the conditions (7.31) contain $2k$ real relations. The fulfilment of one of them, namely the condition $\operatorname{Im} a_k = 0$, can be ensured by means of a suitable choice of the constant c_0. Con-

sequently, there still remain $2k-1$ conditions which lead to the relations of the form (for the original Problem **A**)

$$\mathrm{Re}\left\{\iint_G \chi_j'(\zeta)F(\zeta)\,d\xi\,d\eta\right\} + \int_\Gamma \chi_j''(\zeta)\gamma(\zeta)\,ds = 0 \qquad (7.35)$$
$$(j = 1, 2, \ldots, 2k-1)\,,$$

where χ_j' and χ_j'' are linearly independent functions which are independent of the functions F and γ. The relations (7.35) are the necessary and sufficient conditions of solubility of Problem **A** (for $n < 0$). Therefore, they contain the necessary condition (2.5) (we assume that its sufficiency is not proved)

$$\mathrm{Re}\left\{\iint_G w'(z)F(z)\,dx\,dy\right\} - \frac{1}{2i}\int_\Gamma \lambda(t)w'(t)\gamma(t)\,dt = 0\,, \qquad (7.36)$$

where w' is an arbitrary solution of the adjoint homogeneous Problem $\mathring{\mathbf{A}}'$

$$\mathfrak{C}'(w') \equiv \partial_{\bar z}w' - \bar{B}\overline{w}' = 0 \qquad (\text{in } G)\,, \qquad (7.37)$$
$$\mathrm{Re}[\lambda(z)z'(s)w'(z)] = 0 \qquad (\text{on } \Gamma)\,. \qquad (7.38)$$

The index of this problem $n' = -n+1 = k+1 > 0$ and according to the results of the preceding subsection the problem has exactly $2k-1$ linearly independent sclutions $w_j', \ldots, w_{2k-1}'$. Now it is easy to prove that we may set

$$\chi_j'(z) = w_j'(z)\,, \qquad \chi_j''(z) = -\frac{1}{2i}\lambda(z)z'(s)w_j'(z) \qquad (7.39)$$
$$(j = 1, \ldots, 2k-1)\,.$$

Thus, the conditions (7.35) and (7.36) are identical. We have obtained here a new proof of the sufficiency of the condition (7.36).

The results of this and the preceding subsections fully coincide with Theorem 4.11 proved above (§4) in a different way.

7.4. We shall now make a remark which will be used in the next section. The integral equations (7.14)

and (7.32) enable us to solve Problem **A** under weaker restrictions on the right-hand side of the equation $\mathfrak{C}(w) = F$.

We assume that $F \in L_q(G+\Gamma)$, $q \geqslant 1$, preserving the remaining assumptions of the condition I. Then it is reasonable to seek the solution of Problem **A** in the class $D_{1,q}(G+\Gamma)$, bearing in mind that the equation and the boundary condition will be satisfied in the generalized sense, since the unknown solution is not necessarily continuous in $G+\Gamma$. It can easily be proved that also in this case the solution of Problem **A** is reduced to the integral equations (7.14) (for $n \geqslant 0$) and (7.32) (for $n < 0$). Since in view of Theorem 1.29 $P_n F$ and $P_k^* F (k = -n)$ belong to any class $L_r(G+\Gamma)$ where r is an arbitrary number smaller than two, we may set $r \geqslant \dfrac{p}{p-1}$. But it was indicated above that the operators Q_n and Q_n^* are completely continuous in an arbitrary L_r if $r \geqslant \dfrac{p}{p-1}$. Therefore the Fredholm theorems may be applied to the equations (7.14) and (7.32) and consequently Theorem 4.11 remains valid also in the present case, when $F \in L_q(G+\Gamma)$, $q \geqslant 1$.

Apparently, the same may be said with respect to other theorems of §4 as well, which concern multiply-connected domains.

It is also easy to verify that the integral equations (7.14) and (7.32) allow a considerable relaxation of the requirements also with respect to the right-hand side γ of the boundary condition of Problem **A**. It is for instance sufficient to demand that $\gamma \in L_r(\Gamma)$, $r > 2$. Obviously, at the same time the demands concerning the solution of the problem should be relaxed in a suitable way. The mathematical apparatus which will here be required may be found in the paper of Khvedelidse, [91a].

7.5. The integral equations constructed in the preceding subsections contain the unknown complex function w and have complex kernels. They in fact represent systems

of two real equations containing the real and imaginary parts of the required solution of Problem **A**. In many cases, however, the explicit determination of w is not required, and it is sufficient to know only its real or imaginary part. In these cases it is expedient to have equations making possible the determination only of either the real or the imaginary part of the required solution of Problem **A**. We shall prove in this article that such equations can be derived by making use of the results of §16, Ch. III ([14a], §8.10).

It was established in §16, Ch. III, that the real part $u(x, y)$ of the solution of the equation $\mathfrak{E}(w) \equiv \partial_{\bar{z}} w + Aw + B\bar{w} = F$ satisfies the integral equation

$$u(z) - \frac{2}{\pi} \iint_G u(\zeta) \operatorname{Re}\left[\frac{B(\zeta)e^{\omega(\zeta)}}{(\zeta - z)e^{\omega(z)}}\right] d\zeta\, d\eta = f(z)\,, \quad (7.40)$$

where

$$w(z) = \Phi_0(z) - \frac{1}{\pi} \iint_G \frac{A(\zeta) - B(\zeta)}{\zeta - z} d\xi\, d\eta\,, \quad (7.41)$$

$$f(z) = \operatorname{Re}[e^{-\omega(z)}\Phi(z)] - \operatorname{Re}\left[\frac{1}{\pi} \iint_G \frac{e^{\omega(\zeta)}F(\zeta)}{e^{\omega(z)}(\zeta - z)} d\xi\, d\eta\right], \quad (7.42)$$

where Φ_0 and Φ are arbitrary functions holomorphic in G.

We shall hereafter assume that G is the circle $|z| < 1$. In addition, we shall consider only the case when the index of the boundary value Problem **A** is a non-negative integer, i.e. $n \geqslant 0$.

Taking for Φ_0 the analytic function

$$\Phi_0(z) = \frac{z}{\pi} \iint_G \frac{\overline{A(\zeta)} - \overline{B(\zeta)}}{1 - z\bar{\zeta}} d\xi\, d\eta\,,$$

we obtain

$$\omega(z) = \frac{1}{\pi} \iint_G \left(\frac{B(\zeta) - A(\zeta)}{\zeta - z} - \frac{z\left(\overline{B(\zeta)} - \overline{A(\zeta)}\right)}{1 - z\bar{\zeta}}\right) d\xi\, d\eta\,. \quad (7.43)$$

It is evident that this function is continuous in $G+\Gamma$ and has real values on Γ.

The second analytic function Φ can always be represented in the form

$$\Phi(z) = \Phi_1(z) - \frac{z^{2n+1}}{\pi} \iint_G \frac{e^{\overline{\omega(\zeta)}} \left(\overline{F(\zeta)} - 2\overline{B(\zeta)}\right)}{1 - z\bar{\zeta}} d\xi \, d\eta \,, \quad (7.44)$$

where Φ_1 is a new function holomorphic in G.

Introducing this result into the right-hand side of (7.42) we obtain the following (real) integral equation:

$$u(x, y) - \iint_G K(z, \zeta) u(\xi, \eta) d\xi \, d\eta = \mathrm{Re}[e^{-\omega(z)} \Phi_1(z)] + F_1(z) \,, \quad (7.45)$$

where

$$K(z, \zeta) = \frac{2}{\pi} \mathrm{Re} \left\{ e^{-\omega(z)} \left[\frac{B(\zeta) e^{\omega(\zeta)}}{\zeta - z} + \frac{z^{2n+1} \overline{B(\zeta)} e^{\overline{\omega(\zeta)}}}{1 - z\bar{\zeta}} \right] \right\}, \quad (7.46)$$

$$F_1(z) = -\frac{1}{\pi} \mathrm{Re} \left\{ e^{-\omega(z)} \iint_G \left(\frac{F(\zeta) e^{\omega(\zeta)}}{\zeta - z} + \frac{z^{2n+1} \overline{F(\zeta)} e^{\overline{\omega(\zeta)}}}{1 - z\bar{\zeta}} \right) d\xi \, d\eta \right\}. \quad (7.47)$$

Let $u(x, y)$ be the solution of the equation (7.45). Let us consider the function

$$\begin{aligned} w(z) &\equiv u(x, y) + iv(x, z) \\ &= \frac{2}{\pi} e^{-\omega(z)} \iint_G \left(\frac{B(\zeta) e^{\omega(z)}}{\zeta - z} + \frac{z^{2n+1} \overline{B(\zeta)} e^{\omega(\zeta)}}{1 - z\bar{\zeta}} \right) u(\xi, \eta) d\xi \, d\eta + \\ &\quad + e^{-\omega(z)} \Phi_1(z) - \frac{e^{-\omega(\zeta)}}{\pi} \times \\ &\quad \times \iint_G \left(\frac{F(\zeta) e^{\omega(\zeta)}}{\zeta - z} + \frac{z^{2n+1} \overline{B(\zeta)} e^{\overline{\omega(\zeta)}}}{1 - z\bar{\zeta}} \right) d\xi \, d\eta ; \quad (7.48) \end{aligned}$$

It is readily observed that it is continuous in $G+\Gamma$, satisfies the equation $\mathfrak{C}(w) = F$ and the boundary condition

$$\mathrm{Re}[z^{-n} w] = e^{-\omega(z)} \mathrm{Re}[z^{-n} \Phi_1(z)] \quad (\text{on } \Gamma) \,. \quad (7.49)$$

It should be here taken into account that $\omega(z) = \overline{\omega(z)}$ on Γ. If the function Φ_1 is subjected to the boundary condition

$$\operatorname{Re}[z^{-n}\Phi_1(z)] = \gamma(z)\, e^{\omega(z)}\,, \qquad z \in \Gamma; \tag{7.50}$$

the function $w(z)$ given by the formula (7.48) will, by (7.49), be the solution of the boundary value Problem **A**, i.e. $\mathfrak{C}(w) = F$ (in G) and $\operatorname{Re}[z^{-n}w(z)] = \gamma(z)$ (on Γ). We have seen already (p. 295) that a function holomorphic in G and satisfying the condition (7.50) has the form

$$\Phi_1(z) = \frac{z^n}{2\pi i}\int_\Gamma \gamma(t)\, e^{\omega(t)}\frac{t+z}{t-z}\frac{dt}{t} + ic_0 z^n + \\ + \sum_{k=0}^{n-1} \alpha_k(z^k - z^{2n-k}) + i\beta_k(z^k + z^{2n-k})\,, \tag{7.51}$$

where c_0, α_k and β_k $(k = 0, 1, \ldots, n-1)$ are arbitrary real constants.

Thus, the solution of Problem **A** is equivalent to the solution of the integral equation (7.45) with a real kernel and real right-hand side, which determines the real part of the unknown function. By a nearly literal reconstruction of the argument of §7.2 concerning the solubility of the integral equation (7.15) we find that the equation (7.45) has a solution for an arbitrary continuous right-hand side. Setting $\gamma \equiv 0, F \equiv 0$ we obtain the non-homogeneous integral equation

$$u(x, y) - \iint_G K(z, \zeta)\, u(\xi, \eta) \\ = \operatorname{Re}\Big\{e^{-\omega(z)}\Big[ic_0 z^n + \sum_{k=0}^{n-1} \alpha_k(z^k - z^{2n-k}) + i\beta_k(z^k + z^{2n-k})\Big]\Big\}, \tag{7.52}$$

yielding the real parts of the solutions of the homogeneous Problem $\mathring{\mathbf{A}}$.

Following the method indicated in the present section, §7.3, we can construct an integral equation for

the real part of the unknown solution of Problem **A** also in the case of a negative index, $n < 0$. Since there are no important difficulties we shall not consider this case in detail.

7.6. The solution of Problem **A** can be represented by the resolvents of the integral equations derived in the present section. This will lead us to integral formulae representing the solution of the problem by the functions F and γ. Thus, we shall obtain formulae constituting generalizations of the celebrated integral formulae of Poisson and Schwarz.

Below we shall confine ourselves to the case $n \geqslant 0$.

Assume that $n \geqslant 0$, and let us consider the integral equations

$$w(z) + Q_n w = \frac{1}{2\pi}\left(\frac{t^{n+1}}{t-z} + \frac{z^{2n+1}\bar{t}^{n+1}}{1-z\bar{t}}\right), \tag{7.53}$$

$$w(z) + Q_n w = -\frac{1}{\pi}\left(\frac{1}{t-z} + \frac{z^{2n+1}}{1-z\bar{t}}\right), \tag{7.54}$$

$$w(z) + Q_n w = -\frac{1}{\pi i}\left(\frac{1}{t-z} - \frac{z^{2n+1}}{1-z\bar{t}}\right), \tag{7.55}$$

where t is an arbitrary fixed point of the plane. Since the right-hand sides of these equations belong to an arbitrary $L_q(G+\Gamma)$, $q < 2$, as it was already established in §7.2 they have solutions which we shall denote by $X_n(z, t)$, $X'_n(z, t)$ and $X''_n(z, t)$, respectively.

Let

$$\begin{aligned} \Omega'_n(z, t) &= \tfrac{1}{2}(\mathrm{X}'_n + i\mathrm{X}''_n)\,, \\ \Omega''_n(z, t) &= \tfrac{1}{2}(\mathrm{X}'_n - i\mathrm{X}''_n)\,. \end{aligned} \tag{7.56}$$

Let us call the functions X_n, Ω'_n and Ω''_n the resolvents of Problem **A** in the case of a circle. We shall now prove that the formula

$$w_*(z) = \int_0^{2\pi} \mathrm{X}_n(z, e^{i\psi})\gamma(\psi)\,d\psi + $$
$$+ \iint_G \Omega'_n(z, \zeta)F(\zeta)\,d\xi\,d\eta + \iint_G \Omega''_n(z, \zeta)\overline{F(\zeta)}\,d\xi\,d\eta \tag{7.57}$$

provides us with the solution of Problem **A**, i.e. $\mathfrak{C}(w_*) = F$ in G and

$$\mathrm{Re}[z^{-n}w_*(z)] = \gamma \quad (\text{on } \Gamma) . \tag{7.58}$$

Since for $t \in \Gamma$ in view of (7.53)

$$X_n(z, t) = -Q_n X_n + \frac{1}{2\pi} \frac{t^{2n+1}+z^{2n+1}}{(t-z)t^n},$$

we have

$$\lim_{z\to\zeta\in\Gamma}\left[z^{-n}\int_0^{2\pi} X_n(z, e^{i\psi})\gamma(\psi)\,d\psi\right] = \gamma(\zeta)+\int_0^{2\pi} \zeta^{-n}X_n(\zeta, e^{i\psi})\gamma(\psi)\,d\psi .$$

Taking into account that $\mathrm{Re}[\zeta^{-n}X_n(\zeta, e^{i\psi})] = 0$ as $\zeta \in \Gamma$, we find that

$$w_*'(z) = \int_0^{2\pi} X_n(z, e^{i\psi})\gamma(\psi)\,d\psi \tag{7.59}$$

satisfies the boundary condition (7.58). Moreover, it is evident that $\mathfrak{C}(w_*') = 0$ since $\mathfrak{C}\big(X_n(z, e^{i\psi})\big) = 0$ for $z \in G$.

Denoting by f and $-g$ the real and imaginary parts of the function F we have

$$w_*''(z) = \iint_G (\Omega_n' F + \Omega_n'' \overline{F})\,d\xi\,d\eta$$
$$= \iint \big(X_n'(z, \zeta)f(\zeta) + X_n''(z, \zeta)g(\zeta)\big)\,d\xi\,d\eta . \tag{7.60}$$

Since for $z \in \Gamma$, $\zeta \in G$,

$$\mathrm{Re}[z^{-n}X_n'(z, \zeta)] = 0 , \qquad \mathrm{Re}[z^{-n}X_n''X(z, \zeta)] = 0 ,$$

the function w_*'' satisfies the homogeneous boundary condition $\mathrm{Re}[z^{-n}w_*''] = 0$ on Γ. Furthermore, since

$$X_n'(z, t) = -Q_n X_n' - \frac{1}{\pi(t-z)} - \frac{z^{2n+1}}{\pi(1-z\bar{t})},$$
$$X_n''(z, t) = -Q_n X_n'' - \frac{1}{\pi i(t-z)} + \frac{z^{2n+1}}{\pi i(1-z\bar{t})},$$

we have

$$w_*''(z) = -\iint_G (fQ_n X_n' + gQ_n X_n'')\,d\xi\,d\eta -$$
$$-\frac{1}{\pi}\iint_G \frac{F(\zeta)\,d\xi\,d\eta}{\zeta - z} - \frac{z^{2n+1}}{\pi}\iint_G \frac{\overline{F(\zeta)}\,d\zeta\,d\eta}{1 - z\bar{\zeta}}.$$

Applying to both sides of the above relation the operation $\partial_{\bar{z}}$, we obtain

$$\partial_{\bar{z}} w_*'' = -\iint_G (f\partial_{\bar{z}} Q_n X_n' + g\partial_{\bar{z}} Q_n X_n'')\,d\xi\,d\eta + F(z)$$
$$= F(z) - \iint_G \{B(z)[\overline{X_n'(z,\zeta)} f(\zeta) + \overline{X_n''(z,\zeta)}\, g(\zeta)]\}\,d\xi\,d\eta$$
$$= F(\zeta) - B(z)\overline{w_*''(z)}\,.$$

Therefore, the function $w_* = w_*' + w_*''$ given by the relation (7.57) is the particular solution of Problem **A**. In order to obtain the general solution of this problem we have to add to w_* the solution of the integral equation (7.21).

For $n = 0$ and $F \equiv 0$ the general solution of Problem **A** has the form

$$w(z) = \int_0^{2\pi} X_0(z, e^{i\varphi})\,\gamma(\psi)\,d\psi + c_0\hat{w}(z)\,. \tag{7.61}$$

Here c_0 is an arbitrary real constant and $\hat{w}$ is a solution of the homogeneous Problem $\mathring{\mathbf{A}}$; $\hat{w}$ satisfies the integral equation

$$\hat{w} + Q_0\hat{w} = i\,.$$

The formula (7.61) will be called *the generalized Schwarz integral.* It gives the expression of a generalized analytic function in the circle $|z| < 1$ by the boundary values of its real part. The imaginary part of the function w on the circumference $|z| = 1$ is given by the formula

$$\gamma_*(z) = -i\int_0^{2\pi} X_0(z, e^{i\varphi})\,\gamma(\psi)\,d\psi - ic_0\hat{w}(z)\,, \qquad z \in \Gamma\,. \tag{7.62}$$

7.7. The formula (7.61) can also be derived with the aid of the generalized principle of symmetry (the Riemann–Schwarz principle).

Continuing outside the circle G ($|z| < 1$) the coefficient of the equation $\mathfrak{C}(w) \equiv \partial_{\bar{z}} w + B\bar{w} = 0$ according to the rule (Ch. III, §11)

$$B_0(z) = -\frac{1}{\bar{z}^2} \overline{B\left(\frac{1}{\bar{z}}\right)} \qquad (|z| \geqslant 1),$$

we shall consider this equation on the entire plane; evidently, $B \in L_{p,2}(E)$, $p > 2$. Let $\Omega_1(z, \zeta)$ and $\Omega_2(z, \zeta)$ be the normal kernels of the equation thus obtained. If $w(z)$ satisfies the equation $\mathfrak{C}(w) = 0$ inside G and is continuous in $\bar{G}$, then the function

$$w_*(z) = \overline{w\left(\frac{1}{\bar{z}}\right)}$$

is continuous outside G and satisfies the equation $\mathfrak{C}(w) = 0$ outside $G+\Gamma$. Therefore, according to the generalized Cauchy formula (10.6), Ch. III, we have

$$w(z) = \frac{1}{2\pi i} \int_\Gamma \Omega_1(z,\zeta) w(\zeta) d\zeta - \Omega_2(z,\zeta) \overline{w(\zeta)} d\bar{\zeta}, \quad z \in G, \quad (7.63)$$

$$0 = \frac{1}{2\pi i} \int_\Gamma \Omega_1(z,\zeta) \overline{w(\zeta)} d\zeta - \Omega_2(z,\zeta) w(\zeta) d\bar{\zeta} -$$

$$-\frac{1}{2\pi i} \int_{\Gamma_R} \Omega_1(z,\zeta) w_*(\zeta) d\zeta - \Omega_2(z,\zeta) \overline{w_*(\zeta)} d\bar{\zeta}, \qquad z \in G, \quad (7.64)$$

where Γ_R is a circle of sufficiently large radius R with centre at the point $z = 0$. If we pass to the limit i.e. $R \to \infty$, in the relation (7.64) bearing in mind that

$$w_*(\infty) = \overline{w(0)} = c_0 - ic_1,$$

we obtain

$$c_0 w_0(z) - c_1 w_1(z)$$
$$= \frac{1}{2\pi i} \int_\Gamma \Omega_1(z,\zeta) \overline{w(\zeta)} d\zeta - \Omega_2(z,\zeta) w(\zeta) d\bar{\zeta}, \quad (7.65)$$

where w_0 and w_1 are constant solutions of the equation $\mathfrak{C}(w) = 0$ satisfying the conditions (Ch. III, §4.9, §6.1)

$$w_0(\infty) = 1\,, \qquad w_1(\infty) = i\,.$$

Adding the relations (7.63) and (7.65) we obtain

$$w(z) = \int_0^{2\pi} \tilde{X}_0(z, \zeta)\gamma(\psi)\,d\psi - c_0 w_0(z) + c_1 w_1(z)\,, \quad (7.66)$$

where

$$\gamma(\psi) = \operatorname{Re} w(e^{i\psi})\,,$$

$$\tilde{X}_0(z, \zeta) = \frac{1}{\pi}[\Omega_1(z, \zeta)\zeta + \Omega_2(z, \zeta)\bar{\zeta}]\,, \qquad \zeta = e^{i\psi}\,. \quad (7.67)$$

Setting in (7.66) $z = 0$ we have

$$c_0\big(1 + w_0(0)\big) + c_1\big(i - w_1(0)\big) = \int_\Gamma \tilde{X}_0(z, e^{i\psi})\gamma(\psi)\,d\psi\,. \quad (7.68)$$

But the following relations cannot hold at the same time

$$w_0(0) = -1\,, \qquad w_1(0) = i\,,$$

because it would contradict the inequality (equation (6.6) on p. 162)

$$\operatorname{Im}[\overline{w_0(z)}\,w_1(z)] > 0\,.$$

Consequently, putting for instance $w_0(0) \neq -1$, we obtain, by (7.68),

$$c_0 = \int_0^{2\pi} \frac{\tilde{X}_0(0, e^{i\psi})}{1 + w_0(0)}\gamma(\psi)\,d\psi + c_1\frac{w_1(0) - i}{w_0(0) + 1}\,.$$

Introducing this expression into the right-hand side of (7.66) we have

$$w(z) = \int_0^{2\pi} X_0(z, e^{i\psi})\gamma(\psi)\,d\psi + c\hat{w}(z) \qquad (c = c_1)\,, \quad (7.69)$$

where

$$X_0(z, e^{i\psi}) = \tilde{X}_0(z, e^{i\psi}) - \frac{w_0(z)\tilde{X}_0(0, e^{i\psi})}{w_0(0) + 1}\,, \quad (7.70)$$

$$\hat{w}(z) = w_1(z) + \frac{i - w_1(0)}{1 + w_0(0)}w_0(z)\,.$$

Thus, we have again arrived at the formula (7.61) which is a generalization of the Schwarz formula. The function $X_0(z, \zeta)$ appearing in the formula (7.69) will hereafter be called *the kernel of the generalized Schwarz formula* for the equation $\mathfrak{C}(w) = 0$.

7.8. We shall now demonstrate how for $n > 0$ the solution of Problem **A** can be obtained with the aid of the formula (7.69). Let us represent the unknown solution in the form

$$w(z) = \widetilde{w}(z) + \sum_{k=1}^{2n} c_k \widetilde{w}_k(z) \qquad (c_k\text{—real constant})\,, \tag{7.71}$$

where

$$\widetilde{w}_{2k-1}(z) = \mathfrak{R}_0(z^{k-1}); \qquad \widetilde{w}_{2k}(z) = \mathfrak{R}_0(iz^{k-1}) \tag{7.72}$$
$$(k = 1, \ldots, 2n)\,,$$
$$\widetilde{w}(z) = \mathfrak{R}_0\big(z^n \Phi(z)\big)\,, \tag{7.73}$$

In the above relations $\Phi(z)$ is an unknown function holomorphic in G. $\mathrm{R}_0(\Phi)$ is the operator introduced in §7, Ch. III associating with the analytic function Φ and the point $z = \infty$, a solution of the equation $\mathfrak{C}(w) = 0$ of the form

$$w(z) = \Phi(z) e^{\omega(z)}\,, \tag{7.74}$$

where ω is continuous on the entire plane, it is holomorphic outside $G + \Gamma$ and vanishes at infinity. It is evident that the function $w_* = z^{-n}\widetilde{w}(z)$ is continuous in G, satisfies the equation

$$\mathfrak{C}_n(w^*) \equiv \partial_{\bar{z}} w^* + B_n \overline{w}_* = 0\,, \qquad B_n = B\left(\frac{\bar{z}}{z}\right)^n \tag{7.75}$$

and the boundary condition

$$\mathrm{Re}[w_*(z)] = \gamma(z) - \sum_{k=1}^{2n} c_k \mathrm{Re}[z^{-n}\widetilde{w}_k(z)]\,, \qquad z \in \Gamma\,. \tag{7.76}$$

Applying the formula (7.69) we have

$$w_*(z) = \int_0^{2\pi} X_0(z, \zeta)\gamma(\psi)\,d\psi -$$
$$- \sum_{k=1}^{2n} \int_0^{2\pi} X_0(z, \zeta)\,\mathrm{Re}[\zeta^{-n}\widetilde{w}_k(\zeta)]\,d\psi + c_0 \hat{w}_0(z)\,. \tag{7.77}$$

where X_0 is the kernel of the generalized Schwarz formula for the equation (7.75), $\hat{w}_0$ is a solution of the equation $\mathfrak{C}_n(w) = 0$ satisfying the boundary condition $\mathrm{Re}[\hat{w}] = 0$ (on Γ), and c_0 is an arbitrary real constant. Since $\widetilde{w} = z^n w_*$ the formula (7.71) takes the form

$$w(z) = \int_\Gamma X_n(z, \zeta)\gamma(\psi)\,d\psi + \sum_{k=0}^{2n} c_k w_k(z)\,, \tag{7.78}$$

where

$$X_n(z, \zeta) = z^n X_0(z, \zeta)\,, \tag{7.79}$$

and $w_0, w_1, \dots, w_{2n}$ are linearly independent solutions of the homogeneous Problem $\mathring{\mathbf{A}}$. Also

$$w_k(z) = \widetilde{w}_k(z) - \int_\Gamma z^n X_0(z, \zeta)\,\mathrm{Re}[\zeta^{-n}\widetilde{w}_k(\zeta)]\,d\psi \tag{7.80}$$

$$(\zeta = e^{i\psi}\,, \quad k = 1, \dots, 2n);$$

$$w_0(z) = z^n \hat{w}_0(z)\,. \tag{7.81}$$

The formula (7.78) gives the general solution of Problem **A** for $n \geqslant 0$ and $F \equiv 0$.

7.9. In this section we shall investigate the differential properties of the solution of Problem **A** in the closed domain. It is evident that these properties depend on the differential properties of the coefficients and the free terms of the problem, and on the smoothness properties of the contour of the domain.

We have the following

THEOREM 4.16a. *If* (1) $G \in C_\mu^{k+1}$, (2) $A, B, F \in C_\mu^k(G+\Gamma)$ *and* (3) λ *and* $\gamma \in C_\mu^{k+1}(\Gamma)$, *then the solution of Problem* $\mathring{\mathbf{A}}$, *if it exists, belongs to the class* $C_\mu^{k+1}(G+\Gamma)$ $(k \geqslant 0,\ 0 < \mu < 1)$

PROOF. Let us first consider the case of the circular domain G $(|z| < 1)$.

Let n be the index of the function $\lambda(z)$ with respect to the circumference Γ. We have then

$$\overline{\lambda(z)} = z^{-n} e^{\chi(z)} e^{-p(z)}\,, \tag{7.82}$$

where $\chi = p+iq$ is a function holomorphic in G which is represented by the Schwarz integral

$$\chi(z) = \frac{1}{2\pi}\int_0^{2\pi} q(t)\frac{t+z}{t-z}\frac{dt}{t},$$

where $q(t) = -\arg\lambda(t)+n\arg t$, $t \in \Gamma$. Since according to the assumption $q \in C_\mu^{k+1}(\Gamma)$, in view of Theorem 1.10 $\chi(z) \in C_\mu^{k+1}(G+\Gamma)$. According to (7.82) the boundary condition of Problem A is reduced to the form

$$\operatorname{Re}[z^{-n}w_1(z)] = \gamma_1(z), \tag{7.83}$$

where

$$w_1(z) = e^{\chi(z)}w(z), \qquad \gamma_1(z) = \gamma(z)e^{p(z)}.$$

Obviously, $\gamma_1 \in C_\mu^{k+1}(\Gamma)$. For w_1 we have the equation

$$\partial_{\bar z}w_1 + A_1w_1 + B_1\overline{w}_1 = F_1,$$

where

$$A_1 = A, \qquad B_1 = Be^{\chi-\bar\chi}, \qquad F_1 = e^{\chi}F.$$

Consequently, A_1, B_1 and $F_1 \in C_\mu^k(G+\Gamma)$.

We first consider the case $n \geqslant 0$. Then, as was proved in §7.2 w_1 satisfies the integral equation

$$w_1 + Q_nw_1 = P_nF_1 + \frac{z^n}{2\pi i}\int_\Gamma \gamma_1(t)\frac{t+z}{t-z}\frac{dt}{t} + \sum_{l=0}^{2n} c_lz^l \tag{7.84}$$

$$(c_{2n-l} = -\overline{c_l}, \qquad l = 0, 1, \ldots, n).$$

In accordance with the formulae (7.7) and (7.9),

$$Q_nw_1 = \mathrm{T}_G(A_1w_1 + B_1\overline{w_1}) + z^{2n+1}\mathrm{T}_{G'}(A_*w_* + B_*\overline{w_*}),$$

where G' is the exterior of the circle $|z| < 1$ and

$$A_*(\zeta) = \overline{A_1\left(\frac{1}{\bar\zeta}\right)}, \qquad B_*(\zeta) = \overline{B_1\left(\frac{1}{\bar\zeta}\right)},$$

$$w^*(\zeta) = \frac{\overline{w_1\left(\frac{1}{\bar\zeta}\right)}}{\zeta\bar\zeta^2}.$$

If we now take into account the corollary of Theorem 1.32 (p. 61 above) we easily find that $Q_n w$ is a completely continuous operator mapping $C^k_\mu(G+\Gamma)$ onto $C^{k+1}_\mu(G+\Gamma)$. On the same basis $P_n F_1 \in C^{k+1}_\mu(G+\Gamma)$. Moreover, in view of Theorem 1.10 the integral over the circle entering the right-hand side of (7.84) also belongs to $C^{k+1}_\mu(G+\Gamma)$. Consequently, the right-hand side of the equation (7.84) belongs to the class $C^{k+1}_\mu(G+\Gamma)$. Since the inverse operator $(\mathrm{I}+Q_n)^{-1}$ exists and it is, obviously, an operator mapping $C^k_\mu(G+\Gamma)$ onto $C^{k+1}_\mu(G+\Gamma)$, the solution $w_1(z)$ of the equation (7.84) belongs to the class $C^{k+1}_\mu(G+\Gamma)$. We have now for the function $w(z) = w_1 e^{-\chi}$ which is the solution of the original Problem **A**, $w(z) \in C^{k+1}_\mu(G+\Gamma)$. The same result is also obtained in the case $n < 0$, by considering the integral equation (7.32). Thus, the theorem is proved in the case of a circular domain.

Let us now proceed to the case of an arbitrary simply-connected domain of the class C^{k+1}_μ. Mapping this domain onto the unit circle G', $|\zeta| < 1$, we find that the function $w^*(\zeta) = w[\varphi(\zeta)]$ (w is the solution of the original Problem **A**, $\varphi(\zeta)$ is the function establishing the conformal mapping) satisfies the equation

$$\partial_{\bar\zeta} w_* + A_*(\zeta) w_* + B_*(\zeta)\overline{w}_* = F_*(\zeta) \qquad (\text{in } G')$$

and the boundary condition

$$\operatorname{Re}[\lambda_*(\zeta)\overline{w_*(\zeta)}] = \gamma_*(\zeta) \qquad (\text{on } \Gamma)\,,$$

where

$$A_* = \overline{\varphi'}A[\varphi(\zeta)]\,, \qquad B_* = \overline{\varphi'}B[\varphi(\zeta)]\,, \qquad F_* = \overline{\varphi'}F[\varphi(\zeta)]$$
$$\lambda_* = \lambda\big(\varphi(\zeta)\big)\,, \qquad \gamma_* = \gamma[\varphi(\zeta)]\,.$$

Since according to Theorem 1.8 $\varphi \in C^k_\mu(G'+\Gamma')$, it is readily observed that

$$A_*\,, B_*\,, F_* \in C^k_\mu(G'+\mathrm{I})\,, \qquad \lambda_* \text{ and } \gamma^* \in C^{k+1}_\mu(\mathrm{I})\,.$$

Hence, in view of Theorem 4.13 which was already proved for a circular domain, $w_*(\zeta) \in C^{k+1}_\mu(G'+\Gamma')$. It is therefore evident that $w(z) = w_*\big(\psi(z)\big) \in C^{k+1}_\mu(G+\Gamma)$. It should be

here taken into account that the function $\psi(z)$ inverse to $\varphi(\zeta)$ belongs to $C_\mu^{k+1}(G+\Gamma)$ (Theorem 1.8). Thus, Theorem 4.16 is now completely proved for a simply-connected domain.

Let us now proceed to the proof of the theorem for a multiply-connected domain. Taking into account that the contours belong to the class C_μ^{k+1}, the functions A, B, F, belonging according to the assumption to the class $C_\mu^k(G+\Gamma)$, can be continued outside $G+\Gamma$ preserving the class on the entire plane, and this can be done in such a way that $A = B = F = 0$ near infinity. By means of the generalized Cauchy formula the solution of Problem **A** can be represented in the form

$$w(z) = w_0(z) + w_1(z) + \dots + w_m(z) ,$$

where

$$w_j(z) = \frac{1}{2\pi i} \int_{\Gamma_j} \Omega_1(z, \zeta) w(\zeta) d\zeta - \Omega_2(z, \zeta) \overline{w(\zeta) d\zeta}$$

$$j = 0, 1, \dots, m) .$$

Here Ω_1 and Ω_2 are the normal kernels of the equation $\mathfrak{C}(w) = 0$, corresponding to the entire plane. By virtue of Theorem 3.3 w and $w_j (j = 1, 2, \dots, m)$ belong to the class C_μ^{k+1} inside Γ_0 and outside Γ_j, respectively. Besides, $w_j(z)$ $(j = 0, 1, \dots, m)$ satisfies the boundary condition

$$\operatorname{Re}[\lambda(z) w_j(z)] = \gamma_j(z) \equiv \gamma - \sum_{\substack{i=0 \\ i \neq j}}^{m} \operatorname{Re}(\bar{\lambda} w_i) \quad (\text{on } \Gamma_j) .$$

Obviously, $\gamma_j \in C_\mu^{k+1}(\Gamma_j)$. Therefore, according to the result proved above, $w_j(z) \in C_\mu^{k+1}(G_j + \Gamma_j)$ $(j = 0, 1, \dots, m)$. Hence, $w(z) \in C_\mu^{k+1}(G+\Gamma)$. This completes the proof of the theorem.

§8. The boundary value problem of inclined derivative for an elliptic equation of the second order

In this section we shall investigate some problems for elliptic equations of the second order with boundary conditions containing the unknown function and its first

derivatives; these problems are usually called *the problems of inclined derivative*. Similar problems were first considered by Poincaré in connection with the theory of tides, [73]. It should be observed that an investigation of these problems is of a fundamental importance for various reasons; they cannot, for instance be regarded as classical problems. An application of the classical method of potential to these problem leads, as a rule, to singular integral equations for which the ordinary Fredholm alternative is not valid. Accordingly, in the last two or three decades great interest has arisen in connection with the problem of inclined derivative and singular integral equations. The problem was investigated both by means of the classical methods of potential theory and integral equations, [54], and particularly in the last years by the methods of functional analysis [16], [17]. An application of the theory of singular integral equations led to most complete results in the case of two independent variables, especially for equations with analytic coefficients, [14b], [91].

These results will here be generalized to the case of equations with non-analytic coefficients [14h] in a two-dimensional domain, the coefficients being subjected to comparatively weak conditions. We succeded in obtaining more general results that the classical ones in the investigation of wide classes of problems for which the Fredholm alternative is not valid. In many dimensional problems, however, more or less definite results were obtained at the expense of very strong restrictions which in fact exclude all non-Fredholm cases. The basic restriction in this case concerns the direction of the differentiation in the boundary condition. In the two-dimensional case these restrictions are much weaker and exactly for this reason there are cases in the two-dimensional problems in which the classical Fredholm alternative is not valid. At the same time numerous examples from many branches of mathematics and its applications indicate that these

more general problems are of no less interest than the classical ones. It is therefore of a great importance to achieve further progress in many-dimensional problems in non-Fredholm cases.

8.1. To begin with we shall investigate the boundary value problem which can at once be reduced to Problem **A** investigated in the preceding paragraphs.

Problem **B**. *It is required to find in the domain G the solution $U(x, y)$ of the equation*

$$\Delta U + a(x, y)\, U_x + b(x, y)\, U_y = f(x, y)\,, \tag{8.1}$$

satisfying the boundary condition

$$\alpha U_x - \beta U_y = \gamma\,. \tag{8.2}$$

Introducing the notation

$$u = U_x\,, \qquad v = -U_y\,, \tag{8.3}$$

the equation (8.1) can be reduced to the system

$$u_x + v_y = 0\,, \qquad u_x - v_y + au - bv = f\,, \tag{8.4}$$

The boundary condition now has the form

$$\alpha u + \beta v = \gamma\,, \tag{8.5}$$

or, in complex notation,

$$\partial_{\bar z} w + \tfrac{1}{4}(a + ib)\, w + \tfrac{1}{4}(a - ib)\overline{w} = \tfrac{1}{2} f\,, \tag{8.6}$$

$$\operatorname{Re}[\overline{\lambda(z)}\, w(z)] = \gamma\,, \qquad \lambda = \alpha + i\beta\,, \tag{8.7}$$

where $w = u + iv$. We have thus arrived at Problem **A** which was investigated in the preceding section.

If the function $w = u + iv$ is the solution of Problem **A**, the solution of Problem **B** is obtained by means of the following curvilinear integral:

$$U(x, y) = c_0 + \operatorname{Re} \int_{z_0}^{z} w(\zeta)\, d\zeta\,, \qquad c_0 = \text{const}\,. \tag{8.8}$$

If G is a simply-connected (bounded) domain the right-hand side of the relation (8.8) is a single-valued function of the point z (z_0 and c_0 being fixed).

Thus, in the case of a simply-connected domain G the solution of Problem **B** is entirely equivalent to the solution of the corresponding Problem **A**. In other words, Problem **B** is soluble if and only if Problem **A** is soluble, and the solution of Problem **B** is constructed according to the formula (8.8). It should be observed that the homogeneous Problem $\mathring{\mathbf{B}}$ ($\gamma \equiv f \equiv 0$) has always a solution equal to a constant. Hence, if the homogeneous Problem $\mathring{\mathbf{A}}$ has l linearly independent solutions the homogeneous Problem $\mathring{\mathbf{B}}$ has $l+1$ linearly independent solutions.

Speaking of the solution of the equation (8.1) we have in mind the generalized solution which is defined as follows. Let $w(z) = u(x, y)+iv(x, y)$ be the generalized solution of the equation (8.6). *Then the function* $U(x, y)$ *will be called the generalized solution of the equation* (8.1) *in the vicinity of the point* z_0 *if it is represented in this vicinity by the formula* (8.8). *The function* $U(x, y)$ *will be called the solution of the equation* (8.1) *in the domain* G *if it is the solution of this equation in the vicinity of every point of the domain.* In what follows, when speaking of the solution of the equation (8.1) in the domain we shall understand continuous single-valued solutions.

Let $a, b, f \in L_p(\bar{G})$, $p > 2$. Then continuous solutions of the equation (8.6) belong to the class $D_{1,p}$ inside G. Consequently in this case continuous solutions of the equation of the second order (8.1) belong to the class $D_{2,p}$ inside the domain. If $a, b, f \in C_\nu^m$, $0 < \nu < 1$, then $w \in C_\nu^{m+1}$ and $U(x, y) \in C_\nu^{m+2}$ inside G.

If the domain G is multiply-connected, the right-hand side of the relation (8.8) will in general be a multi-valued function in the domain G. Let $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$ be simple closed rectifiable contours bounding the domain G, $\Gamma_1, \ldots, \Gamma_m$ being situated inside Γ_0. Then for the single-

valuedness of the right-hand side of (8.8) it is necessary and sufficient that the following relations be satisfied

$$\mathrm{Re} \int_{\Gamma_j} w\,d\zeta \equiv \int_{\Gamma_j} u\,dx - v\,dy = 0 \qquad (j = 1, 2, \ldots, m). \qquad (8.9)$$

Thus, in the case of a multiply-connected domain Problem **B** in general is not equivalent to the corresponding Problem **A**. In order to obtain the solution of Problem **B** the solution of Problem **A** should satisfy the additional relations (8.9), i.e. to the conditions of solubility of Problem **A** m relations (8.9) are added.

8.2. The pair of functions α, β defines at every point of the contour Γ a definite direction $\boldsymbol{l}$,

$$\cos(l, x) = \alpha, \qquad \cos(l, y) = -\beta \qquad (\alpha^2 + \beta^2 = 1). \qquad (8.10)$$

The condition (8.2) therefore can be written in the form

$$\frac{\partial U}{\partial l} = \gamma. \qquad (8.11)$$

The direction $\boldsymbol{l}$ varies along the contour and makes an angle ϑ with the normal ν to the curve Γ, which in general is a variable function of the point of the contour. Problem **B** is therefore called the problem of the inclined derivative. If $\boldsymbol{l}$ coincides at all points of Γ with the normal we have *the second fundamental problem* for the equation (8.1), usually called *the Neumann problem.* If $\boldsymbol{l}$ coincides at all points of the boundary Γ with the tangent, the relation (8.11) is equivalent to the condition

$$U = \int_{s_0}^{s} \gamma(s)\,ds + c_0, \qquad c_0 = \text{const}. \qquad (8.12)$$

Consequently, we have *the first fundamental problem* for the equation (8.1), usually called *the Dirichlet problem.* In this case, in order to ensure the single-valuedness and continuity of the unknown function U on all boundary contours, it is necessary to require that the following conditions be satisfied:

$$\int_{\Gamma_j} \gamma\,ds = 0 \qquad (j = 0, 1, \ldots, m). \qquad (8.13$$

Thus, the celebrated classical boundary value problems of Dirichlet and Neumann are particular cases of Problem **B** and, consequently, can be reduced to the boundary value Problem **A** investigated in the preceding sections.

8.3. Let us now consider a more general boundary value problem.

Problem **C**. *It is required to find in the domain G the solution $U(x, y)$ of the equation*

$$\Delta U + a(x, y) U_x + b(x, y) U_y + \varepsilon c(x, y) U = f(x, y) , \quad (8.14)$$

satisfying the boundary condition

$$\alpha U_x - \beta U_y + \varepsilon\gamma U = \delta , \quad \alpha^2 + \beta^2 \equiv 1 , \quad (8.15)$$

where a, b, c, f are known real functions of the point $z = x + iy$ of the domain G, α, β, γ, δ are known real functions of the point on the contour Γ bounding the domain G, and ε is a constant parameter. The boundary condition (8.15) can also be written thus:

$$\frac{dU}{dl} + \varepsilon\gamma U = \delta , \quad (8.16)$$

where l is the direction defined by the relations (8.10).

Confining ourselves to the case of a simply-connected domain we shall give here a method of solution of Problem **C** closely related to the method of solution of Problem **A** developed in the preceding sections. This method was proposed by the author in the paper [14h].

With the aid of the conformal mapping of the domain the problem can be reduced to an analogous problem for the unit circle $|z| < 1$. We shall assume therefore that the domain G under consideration is the circle $|z| < 1$ and the contour Γ is the circumference $|z| = 1$. In respect of the other data of the problem the following assumptions will be made: (1) $a, b, c, f \in L_p(G+\Gamma)$, $p > 2$, (2) α, β, γ and $\delta \in C_\sigma(\Gamma)$, $0 < \sigma < 1$, (3) $\Gamma \in C^k_\mu$, $k \geqslant 1$, $0 < \mu \leqslant 1$. The unknown solution will be understood in the generalized sense. It belongs to $C^1(\bar{G})$ and $D_{2,p}(G)$, $p > 2$.

Let us introduce the complex function

$$w(z) = U_x - iU_y \equiv 2\partial_z U\,, \tag{8.17}$$

then the equation (8.14) and the boundary conditions (8.15) have the form

$$\partial_{\bar z} w + \tfrac{1}{4}(a+ib)w + \tfrac{1}{4}(a-ib)\overline{w} + \tfrac{1}{2}\varepsilon c U = \tfrac{1}{2}f\,, \tag{8.18}$$

$$\operatorname{Re}[\overline{\lambda(z)}w] + \varepsilon\gamma U = \delta\,, \qquad \lambda = \alpha + i\beta\,. \tag{8.19}$$

We shall now prove that the unknown (real) function $U(x, y)$ which is given by the formula (8.8) can be also represented in the form

$$\Delta U + a(x, y)\,U_x + b(x, y)\,U_y + \varepsilon c(x, y)\,U = f(x, y)\,, \tag{8.14}$$

where c_0 is a constant. According to the formula (6.10) of Ch. I,

$$U(x, y) = -\frac{1}{2\pi}\iint\limits_G \frac{\overline{w(\zeta)}}{\zeta - z}\,d\xi\,d\eta + \Phi(z)\,, \tag{8.21}$$

$$\Phi(z) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{U(t)\,dt}{t-z}\,.$$

Since $U = \overline{U}$ on the circumference Γ we have

$$\Phi(z) - \overline{\Phi(z)} = \frac{1}{2\pi}\iint\limits_G \frac{\overline{w}(\zeta)\,d\xi\,d\eta}{\zeta - z} + \frac{1}{2\pi}\iint\limits_G \frac{zw(\zeta)}{1 - z\zeta}\,d\xi\,d\eta\,,$$

$$z \in \Gamma\,.$$

From this relation, the Cauchy theorem and Cauchy formula it follows that

$$\Phi(z) - \overline{\Phi(0)} = \frac{1}{2\pi}\iint\limits_G \frac{zw(\zeta)}{1 - z\bar\zeta}\,d\xi\,d\eta\,, \qquad z \in G\,, \tag{8.22}$$

where

$$\overline{\Phi(0)} = \frac{1}{2\pi i}\int\limits_\Gamma \frac{U(t)\,dt}{t} = \frac{1}{2\pi}\int\limits_0^{2\pi} U(e^{i\psi})\,d\psi = c_0\,, \quad c_0 = \bar c_0\,. \tag{8.23}$$

Substituting in the right-hand side of (8.21) for Φ the expression (8.22) and remembering that $U = \overline{U}$ we arrive at the relation (8.20).

It follows from (8.17) that w satisfies the relation

$$\partial_{\bar{z}} w - \partial_z \overline{w} = 0 \tag{8.24}$$

Therefore, it remains to prove that if w satisfies the relation (8.24) in the domain G and is continuous in $\overline{G}$, the real function of the form (8.20) satisfies the relation (8.17). Applying to (8.20) the operation ∂_z we obtain

$$\partial_z U = \frac{1}{4} w - \frac{1}{4\pi} \partial_z \iint_G \frac{\overline{w(\zeta)}}{\zeta - z} d\xi\, d\eta + \\ + \frac{1}{4\pi} \partial_z \iint_G \frac{z w(\zeta)}{1 - \bar{\zeta} z} d\xi\, d\eta . \tag{8.25}$$

Making use of Green's identity and the relation (8.24) we easily derive the formulae

$$\iint_G \frac{z w(\zeta)\, d\xi\, d\eta}{1 - \bar{\zeta} z} \\ = -\frac{1}{2i} \int_\Gamma w \ln(1 - \bar{\zeta} z)\, d\zeta - \frac{1}{2i} \int_\Gamma \overline{w} \ln(1 - \bar{\zeta} z)\, d\bar{\zeta} ,$$

$$\iint_G \frac{\overline{w}\, d\xi\, d\eta}{\zeta - z} = -\frac{1}{2i} \int_\Gamma \overline{w} \ln[(\zeta - z)(\bar{\zeta} - \bar{z})]\, d\bar{\zeta} - \\ - \iint_G \partial_{\bar{\zeta}} w \ln[(\zeta - z)(\zeta - z)]\, d\xi\, d\eta .$$

Differentiating these relations with respect to z we have

$$\partial_z \iint \frac{z w(\zeta)\, d\xi\, d\eta}{1 - \bar{\zeta} z} = \frac{1}{2i} \int_\Gamma \frac{w\, d\zeta}{\zeta - z} + \frac{1}{2i} \int_\Gamma \frac{\overline{w}\, d\bar{\zeta}}{\zeta - z} ,$$

$$\partial_z \iint_G \frac{\overline{w}\, d\xi\, d\eta}{\zeta - z} = \frac{1}{2i} \int_\Gamma \frac{\overline{w}\, d\bar{\zeta}}{\zeta - z} + \iint_G \frac{\partial_{\bar{\zeta}} w}{\zeta - z}\, d\xi\, d\eta \\ = \frac{1}{2i} \int_\Gamma \frac{\overline{w}\, d\zeta}{\zeta - z} - \pi w + \frac{1}{2i} \int_\Gamma \frac{w\, d\zeta}{\zeta - z} .$$

In view of the last formulae the right-hand side of the relation (8.25) is equal to $\frac{1}{2}w$. This completes the proof.

As we have often done before, let us make use of the substitutions

$$\overline{\lambda(z)} = z^{-n}e^{\chi(z)}e^{-p(z)}, \qquad w_*(z) = e^{\chi(z)}w(z), \qquad (8.25a)$$

where $\chi(z) = p+iq$ is a function holomorphic in G represented by the Schwarz integral

$$\chi(z) = \frac{1}{2\pi}\int_\Gamma q(t)\frac{t+z\,dt}{t-zt}, \qquad q = -\arg\lambda + n\arg z,$$

Then we can reduce the equation (8.18) and the boundary condition (8.19) to the form

$$\partial_{\bar z}w_* + Aw_* + B\overline{w}_* + \varepsilon CU = F \quad (\text{in } G), \qquad (8.26)$$

$$\operatorname{Re}[z^{-n}w_*(z)] + \varepsilon\gamma_* U = \delta_* \quad (\text{on } \Gamma), \qquad (8.27)$$

where

$$A = \tfrac{1}{4}(a+ib), \qquad B = \tfrac{1}{4}(a-ib)e^{2iq},$$

$$C = \tfrac{1}{2}ce^{\chi}, \qquad F = \tfrac{1}{2}e^{\chi}f, \qquad \gamma_* = \gamma e^{p}, \qquad \delta_* = \delta e^{p}.$$

The formula (8.20) yielding the required solution of Problem C can now be written thus:

$$U(x,y) = c_0 + \mathrm{P}w_*, \qquad (8.28)$$

where c_0 is a real constant,

$$\mathrm{P}w_* = \operatorname{Re}\left[-\frac{1}{2\pi}\iint_G\left(\frac{\overline{e^{-\chi(\zeta)}w_*(\zeta)}}{\zeta-z} - \frac{ze^{-\chi(\zeta)}w_*(\zeta)}{1-\bar\zeta z}\right)d\xi\,d\eta\right]. \qquad (8.29)$$

8.4. Let us first consider the case $n \geqslant 0$. Setting the equation (8.26) and the boundary condition (8.27) in the form

$$\partial_{\bar z}w_* + Aw_* + B\overline{w}_* = F - \varepsilon CU \quad (\text{in } G), \qquad (8.30)$$

$$\operatorname{Re}[z^{-n}w_*(z)] = \delta_* - \varepsilon\gamma_* U \quad (\text{on } \Gamma) \qquad (8.31)$$

and regarding the right-hand sides for the time as being known, we have the problem investigated in the preceding section. Its solution can be written in the form

$$w_*(z) = \hat{w}(z) + \varepsilon\mathrm{P}'U + \sum_{k=1}^{2n+1} c_k w_k(z), \qquad (8.32)$$

where c_k are arbitrary real constants, $w_1, \ldots, w_{2n+1}$ are linearly independent solutions of the homogeneous problem

$$\partial_{\bar{z}} w + Aw + B\bar{w} = 0 \quad (\text{in } G)\,, \tag{8.33}$$
$$\operatorname{Re}[z^{-n}w] = 0 \quad (\text{on } \Gamma)\,,$$

and $\hat{w} + \varepsilon \mathrm{P}' U$ is the particular solution of the problem (8.30)–(8.31). By formula (7.57) we have the expressions

$$\hat{w}(z) = \int_0^{2\pi} \hat{X}_n(z, e^{i\varphi})\,\delta(\varphi)\,d\varphi + \iint_G \hat{\Omega}_n(z, \zeta) f(\xi, \eta)\,d\xi\,d\eta\,,$$

$$\mathrm{P}' U = -\int_0^{2\pi} \hat{X}_n(z, e^{i\varphi})\,\gamma(\varphi)\,U\,d\varphi -$$
$$- \iint_G [\Omega_n'(z, \zeta)\,C(\zeta) + \Omega_n''(z, \zeta)\,\overline{C(\zeta)}]\,U(\xi, \eta)\,d\xi\,d\eta\,,$$

for $\hat{w}$ and $\mathrm{P}' U$, where

$$\hat{X}_n(z, e^{i\varphi}) = X_n(z, e^{i\varphi})\,e^{p(\varphi)}\,,$$

$$\Omega_n(z, \zeta) = \tfrac{1}{2}\Omega_n'(z, \zeta)\,e^{\chi(\zeta)} + \tfrac{1}{2}\Omega_2''(z, \zeta)\,e^{\overline{\chi(\zeta)}}\,.$$

Thus, $\hat{w}$ is a known function in z and $\mathrm{P}' U$ depends on the as yet unknown function U. Substituting in the right-hand side of the relation (8.28) for w_* the expression (8.32) we obtain the following integral equation for U:

$$U - \varepsilon \hat{\mathrm{P}} U = \mathrm{P}\hat{w}(z) + c_0 + \sum_{k=0}^{2n+1} c_k \mathrm{P} w_k(z)\,, \tag{8.34}$$

It is readily seen that $\hat{\mathrm{P}} = \mathrm{P}\mathrm{P}'$ is a completely continuous operator in $C(G+\Gamma)$ and in an arbitrary $L_q(G+\Gamma)$, $q \geqslant \dfrac{p}{p-1}$. Hence, we may apply the Fredholm theorems to the equation (8.34).

Thus, if Problem **C** has the solution $U(x, y)$, it will be a solution of the equation (8.34) for some fixed values of the constants $c_0, c_1, \ldots, c_{2n+1}$. Conversely, if for some

fixed values of the constants $c_0, c_1, \dots, c_{2n+1}$ the equation (8.34) has a solution U, then it is the solution of Problem **C**.

Let $\varepsilon_1, \varepsilon_2, \dots$ $(0 < |\varepsilon_1| \leqslant |\varepsilon_2| \leqslant \dots)$ be the eigenvalues of the homogeneous equation

$$w - \varepsilon \hat{\mathrm{P}} w = 0\,. \tag{8.35}$$

If $\varepsilon \neq \varepsilon_k$ the integral equation (8.34) has a solution for arbitrary values of the constants $c_0, c_1, \dots, c_{2n+1}$. Consequently, in this case the solution of Problem **C** exists for arbitrary functions $f(x, y)$ and $\delta(z)$ appearing in the right-hand sides of the equation (8.14) and the boundary condition (8.15), and the solution is constructed according to the formula

$$U(x, y) = \int_0^{2\pi} S_{n,\varepsilon}(x, y, \vartheta)\gamma(\vartheta)\,d\vartheta +$$

$$+ \iint_G S'_{n,\varepsilon}(x, y, \xi, \eta) f(\xi, \eta)\,d\xi\,d\eta + \sum_{k=0}^{2n+1} c_k U_k(x, y)\,, \tag{8.36}$$

where $S_{n,\varepsilon}$ and $S'_{n,\varepsilon}$ are fully determined functions which depend only on the coefficients of the equation under consideration (8.14), on the functions α, β, γ appearing in the boundary condition (8.15) and on the parameter ε; $U_0(x, y), U_1(x, y), \dots, U_{2n+1}(x, y)$ are linearly independent solutions of the homogeneous Problem $\mathring{\mathbf{C}}$ $(f \equiv \delta \equiv 0)$. Thus, we have proved

THEOREM 4.17. *If $n \geqslant 0$ Problem **C** has a solution for all values of the parameter ε except for, perhaps, a discrete set of values $\varepsilon_1, \varepsilon_2, \dots$ $(0 < |\varepsilon_1| \leqslant |\varepsilon_2| \leqslant \dots)$ which are the eigenvalues of the homogeneous equation* (8.35). *The homogenous Problem $\mathring{\mathbf{C}}$ $(f \equiv \delta \equiv 0)$ has exactly $2n+2$ linearly independent solutions if $\varepsilon \neq \varepsilon_k$ $(k = 1, 2, \dots)$.*

In particular, this theorem implies that *for sufficiently small values of the parameter ε $(0 < |\varepsilon| < \varepsilon_1)$ the homogeneous Problem $\mathring{\mathbf{C}}$ has exactly $2n+2$ linearly independent*

solutions and the non-homogeneous problem is always soluble.

If $\varepsilon \neq \varepsilon_k$ $(k = 1, 2, \ldots)$ it follows from the formula (8.36) that Problem **C** has the particular solution

$$U(x, y) = \int_\Gamma S_{n,\varepsilon}(x, y, \vartheta)\gamma(\vartheta)\,d\vartheta + \\ + \iint_G S'_{n,\varepsilon}(x, y, \xi, \eta)f(\xi, \eta)\,d\xi\,d\eta\,, \qquad (8.37)$$

which depends continuously on the given functions. In this case adding to the boundary condition (8.15) $2n+2$ point conditions (§6) we can obtain the correctness of the problem. Therefore, for $\varepsilon \neq \varepsilon_k$ and $n \geqslant 0$ Problem **C** may be called *quasi-correct.*

Let ε be an eigenvalue of rank p of the equation (8.35). Constructing, in accordance with the third Fredholm theorem, the condition of solubility of the non-homogeneous equation (8.34) we obtain an algebraic system of p linear equations for the determination of $2n+2$ constants $c_0, c_1, \ldots, c_{2n+1}$. Let r be the rank of the corresponding matrix, $0 \leqslant r \leqslant \min(p, 2n+2)$.

Evidently, we have

THEOREM 4.18. *If $n \geqslant 0$ the homogeneous Problem $\mathring{\mathbf{C}}$ has $N = 2n+2+p-r$ linearly independent solutions and the non-homogeneous Problem **C** will be soluble if and only if the following relations are satisfied:*

$$\int_\Gamma \delta j\,ds + \iint_G fh_j\,dx\,dy = 0 \qquad (j = 1, \ldots, p-r)\,, \qquad (8.38)$$

where g_j and h_j are linearly independent functions which depend only on the coefficients of the equation (8.14) *and on the functions α, β, γ entering the boundary condition* (8.15).

In particular, we have proved

THEOREM 4.18a. *If $n \geqslant 0$ the homogeneous boundary value Problem $\mathring{\mathbf{C}}$ has a finite number N of linearly independent solutions, and $N \geqslant 2n+2$.*

8.5. We now proceed to the case $n < 0$. Let $k = -n$. Then according to the results of §7.3 the solution of the

equation (8.30) satisfying the boundary condition (8.31) (U is for the time being regarded as known) satisfies the integral equation

$$w_*(z)+Q_k^* w_* = -\varepsilon P_k^*(CU)-\frac{\varepsilon}{\pi i}\int_\Gamma \frac{\gamma_*(t)\,U(t)\,dt}{t^k(t-z)}+ \\ +P_k^* F_1+\frac{1}{\pi i}\int_\Gamma \frac{\delta_*(t)\,dt}{t^k(t-z)}. \qquad (8.39)$$

Since the operator $\mathrm{I}+Q_k^*$ is invertible, by solving the equation (8.39) and substituting w_* into the right-hand side of (8.28) we obtain for U an equation of the form

$$U-\varepsilon\hat{\mathrm{P}}^* U = \int_0^{2\pi} \hat{X}_{-n}^*(z, e^{i\varphi})\,\delta(\varphi)\,d\varphi+ \\ +\iint_G \hat{\Omega}_{-n}^*(z,\zeta) f(\xi,\eta)\,d\xi\,d\eta+c_0\,, \qquad (8.40)$$

where $\hat{P}^*$ is a completely continuous operator. Besides, the fulfilment of the relations (7.31) leads to $2k-1$ relations

$$\varepsilon\iint_G \chi_j U\,dx\,dy+\iint_G f\chi_j'\,dx\,dy+ \\ +\int_\Gamma \delta\chi_j''\,ds+\varepsilon\int_\Gamma U\tilde{\chi}_j\,ds=0 \qquad (8.41)$$

$$(j=1,2,\ldots,2k-1)\,,$$

where $\chi_j, \chi_j', \chi_j'', \tilde{\chi}_j$ are linearly independent functions, independent of the choice of f and δ.

Assume that ε is not an eigenvalue of the homogeneous integral equation $w-\varepsilon\hat{P}^* w=0$. Then the equation (8.40) is soluble for an arbitrary right-hand side. Imposing upon this solution the conditions (8.41) we obtain $2k-1$ relations

$$\iint_G f\hat{\chi}_j'\,dx\,dy+\int_\Gamma \delta\hat{\chi}_j''\,ds+c_0\hat{H}_j\varepsilon=0 \qquad (8.42)$$

$$(j=1,\ldots,2k-1)\,.$$

It should be observed that the linearly independent functions $\hat{\chi}_j', \hat{\chi}_j''$ and the constants $\hat{H}_j(\varepsilon)$ are independent of the choice of f and δ.

If all the constants $\hat{H}_j$ vanish, the necessary and sufficient conditions of solubility of Problem **C** are the following:

$$\iint_G f\hat{\chi}'_j dx dy + \int_\Gamma \delta\hat{\chi}''_j ds = 0 \qquad (j = 1, \ldots, 2k-1). \tag{8.43}$$

If at least one of the constants $\hat{H}_j$ does not vanish, the constant c_0 is determined uniquely by f and δ from the appropriate relation (8.42). Substituting this value into the remaining relations (8.42) we obtain the necessary and sufficient conditions of solubility of Problem **C** in the form

$$\iint_G f\hat{\chi}'_{*j} dx dy + \int_\Gamma \delta\hat{\chi}''_{*j} ds = 0 \qquad (j = 1, \ldots, 2k-2). \tag{8.44}$$

Thus, we have proved

THEOREM 4.19. *If $n < 0$ the homogeneous Problem $\mathring{\mathbf{C}}$, for all values of the parameter ε, except, perhaps for a discrete set of values $\varepsilon_1, \varepsilon_2, \ldots$ $(0 < |\varepsilon_1| \leqslant |\varepsilon_2| \leqslant \ldots)$, either has no non-trivial solution at all, and then the non-homogeneous Problem **C** has a solution only if $2k-2$ relations* (8.44) *are satisfied, or Problem $\mathring{\mathbf{C}}$ has one non-trivial solution and then the non-homogenous Problem **C** has a solution only if $2k-1$ relations* (8.43) *are satisfied.*

If ε is an eigenvalue of the homogeneous equation $w - \varepsilon\hat{\mathrm{P}}^*w = 0$, an examination of the conditions of solubility of the equation (8.40) and the relations (8.42) yields the following result.

THEOREM 4.20. *If $n < 0$ and p is the rank of the eigenvalue ε of the equation $w - \varepsilon\hat{P}^*w = 0$, the homogeneous Problem $\mathring{\mathbf{C}}$ has l linearly independent solutions, $l \leqslant p+1$, and the non-homogeneous Problem **C** has a solution if and only if $2k-2+l$ relations are satisfied*

$$\iint_G f\chi'_j dx dy + \int_\Gamma \delta\chi''_j ds = 0 \tag{8.45}$$

$$(j = 1, \ldots, 2k-2+l),$$

where χ_j' *and* χ_j'' *are linearly independent functions which are independent of the choice of* f *and* δ.

Theorem 4.20 has two important corollaries which will now be stated in the form of theorems.

THEOREM 4.21. *Problem* **C** *is correct, i.e. it is always soluble and has a unique solution if and only if the following conditions are satisfied:* (1) *the homogeneous Problem* $\mathring{\mathbf{C}}$ *has no non-trivial solution, and* (2) *the index* $n = -1$.

In this case it is evident that the solution of Problem **C** depends continuously on the given functions f and δ.

THEOREM 4.22. *If* $n < -1$ *the non-homogeneous Problem* $\mathring{\mathbf{C}}$ *cannot have a solution for arbitrary given* f *and* δ, *and the number of conditions of solubility is not smaller than* $2k-2$ $(k = -n)$.

8.6. We now investigate the degree of smoothness and the differential properties of the solution of Problem **C** taking into account the smoothness properties of the coefficients and the free terms of the equation and the boundary condition, and of the boundary of the domain.

THEOREM 4.23. *If* (1) $\Gamma \in C_\mu^1 (0 < \mu \leqslant 1)$, (2) $a, b, c, f \in L_p(G+\Gamma), p > 2$, *and* (3) $\alpha, \beta, \gamma, \delta \in C_\nu(\Gamma), 0 < \nu < 1$, *then the solution* $U(x, y)$ *of Problem* **C** *belongs to the class* $C_\tau^1(G+\Gamma)$ *where*

$$\tau = \min\left(\nu, \frac{p-2}{p}\right).$$

PROOF. It is sufficient to consider a simply-connected domain and prove the result for $n \geqslant 0$. A similar argument is valid also for the case $n < 0$. As was shown in §8.4 the solution of Problem **C** is given by the formula (8.28), i.e. $U = c_0 + Pw^*$ where w_* is the solution of the boundary value problem (8.30)–(8.31). But U is a solution of the equation (8.34) and consequently it is continuous in the Hölder sense in $\bar{G}$. Therefore, w_* is continuous in $G+\Gamma$ and according to (8.28) $U \in C_{\frac{p-2}{p}}(G+\Gamma)$. Such being the case, in view of Theorem 3.1 w_* belongs to $C_\tau(G+\Gamma)$. Now

$$U_z = \tfrac{1}{2}w(z) = \tfrac{1}{2}e^{-\chi(z)}w_*(z) .$$

This means that $U \in C^1_\tau(G+\Gamma)$, which was to be proved. It is also evident that $U \in D_{2,p}$.

By means of Theorem 3.1 we can prove in a similar way:

THEOREM 4.23a. *If* (1) $\Gamma \in C^{k+1}_\mu$, (2) $a, b, c, f \in C^k_\mu(G+\Gamma)$, *and* (3) $\alpha, \beta, \gamma, \delta \in C^{k+1}_\mu(\Gamma)$, *then the solution of Problem* **C**, *if it exists, belongs to the class* $C^{k+2}_\mu(G+\Gamma)(k \geqslant 0, 0 < < \mu < 1)$.

8.7. To the preceding problem we can also reduce the boundary value problem of inclined derivative for a more general second order equation of elliptic type

$$a(x, y)\frac{\partial^2 U}{\partial x^2} + 2b(x, y)\frac{\partial^2 U}{\partial x \partial y} + c(x, y)\frac{\partial^2 U}{\partial y^2} +$$

$$+ d(x, y)\frac{\partial U}{\partial x} + e(x, y)\frac{\partial U}{\partial y} + d(x, y)\, U = g(x, y)\,, \qquad (8.46)$$

$$ac - b^2 \geqslant \Delta_0 > 0 \qquad (\text{in } G+\Gamma)\,, \qquad \Delta_0 = \text{const}\,.$$

Assume that $a, b, c \in D_{k+1,p}(G+\Gamma)$, $k \geqslant 0, p > 2$, $\Gamma \in C^{k+1}_\sigma$, $0 < \sigma \leqslant 1$. Then $a, b, c \in C^k_\nu(G+\Gamma)$, $k \geqslant 0$, $\nu = \dfrac{p-2}{p}$, and we may continue these functions onto the entire plane the class being preserved, and near infinity the following conditions can always be ensured: $a = c = = 1, b = 0$. Then there exists the complete homeomorphism $\zeta(z) = \xi(x, y) + i\eta(x, y)$ of the quadratic form $a\,dx^2 - 2b\,dx\,dy + c\,dy^2$ and in consequence of the change of variables

$$\xi = \xi(x, y)\,, \qquad \eta = \eta(x, y) \qquad (8.47)$$

equation (8.46) takes the form (Ch. II, §7.2)

$$\Delta U + p(\xi, \eta)\, U_\xi + q(\xi, \eta)\, U_\eta + r(\xi, \eta)\, U = h(\xi, \eta)\,. \qquad (8.48)$$

Since $\zeta(z) \in D_{k+2,p}(E)$, $\zeta(z) \in C^{k+1}_\nu(E)$, $\nu = \dfrac{p-2}{p}$. It is therefore readily observed that the image Γ' of the contour Γ also belongs to the class $C^{k+1}_{\sigma'}$, $\sigma' = \min\left(\sigma, \dfrac{p-2}{p}\right)$.

Let $d, e, f, g \in D_{k,p}(G+\Gamma)$. Then the formulae (7.29) of Ch. II, indicate that

$$p, q, r, h \in D_{k,p}(G'+\Gamma'), \qquad G' = \zeta(G).$$

The boundary condition

$$\alpha U_x - \beta U_y + \gamma U = \delta \quad (\text{on } \Gamma) \tag{8.49}$$

in consequence of the change of the variables (8.47) takes the form

$$\alpha' U_\xi - \beta' U_\eta + \gamma U = \delta \quad (\text{on } \Gamma), \tag{8.50}$$

where

$$\alpha' = \alpha\xi_x - \beta\xi_y, \qquad \beta' = -\alpha\eta_x + \beta\eta_y. \tag{8.51}$$

Since $\xi_x, \xi_y, \eta_x, \eta_y \in C_\nu^k(E)$, then α' and β' belong to the class $C_\nu^{k'}(\Gamma)$ if $\alpha, \beta \in C_\nu^{k_0}(\Gamma)$; also $k' = \min(k_0, k)$. It follows from (8.51) that

$$\alpha' - i\beta' = (\alpha - i\beta)\zeta_z + (\alpha + i\beta)\zeta_{\bar{z}}.$$

But it was shown in Ch. II (p. 127) that $\zeta(z)$ satisfies the equation

$$\frac{\partial\zeta}{\partial\bar{z}} - q(z)\frac{\partial\zeta}{\partial z} = 0, \qquad |q(z)| \leqslant q_0 < 1. \tag{8.52}$$

whence

$$\alpha' - i\beta' = (\alpha - i\beta)\left(1 + \frac{\alpha + i\beta}{\alpha - i\beta}q(z)\right)\frac{\partial\zeta}{\partial z}. \tag{8.53}$$

But the Jacobian

$$J = \left|\frac{\partial\zeta}{\partial z}\right|^2 - \left|\frac{\partial\zeta}{\partial\bar{z}}\right|^2 = (1 - |q|^2)\left|\frac{\partial\zeta}{\partial z}\right|^2 > 0,$$

Consequently, $\dfrac{\partial\zeta}{\partial z} \neq 0$ on the entire plane.

On the other hand, differentiating the equation (8.52) with respect to z we find that $\dfrac{\partial\zeta}{\partial z}$ satisfies the equation

$$\frac{\partial}{\partial\bar{z}}\left(\frac{\partial\zeta}{\partial z}\right) + q(z)\frac{\partial}{\partial z}\left(\frac{\partial\zeta}{\partial z}\right) = -\frac{\partial q}{\partial z}\frac{\partial\zeta}{\partial z}$$

which is of the form (17.15), Ch. III. Now, according to Theorem 3.28 the principle of the argument may be applied to the function $\partial_z\zeta$. Therefore, it follows from (8.53) that the increment of $\arg(\alpha'+i\beta')$ along Γ' is equal to the increment of $\arg(\alpha+i\beta)$ along Γ. This means that the index of the original boundary condition (8.49) is equal to the index of the transformed boundary condition (8.50). We infer, therefore, that the results of the preceding subsection concerning the boundary value problem of inclined derivative are valid also for the equation of a more general form (8.46).

§9. Application of two-dimensional singular integral equations to the boundary value problems

So far in the investigation of boundary value problems we have employed a reduction of the differential equation to canonical form; this procedure requires the construction of a certain homeomorphism of the appropriate Beltrami system. Now, it was shown in Ch. II that the latter problem is solved by means of a singular integral equation of the form

$$f-q\Pi f=p\,,\qquad \Pi f\equiv\partial_z\mathrm{T}f\equiv-\frac{1}{\pi}\iint\limits_E\frac{f(\zeta)\,d\xi\,d\eta}{(\zeta-z)^2}\,, \tag{*}$$

$$|q|\leqslant q_0<1\,.$$

However, after reduction of the differential equation to the canonical form it is necessary to construct new integral equations in order to solve the boundary value problem. Moreover, for the construction of the formula (if it is required) giving a general representation of solutions of the canonical equation we have also to solve other integral equations. It readily follows, therefore, that such a method of solution of the boundary value problem is connected with serious practical difficulties. Naturally, the question arises: is it possible to avoid all the above intermediate stages and to investigate boundary

value problems directly by means of singular integral equations of the form (*)? We shall see in this section that in many cases it is possible. Moreover, we shall find that the indicated method enables us to increase the class of equations under investigation, since many additional assumptions concerning the coefficients of the equation and ensuring the validity of the reduction of the differential equation to the canonical form, become superfluous. Besides, the method can be applied to both linear and quasi-linear equations. This method was given in author's paper [14i]. Further applications were considered in the papers of Vinogradov [15a, b, c, d].

9.1. Let us consider a quasi-linear differential equation of the following form

$$a(x,y,u,u_x,u_y)u_{xx}+2b(x,y,u,u_x,u_y)u_{xy}+$$
$$+c(x,y,u,u_x,u_y)u_{yy}+d(x,y,u,u_x,u_y)=0\,. \qquad (9.1)$$

In what follows we shall assume that the following assumptions are satisfied: (1) $a(x,y,u,p,q)$, $b(\ldots)$, $c(\ldots)$ are bounded and measurable for $(x,y)\in G+\Gamma$, $u^2+p^2+q^2\leqslant\leqslant M$ where M is an arbitrary fixed positive number; (2) for the same values of the arguments

$$ac-b^2\geqslant\Delta_0>0 \qquad (\Delta_0=\Delta_0(M)=\text{const});$$

(3) $d(x,y,u,p,q)$ is a measurable function and $d(x,y,0,0,0)\in L_p(G+\Gamma)$, $p>2$; (4) a,b,c satisfy the Lipschitz condition with respect to the arguments u,p,q:

$$f(x,y,u_1,p_1,q_1)-f(x,y,u_2,p_2,q_2)|$$
$$\leqslant M_1(|u_1-u_2|+|p_1-p_2|+|q_1-q_2|)\,,$$

where M_1 is a constant independent of x and y; (5) $d(x,y,u,p,q)$ satisfies a condition of the form .

$$d(x,y,u_1,p_1,q_1)-d(x,y,u_2,p_2,q_2)|$$
$$\leqslant d_0(x,y)(|u_1-u_2|+|p_1-p_2|+|q_1-q_2|)\,,$$

where

$$d_0(x,y)\in L_p(G+\Gamma)\,, \qquad p>2\,.$$

We shall restrict ourselves to the consideration of the case of a simply-connected domain G bounded by a sufficiently smooth curve Γ. In this case, by a non-singular mapping of the form

$$\xi = \varphi(x, y), \qquad \eta = \psi(x, y) \tag{9.2}$$

the domain G and its boundary Γ can be homeomorphically mapped onto the unit circle $\xi^2+\eta^2<1$ and the circumference $\xi^2+\eta^2=1$, respectively. We shall assume that φ and $\psi \in D_{2,p}(\bar{G})$. $p>2$. To this end it is sufficient that $\Gamma \in C^1_\mu$, $\frac{1}{2} < \mu < 1$. Then the mapping (9.2) can be regarded as conformal. Under such conditions φ and ψ are continuous in $G+\Gamma$ as well as their first derivatives; $\varphi, \psi \in C^1_\beta(G+\Gamma)$, $\beta = \dfrac{p-2}{2}$. In consequence of this mapping the equation (9.1) is transformed into a new equation which satisfies all the indicated above conditions and the domain of the problem is the unit circle ξ^2+ $+\eta^2<1$. In what follows we shall preserve the previous notation and consequently, the domain G is the circle $x^2+y^2<1$. Let us rewrite the equation (9.1) in the complex form

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} + \operatorname{Re}\left[A(z, u, u_z)\frac{\partial^2 u}{\partial z^2}\right] + B(z, u, u_z) = 0\,, \tag{9.3}$$

where

$$A = \frac{a-c+2ib}{a+c}\,, \qquad B = \frac{d}{a+c}\,.$$

These functions satisfy the following conditions: (1) for an arbitrary fixed $M>0$ there exists a constant $q=$ $=q(M)<1$ such that

$$|A\ (z, u, v)| \leqslant q < 1 \quad \text{for} \quad z \in G+\Gamma\,, \ |u|+|v| \leqslant M; \tag{9.4}$$

(2) $B(z, u, v)$ is a measurable function, and

$$B(z, 0, 0) \in L_p(G+\Gamma)\,,\ p>2; \tag{9.5}$$

(3)
$$|A(z\, u_1, v_1) - A(z, u_2, v_2)| \leqslant M_1(|u_1-u_2|+|v_1-v_2|)\,, \tag{9.6}$$

where M_1 is a constant independent of z;

(4) $$|B(z, u_1, v_1) - B(z, u_2, v_2)| \leqslant B_0(z)(|u_1 - u_2| + |v_1 - v_2|), \quad (9.7)$$

where

$$B_0(z) \in L_p(G+\Gamma), \quad p > 2 \quad (9.8)$$

The boundary condition of the problem may be taken in the following form, with no loss of generality:

$$u = 0 \quad (\text{on } \Gamma\colon x^2 + y^2 = 1). \quad (9.9)$$

This problem will be called *Problem* D; we shall seek its solution $u(x, y)$ in the class of functions $D_{2,p}(G+\Gamma)$, $p > 2$. Therefore, $u \in C^1_\alpha(G+\Gamma)$, $\alpha = \frac{p-2}{2}$. We first prove that every function of the class $D_{2,p}(G+\Gamma)$ satisfying the boundary condition (9.9) can be represented in the form

$$u(x, y) = \iint_G g_0(z, \zeta)\varrho(\zeta)\,d\xi\,d\eta \equiv \Pi_0\varrho, \quad (9.10)$$

$$g_0 = \frac{2}{\pi}\ln\left|\frac{z-\zeta}{1-z\bar{\zeta}}\right|,$$

where $p(\zeta)$ is a real function of the point ζ, belonging to the class $L_p(G+\Gamma)$, $p > 2$, and $\frac{1}{4}g_0$ is the Green function for the unit circle.

In fact, if $\varrho \in L_p(G+\Gamma)$, then

$$\Pi_1\varrho \equiv \partial\Pi_0\varrho = \iint_G \frac{\partial g_0(z, \zeta)}{\partial z}\varrho(\zeta)\,d\xi\,d\eta$$

$$\equiv -\frac{1}{\pi}\iint_G \left(\frac{1}{\zeta - z} - \frac{\bar{\zeta}}{1 - z\bar{\zeta}}\right)\varrho(\zeta)\,d\xi\,d\eta. \quad (9.11)$$

Differentiating the above relation with respect to $\bar{z}$ we obtain

$$\frac{1}{4}\Delta\Pi_0\varrho \equiv \frac{\partial^2\Pi_0\varrho}{\partial z\,\partial\bar{z}} = \varrho(\zeta). \quad (9.12)$$

If we therefore set $\varrho = u_{z\bar{z}}$ we have

$$\Delta(u - \Pi_0\varrho) = 0, \quad \text{i.e.} \quad u - \Pi_0\varrho = u_0,$$

where u_0 is a function harmonic in G, and it is evident that it is continuous in $G+\Gamma$. Since $u \equiv 0$ and $\Pi_0\varrho \equiv 0$ on Γ we have $u_0 \equiv 0$ in G, and the formula (9.10) is thus proved.

By continuing the function $\varrho(z)$ on the entire plane according to the law

$$\varrho^*(z) = \begin{cases} \varrho(z) & \text{for} \quad |z| < 1\,, \\ -\dfrac{1}{|z|^4}\,\varrho\left(\dfrac{1}{\bar z}\right) & \text{for} \quad |z| \geqslant 1\,, \end{cases} \tag{9.12a}$$

we can write relation (9.11) in the form

$$\Pi_1\varrho \equiv \mathrm{T}_E\varrho_* = -\frac{1}{\pi}\iint\limits_E \frac{\varrho_*(\zeta)\,d\xi\,d\zeta}{\zeta - z}\,. \tag{9.13}$$

If $\varrho \in L_p(G+\Gamma)$ then $\varrho_* \in L_p(E)$. It was proved in Ch. I, §9.2 that if $\varrho_* \in L_p$, $p > 1$, then $\mathrm{T}_E\varrho_*$ has a derivative with respect to z which is represented by the singular integral *

$$\Pi_2\varrho \equiv \frac{\partial \Pi_1\varrho}{\partial z} = \frac{\partial \mathrm{T}\varrho_*}{\partial z} = -\frac{1}{\pi}\iint\limits_G \frac{\varrho_*(\zeta)}{(\zeta - z)^2}\,d\xi\,d\eta\,, \tag{9.14}$$

$\Pi_2\varrho$ is a linear operator acting from L_p into L_p. Consequently,

$$L_p(\mathrm{II}_2\varrho, \bar G) \leqslant \Lambda_p L_p(\varrho, \bar G)\,. \tag{9.15}$$

We shall prove that $\Lambda_2 = 1$. Assume that $\varrho \in D^0_\infty(G)$. Then

$$\begin{aligned}
(\mathrm{II}_2\varrho, \mathrm{II}_2\varrho) &= \iint\limits_G \frac{\partial T\varrho_*}{\partial z}\,\overline{\frac{\partial T\varrho_*}{\partial \bar z}}\,dx\,dy \\
&= \iint\limits_G \frac{\partial}{\partial \bar z}\left(\overline{\mathrm{T}\varrho_*}\,\frac{\partial T\varrho}{\partial z}\right)dx\,dy - \iint\limits_G \overline{\mathrm{T}\varrho_*}\,\frac{\partial^2 \mathrm{T}\varrho_*}{\partial z\,\partial \bar z}\,dx\,dy \\
&= \frac{1}{2i}\int\limits_\Gamma \overline{T\varrho_*}\,\frac{\partial \mathrm{T}\varrho_*}{\partial z}\,dz - \iint\limits_G \overline{\mathrm{T}\varrho_*}\,\frac{\partial \varrho_*}{\partial z}\,dx\,dy
\end{aligned}$$

$$= \frac{1}{2i}\int\limits_\Gamma \overline{\mathrm{T}\varrho_*}\,\frac{\partial \mathrm{T}\varrho^*}{\partial z}\,dz + \frac{1}{2i}\int\limits_\Gamma \varrho_*\,\mathrm{T}\varrho_*\,d\bar z + \iint\limits_G \varrho^*\bar\varrho^*\,dx\,dy\,.$$

* The representation of $\partial_a T_p$ by means of a singular integral was given in Ch. I, §8 for $\varrho \in C_a$. For the case $\varrho \in L_p$, $p > 1$ it was proved in the papers [36a, b].

We have performed above a few integrations by parts making use of Green's identity. These operations are valid since the functions entering the integrand are continuous and have derivatives of any order in $G+\Gamma\big(\varrho\in D_\infty(G)\big)$.

Since $\varrho_* = \varrho$ (in G) and $\varrho_* = \varrho = 0$ on Γ, we have

$$(\Pi_2\varrho, \Pi_2\varrho) = I_0(\varrho)+(\varrho, \varrho)\,, \tag{9.16}$$

where

$$I_0(\varrho) = \frac{1}{2i}\int \overline{T\varrho_*}\frac{\partial T\varrho_*}{\partial z}\,dz\,. \tag{9.17}$$

Let us now compute the curvilinear integral. Since $\Pi_0\varrho = 0$ on Γ, differentiating this relation with respect to the arc of the curve Γ we obtain

$$\frac{d\Pi_0\varrho}{ds} \equiv iz\Pi_1\varrho - i\bar{z}\overline{\Pi_1\varrho} \equiv izT\varrho^* - i\bar{z}\overline{T\varrho_*} = 0 \quad (\text{on } \Gamma)\,,$$

Thus, $\overline{T\varrho^*} = z^2 T\varrho^*$ on Γ and consequently, in view of (9.17),

$$I_0(\varrho) = \frac{1}{2i}\int_\Gamma z^2 T\varrho_*\frac{\partial T\varrho_*}{\partial z}\,dz\,.$$

According to the condition assumed, $\varrho\equiv 0$ outside a circle $|z|\leqslant r<1$. Therefore,

$$T\varrho^* = -\frac{1}{\pi}\iint_{G_1}\frac{\varrho(\zeta)\,d\xi\,d\eta}{\zeta-z} - \frac{1}{\pi}\frac{\varrho^*(\zeta)\,d\xi\,d\eta}{\zeta-z} \equiv T_1\varrho + T_2\varrho^*\,, \tag{9.18}$$

where G_1 and G_2 are the domains $|z|\leqslant r$ and $z\geqslant\dfrac{1}{r}$, respectively. Bearing in mind that $T_1\varrho$ is holomorphic outside G_1 and vanishes at infinity and that $T_2\varrho^*$ is holomorphic inside the circle $|z|<\dfrac{1}{r}$ we find that $I_0(\varrho) = -\dfrac{1}{\pi}a_0^2 + I(\varrho)$ where

$$I(\varrho) = \frac{1}{2i}\int_\Gamma z^2 T_1\varrho\frac{\partial T_2\varrho}{\partial z}\,dz + \frac{1}{2i}\int_\Gamma z^2 T_2\varrho_*\frac{\partial T_1\varrho}{\partial z}\,dz\,,$$

$$a_0 = \iint_G \varrho(\zeta)\,d\xi\,d\eta\,.$$

Substituting for $T_1\varrho$ and $T_2\varrho$ the expressions from (9.18) we have

$$I(\varrho)=\frac{1}{2i\pi^2}\int_\Gamma z^2dz\iint_{G_1}\frac{\varrho(\zeta_1)d\xi_1d\eta_1}{\zeta_1-z}\iint_{G_2}\frac{\varrho^*(\zeta_2)d\xi_2d\eta_2}{(\zeta_2-z)^2}+$$
$$+\frac{1}{2i\pi^2}\int_\Gamma z^2dz\iint_{G_2}\frac{\varrho^*(\zeta_2)d\xi_2d\eta_2}{\zeta_2-z}\iint_{G_1}\frac{\varrho(\zeta_1)d\xi_1d\eta_1}{(\zeta_1-z)^2}.$$

Changing the order of integration over G_1, G_2 and Γ, which is obviously permissible, we obtain

$$I(\varrho)=\frac{1}{\pi}\iint_{G_1}\varrho(\zeta_1)d\xi_1d\eta_1\iint_{G_2}\varrho_*(\zeta_2)d\xi_2d\eta_2$$
$$\left\{\frac{1}{2\pi i}\int_\Gamma\frac{z^2dz}{(\zeta_1-z)(\zeta_2-z)^2}+\frac{1}{2\pi i}\int_\Gamma\frac{z^2dz}{(\zeta_2-z)(\zeta_1-z)^2}\right\}$$
$$=\frac{1}{\pi}\iint_{G_1}\varrho(\zeta_1)d\xi_1d\xi_2\iint_{G_2}\varrho_*(\zeta_2)\frac{\zeta_2+\zeta_1}{\zeta_2-\zeta_1}d\xi_2d\eta_2$$

or, in view of (9.12a),

$$I(\varrho)=-\frac{1}{\pi}\iint_{G_1}\varrho(\xi,\eta)d\xi d\eta\iint_{G_1}\varrho(x,y)\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}dxdy\,.$$

It follows from the last relation that

$$I(\varrho)=-\frac{1}{\pi}a_0\bar{a}_0-\frac{2}{\pi}\sum_{k=1}^{\infty}a_k\bar{a}_k\leqslant 0\,,$$
$$a_k=\iint_G\varrho(\zeta)\zeta^kd\xi d\eta\,.$$

Hence, $I_0(\varrho)\leqslant 0$, and the equality takes place for functions satisfying the relations

$$\iint_G\varrho(\zeta)\zeta^kd\xi d\eta=0\qquad(k=0,1,\ldots,)\,.\tag{9.19}$$

According to the relation (9.16) we have

$$(\Pi_2\varrho,\Pi_2\varrho)\leqslant(\varrho,\varrho)\,,\tag{9.20}$$

and the equality takes place for functions satisfying the condition (9.19). Since $D_\infty(G)$ is dense in $L_2(G+\Gamma)$ the relation (9.20) is also valid for an arbitrary function such that $\varrho \in L_2(G+\Gamma)$. Hence, the norm of the operator Π_2 in the space L_2 is equal to 1, i.e.

$$\Lambda_2 = L_2(\Pi_2) \equiv \|\Pi_2\|_2 = 1 . \tag{9.21}$$

We now return to the boundary value problem. We may seek its solution in the form (9.10). Substituting this expression into the equation (9.3), taking into account (9.11), (9.12), and (9.14) we obtain the following functional equation for ϱ:

$$\varrho(z) + \mathrm{Re}[A(z, \Pi_0\varrho, \Pi_1\varrho)\Pi_2\varrho] + B(z, \Pi_0\varrho, \Pi_1\varrho) = 0 . \tag{9.22}$$

In general, this equation is non-linear. Let us investigate its solubility in the space $L_p(G+\Gamma)$, $p > 2$. If $\varrho \in L_p(G+\Gamma)$, $p > 2$, then it follows from (9.10) and (9.11) that

$$|\Pi_0\varrho| \leqslant KL_p(\varrho) , \qquad |\Pi_1\varrho| \leqslant KL_p(\varrho) \qquad K = \text{const} . \tag{9.23}$$

$$\Pi_1\varrho \in C_\alpha(G+\Gamma) , \qquad \alpha = \frac{p-2}{p} . \tag{9.24}$$

Therefore, the operator

$$\mathrm{P}\varrho \equiv -\mathrm{Re}[A(z, \Pi_0\varrho, \Pi_1\varrho)\Pi_2\varrho] - B(z, \Pi_0\varrho, \Pi_1\varrho) \tag{9.25}$$

transforms the space $L_p(G+\Gamma)$ onto $L_p(G+\Gamma)$ if $p > 2$. Further, if $\varrho_1, \varrho_2 \in L_p(G+\Gamma)$ we have

$$\begin{aligned}|\mathrm{P}\varrho_1 - \mathrm{P}\varrho_2| \leqslant {} & |A(z, \Pi_0\varrho_1, \Pi_1\varrho_1) - A(z, \Pi_0\varrho_2, \Pi_1\varrho_2)|\,|\Pi_2\varrho_1| + \\ & + |A(z, \Pi_0\varrho_2, \Pi_1\varrho_2)|\,|\Pi_2(\varrho_1 - \varrho_2| + \\ & + |B(z, \Pi_0\varrho_1, \Pi_1\varrho_1) - B(z, \Pi_0\varrho_2, \Pi_1\varrho_2)| .\end{aligned}$$

Making use of the conditions (9.4), (9.6), (9.7) and the inequality (9.23) we obtain

$$\begin{aligned}|\mathrm{P}\varrho_1 - \mathrm{P}\varrho_2| \leqslant {} & q(M)\,|\Pi_2(\varrho_1 - \varrho_2)| + \\ & + [2M_1K\,|\Pi_2\varrho_1| + 2KB_0(z)]L_p(\varrho_1 - \varrho_2) ,\end{aligned} \tag{9.26}$$

where $q(M) < 1$ and M is a constant subjected to the condition

$$|\Pi_0\varrho_2| + |\Pi_1\varrho_2| \leqslant 2KL_p(\varrho_2) \leqslant M .$$

Applying the Minkowski inequality we obtain from (9.26)

$$L_p[\mathrm{P}(\varrho_1-\varrho_2)] \leqslant K_p L_p(\varrho_1-\varrho_2)\,, \tag{9.27}$$

where

$$K_p = q(M)\Lambda_p + 2M_1 K \Lambda_p L_p(\varrho_1) + 2KL_p(B_0)\,. \tag{9.28}$$

According to the Riesz theorem [77] (Ch. I, §9) $\Lambda_p \equiv \equiv L_p(\Pi_2)$ is continuous with respect to p. Since $\Lambda_2 = = L_2(\Pi_2) = 1$, for any fixed $M > 0$ a constant $\varepsilon = \varepsilon(M) > 0$ can be found such that the following inequality is satisfied

$$q(M)\Lambda_p < 1\,, \quad \text{if} \quad 0 < p-2 \leqslant \varepsilon(M)\,. \tag{9.29}$$

Specifying now some p satisfying this condition we take numbers $r > 0$ and $\delta > 0$ satisfying the inequalities

$$2Kr < M \tag{9.30}$$

and

$$\alpha \equiv q(M)\Lambda_p + 2M_1 K \Lambda_p r + 2K\delta < 1\,. \tag{9.31}$$

We also subject the function $B_0(z)$ to the inequality

$$L_p(B_0, \bar{G}) < \delta\,. \tag{9.32}$$

Then, for arbitrary functions ϱ_1 and ϱ_2 belonging to the sphere

$$S(0, r){:} \quad L_p(\varrho) < r\,, \tag{9.33}$$

we have in view of (9.33), (9.32), (9.31), (9.28) and (9.27)

$$L_p[\mathrm{P}(\varrho_1-\varrho_2)] \leqslant \alpha L_p(\varrho_1-\varrho_2)\,, \quad K_p \leqslant \alpha < 1\,. \tag{9.34}$$

Taking into account the fact that the operator P acting on the zero element $\theta = 0$ yields $B(z, 0, 0)$ let us assume that

$$L_p[\mathrm{P}(\theta)] \equiv L_p[B(z, 0, 0)] < (1-\alpha)r\,.$$

Under these conditions we can make use of the so-called generalized principle of contraction mappings, [51], according to which the equation $\varrho - \mathrm{P}\varrho = 0$ has a unique solution $\varrho \in L_p(G+\Gamma)$, $p > 2$ belonging to the sphere $L_p(\varrho) < r$.

In particular, if $B(z, 0, 0) = 0$ the unique solution of the equation $\varrho - \mathrm{P}\varrho = 0$ belonging to the sphere $L_p(\varrho) < r$ is $\varrho \equiv 0$.

Thus, *if the functions $B(z, 0, 0)$ and $B_0(z)$ are sufficiently small in the norm of $L_p(G+\Gamma)$, $p > 2$, the Dirichlet problem for the quasi-linear equation* (9.3) *with the boundary condition* (9.9) *always has a solution. This solution is given by the formula* (9.10) *and consequently belongs to the class $D_{2,p}(G+\Gamma)$. Hence, the solution of the problem has continuous first derivatives belonging to the class $C_\alpha(G+\Gamma)$ where $\alpha = \dfrac{p-2}{p}$.*

Let $|A| \leqslant q < 1$, $|B| \leqslant M_0$ where the constants q and M_0 are independent of x, y, u, u_x, u_y. Then the solution of Problem **D** satisfies the conditions (a priori estimates)

$$D_{2,p}(u, \bar{G}) < M', \qquad C_\alpha^1(u, \bar{G}) < M', \qquad \alpha = \frac{p-2}{p}, \tag{9.35}$$

where the number $p > 2$ depends only on q, and M' depends only on q and M_0.*

9.2. We now consider the case of the linear equation

$$L(u) = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = h. \tag{9.36}$$

We assume that the following conditions are satisfied: (1) a, b, c are measurable bounded functions of the variables x and y in the circle $x^2+y^2 \leqslant 1$; (2) $ac - b^2 \geqslant \Delta_0 > 0$ for $x^2+y^2 \leqslant 1$; (3) $d, e, f, h \in L_p(G+\Gamma)$, $p > 2$. Under these conditions we can, with no loss of generality, assume that

$$a + c = 2 \quad (\text{in } G).$$

Then the operator P has the form

$$\mathrm{P}\varrho = -\tfrac{1}{2}\mathrm{Re}[(a-c+2ib)\Pi_2\varrho + (d+ie)\Pi_1\varrho] - \\ -\tfrac{1}{4}f\Pi_0\varrho + \tfrac{1}{4}h. \tag{9.37}$$

* By other methods, mainly by means of the so-called "a priori" estimates, the Dirichlet problem for quasi-linear equations was investigated by many authors (see for instance [54], [42]).

Consequently it is linear. Taking into account that in the case under consideration

$$A = \tfrac{1}{2}(a-b+2ib)\,,$$
$$B = \tfrac{1}{4}(d+ie)u_z+\tfrac{1}{4}(d-ie)u_{\bar z}+\tfrac{1}{4}fu-\tfrac{1}{4}h\,,$$

we have

$$|A| \leqslant q = \sqrt{1-4\Delta_0} < 1\,, \tag{9.37a}$$
$$B_0(z) = \tfrac{1}{2}\max(\tfrac{1}{2}|f(z)|\,,\ |d+ie|)\,,$$

where q is independent of the choice of the constant M which was introduced previously. Besides, in this case $M_1 = 0$ and the inequality (9.27) takes the form

$$L_p[\mathrm{P}(\varrho_1-\varrho_2)] \leqslant [q\Lambda_p+2KL_p(B_0)]L_p(\varrho_1-\varrho_2)\,,$$

This inequality holds for arbitrary elements ϱ_1 and ϱ_2 belonging to $L_p(G+\Gamma)$. Specifying now some $p > 2$, such that the inequality

$$q\Lambda_p < 1\,,\qquad 0 < p-2 \leqslant \varepsilon\,, \tag{9.37b}$$

holds, and subjecting the function B_0 to the condition

$$\alpha = q\Lambda_p+2KL_p(B_0) < 1\,, \tag{9.38}$$

we obtain

$$L_p[\mathrm{P}(\varrho_1-\varrho_2)] \leqslant \alpha L_p(\varrho_1-\varrho_2)\,,\qquad \alpha < 1\,.$$

Thus, under these conditions the operator P satisfies the principle of contraction mappings and, consequently, the equation $\varrho-\mathrm{P}\varrho = 0$ has a unique solution belonging to $L_p(G+\Gamma)$, $p > 0$. Hence, the Dirichlet problem for the linear equation (9.36) with the homogeneous boundary condition $u = 0$ on Γ always has a unique solution if the condition (9.38) is satisfied. In particular, if $h \equiv 0$ the problem has only the trivial solution $u \equiv 0$. If $d = e = f = 0$ it is evident that the condition (9.38) is satisfied and, consequently, the equation (generalized Laplace equation)

$$L_0(u) = au_{xx}+2bu_{xy}+cu_{yy} = h$$

always has a unique solution satisfying the boundary condition $u = 0$ (on Γ). In particular, for $h = 0$ this problem has only the trivial solution $u \equiv 0$.

In the case of a linear equation the problem can be reduced to an equation of Fredholm type by means of regularization of the corresponding singular integral equation. It follows from (9.37) that the operator $\mathrm{P}\varrho$ has the form

$$\mathrm{P}\varrho = \widetilde{\mathrm{P}}\varrho + \tfrac{1}{4}h \equiv P_2\varrho + \mathrm{P}_1\varrho + \tfrac{1}{4}h\,,$$

where

$$\mathrm{P}_2\varrho = -\operatorname{Re}[A(z)\Pi_2\varrho]\,, \qquad A = \frac{a-c+2ib}{2}\,,$$

$$\mathrm{P}_1\varrho = -\operatorname{Re}[A_0(z)\Pi_1\varrho] - \frac{1}{4}f\Pi_0\varrho\,,\ A_0 = \frac{d+ie}{2}\,.$$

According to (9.37a) and (9.15)

$$L_p(\mathrm{P}_2\varrho) \leqslant q\Lambda_p L_p(\varrho)\,.$$

Therefore, according to the inequality (9.37b) the operator $I - \mathrm{P}_2$ has the inverse $(\mathrm{I}-\mathrm{P}_2)^{-1}$. Applying this operator to both sides of the equation

$$\varrho - \widetilde{\mathrm{P}}\varrho \equiv \varrho - \mathrm{P}_2\varrho - \mathrm{P}_1\varrho = \tfrac{1}{4}h\,, \tag{9.39}$$

we obtain

$$\varrho - (I-\mathrm{P}_2)^{-1}\mathrm{P}_1\varrho = \tfrac{1}{4}(I-\mathrm{P}_2)^{-1}h\,. \tag{9.40}$$

Since the operator $(I-\mathrm{P}_2)^{-1}$ is linear and P_1 is completely continuous, the operator $(I-\mathrm{P}_2)^{-1}\mathrm{P}_1$ is linear and completely continuous in $L_p(G+\Gamma)$. Consequently, the Fredholm theorems can be applied to the equation (9.40) which is equivalent to the singular integral equation (9.39).

Thus, we arrive at the following results:

I. *The Dirichlet problem for the linear equation $L(u) = h$ with the homogeneous boundary condition $u = 0$ on Γ (Problem* **D**) *has a solution for an arbitrary function h of the class $L_p(G+\Gamma)$, $p > 2$, if and only if the corresponding homogeneous problem* $\mathring{\mathbf{D}}$ *has no non-trivial solution. In this case for any $h \in L_p(G+\Gamma)$, $p > 2$, the prob-*

lem has a unique solution of the class $D_{p,2}(G+\Gamma)$, $p>2$, *which belongs also to the class* $C^1_a(G+\Gamma)$, $a=\dfrac{p-2}{p}$.

Thus, *for the solubility of Problem* **D** *it is sufficient to prove the uniqueness theorem*, i.e. to establish that the corresponding homogeneous Problem $\mathring{\mathbf{D}}$ has no solution.

II. *If the homogeneous Problem* $\mathring{\mathbf{D}}$ *has a solution, then the number of its linearly independent solutions is equal to the number of the linearly independent solutions of the homogeneous equation*

$$\varrho-(I-\mathrm{P}_2)^{-1}\mathrm{P}_1\varrho=0\,,$$

and if $\varrho_1, \dots, \varrho_n$ *is the complete system of solutions of this equation, then the functions*

$$u_j(x, y)=\Pi_0\varrho_j\equiv\iint\limits_G g_0(z, \zeta)\,\varrho(\zeta)\,d\xi\,d\eta$$

$$(j=1, \dots, n)$$

constitute the complete system of solutions of the homogeneous Problem $\mathring{\mathbf{D}}$.

III. *If the homogeneous Problem* $\mathring{\mathbf{D}}$ *has a solution, then the non-homogeneous Problem* **D** *is soluble if and only if the following conditions are satisfied*:

$$\iint\limits_G \chi_j(I-\mathrm{P}_2)^{-1}h\,dx\,dy\equiv\iint\limits_G h(I-\mathrm{P}_2^*)^{-1}\chi_j\,dx\,dy=0 \tag{9.41}$$

$$(j=1, \dots, n)\,,$$

where χ_j *is the complete system of solutions of the adjoint homogeneous integral equation*

$$\chi-\mathrm{P}_1^*(I-\mathrm{P}_2^*)^{-1}\chi=0\,, \tag{9.42}$$

where P_1^* *and* P_2^* *are operators adjoint to* P_1 *and* P_2 (they can originally be considered in L_2 and then continued to L_p). If the new function

$$v=(I-\mathrm{P}_2^*)^{-1}\chi\,, \tag{9.43}$$

be introduced, then $\chi = (I - \mathrm{P}_2^*)v$ and the equation (9.42) takes the form

$$v - \widetilde{\mathrm{P}}^* v = 0\,, \quad \widetilde{\mathrm{P}}^* = \mathrm{P}_2^* + \mathrm{P}_1^*\,, \tag{9.44}$$

or

$$v - (I - \mathrm{P}_2^*)^{-1}\mathrm{P}_1^* v = 0\,. \tag{9.44a}$$

Thus, the necessary and sufficient condition (9.41) for the solubility of Problem **D** takes the form

$$\iint_G h(z)v(z)\,dx\,dy = 0\,, \tag{9.45}$$

where v is an arbitrary solution of the homogeneous singular integral equation (9.44) adjoint to (9.39). This equation is equivalent to the Fredholm equation (9.44a) and therefore it has a finite number of linearly independent solutions.

9.3. We now make the following additional assumptions with respect to the coefficients of the equation: (1) $a, b, c \in D_{2,p}(G+\Gamma)$, $p > 2$; (2) $d, e \in D_{1,p}(G+\Gamma)$, $p > 2$. Under these conditions the solution of the equation (9.44) is a solution of the adjoint homogeneous Problem $\mathring{\mathbf{D}}'$:

$$L_*(v) \equiv (av)_{xx} + 2(bv)_{xy} + (cv)_{yy} - (dv)_x - (ev)_y + fv = 0\,,$$
$$v = 0 \quad (\text{on } \Gamma)\,. \tag{9.46}$$

Since

$$\widetilde{\mathrm{P}}\varrho \equiv -\iint_G \left\{\mathrm{Re}\left[A(z)\frac{\partial^2 g_0(z,\zeta)}{\partial z^2} + A_0(z)\frac{\partial g_0(z,\zeta)}{\partial z} + \frac{1}{4}f(z)g_0(z,\zeta)\right]\right\}\varrho(\zeta)\,d\xi\,d\eta\,,$$

we have

$$\widetilde{\mathrm{P}}^* v \equiv -\iint_G \left\{\mathrm{Re}\left[A(\zeta)\frac{\partial^2 g_0(\zeta,z)}{\partial \zeta^2} + A_0(\zeta)\frac{\partial g_0(\zeta,z)}{\partial \zeta} + \frac{1}{4}f(\zeta)g_0(\zeta,z)\right]\right\}v(\zeta)\,d\xi\,d\eta\,.$$

If $A \in D_{2,p}(G)$, $A_0 \in D_{1,p}(G)$ and $f \in L_p(\bar{G})$, $p > 2$, then $\widetilde{\mathrm{P}}^*$ is a linear operator acting from $D_{2,p}$ into $D_{2,p}$. In this

case the solution v of the equation $v-\tilde{\mathrm{P}}^*v = 0$ also belongs to $D_{2,p}(G+\Gamma)$, $p > 2$. Transforming the double integrals by means of Green's identity we obtain

$$\begin{aligned} I_1 &= \iint\limits_G A(\zeta)\frac{\partial^2 g_0(\zeta, z)}{\partial\zeta^2} v(\zeta)\, d\zeta\, d\eta \\ &= \lim_{\varepsilon\to 0} \iint\limits_{|\zeta-z|\geq\varepsilon} A(\zeta)\frac{\partial^2 g_0(\zeta, z)}{\partial\zeta^2} v(\zeta)\, d\xi\, d\eta \\ &= -\frac{1}{2i}\int\limits_\Gamma A(\zeta)\frac{\partial g_0(\zeta, z)}{\partial\zeta} v(\zeta)\, d\bar{\zeta} + \\ &\quad + \frac{1}{2i}\lim_{\varepsilon\to 0}\int\limits_{|\zeta-z|=\varepsilon} A(\zeta)\frac{\partial g_0(\zeta, z)}{\partial\zeta} v(\zeta)\, d\bar{\zeta} - \\ &\quad - \iint\limits_G (Av)_\zeta \frac{\partial g_0(\zeta, z)}{\partial\zeta}\, d\xi\, d\eta\,. \end{aligned} \tag{9.46a}$$

It can readily be proved that

$$\lim_{\varepsilon\to 0}\int\limits_{|\zeta-z|-\varepsilon} A(\zeta)\frac{\partial g_0(\zeta, z)}{\partial\zeta} v(\zeta)\, d\zeta = 0\,, \qquad z\in G\,.$$

Hence, applying once more Green's identity to the double integral on the right-hand side of the relation (9.46a), and taking into account that $g_0 = 0$ on Γ we obtain

$$\begin{aligned} I_1 = -\frac{1}{2i}\int\limits_\Gamma A(\zeta)\frac{\partial g_0(\zeta, z)}{\partial\zeta} v(\zeta)\, d\bar{\zeta} + \\ + \iint\limits_G g_0(\zeta, z)\frac{\partial^2 Av}{\partial\zeta^2}\, d\xi\, d\eta\,. \end{aligned} \tag{9.47}$$

Since

$$\frac{\partial g_0(\zeta, z)}{\partial\zeta} = \frac{1}{2\zeta}\,\frac{\partial g_0(\zeta, z)}{\partial\nu\zeta} \qquad \text{on} \quad \Gamma\,, \qquad z\in G\,,$$

where ν_ζ is the outward normal to Γ at the point ζ, the formula (9.47) can also be written in the form

$$\begin{aligned} &\iint\limits_G A(\zeta)\frac{\partial^2 g_0(\zeta, z)}{\partial\zeta^2} v(\zeta)\, d\xi\, d\eta \\ &= \frac{1}{4}\int\limits_\Gamma \frac{A(\zeta)v(\zeta)}{\zeta^2}\,\frac{\partial g_0(\zeta, z)}{\partial\nu_\zeta}\, ds + \iint\limits_G g_0(\zeta, z)\frac{\partial^2 Av}{\partial\zeta^2}\, d\xi\, d\eta\,. \end{aligned} \tag{9.48}$$

Similarly, we obtain the relation

$$\iint_G A_0(z)\frac{\partial g_0(\zeta, z)}{\partial \zeta} v(\zeta)\,d\xi\,d\eta = -\iint_G g_0(\zeta, z)\frac{\partial A_0 v}{\partial \zeta}\,d\xi\,d\eta\,. \tag{9.49}$$

In view of (9.48) and (9.49) the equation (9.44) can be written in the following form

$$v(z)+\iint_G g_0(\zeta, z)\operatorname{Re}\left[\frac{\partial^2 A(\zeta)v(\zeta)}{\partial \zeta^2}-\frac{\partial A_0(\zeta)v(\zeta)}{\partial \zeta}+\right.$$
$$\left.+\frac{1}{4}f(\zeta)v(\zeta)\right]d\xi\,d\eta$$
$$=\frac{1}{4}\int_\Gamma \operatorname{Re}[A(\zeta)\zeta^{-2}]v(\zeta)\frac{\partial g_0(\zeta, z)}{\partial \nu_\zeta}\,ds_\zeta\,. \tag{9.50}$$

Applying the operator Δ to both sides of the above relation and taking into account that $\frac{1}{4}g_0(z,\zeta)=g(z,\zeta)$ is the Green's function for the circle, we obtain the equation (9.46). Moreover, if in the equation (9.50) the point z tends to the boundary Γ of the domain G we have

$$v(z)=\operatorname{Re}\{A(z)\bar{z}^2\}v(z) \qquad (\text{on } \Gamma)\,.$$

Since $|A|<1$ this relation is possible only when $v\equiv 0$ on Γ. It has, therefore, been established that if v is a solution of (9.44) it is a solution of the adjoint homogeneous Problem $\mathring{\mathbf{D}}$. The inverse assertion is also true. Let us consider the non-homogeneous adjoint boundary value Problem $\mathbf{D}'$:

$$L_*(v)=h_* \quad (\text{in } G)\,; \qquad v=0 \quad (\text{on } \Gamma)\,. \tag{9.51}$$

It is evident that its solution which will be sought also in the class $D_{2,p}$, $p>2$, satisfies the integral equation

$$v(z)=\iint_G g_0(z,\zeta)\operatorname{Re}\left[-\frac{\partial^2 A(\zeta)v(\zeta)}{\partial \zeta^2}+\right.$$
$$\left.+\frac{\partial A_0(\zeta)v(\zeta)}{\partial \zeta}-\frac{1}{4}f(\zeta)v(\zeta)\right]d\xi\,d\eta+$$
$$+\frac{1}{4}\iint_G g_0(z,\zeta)h_*(\zeta)\,d\xi\,d\eta\,.$$

Transforming the first double integral by Green's identity and taking into account the boundary condition we obtain the following integral equation equivalent to the problem under consideration:

$$v(z)-\widetilde{\mathrm{P}}^*v = \frac{1}{4}\iint_G g_0(z,\zeta)h_*(\zeta)\,d\xi\,d\eta\,. \qquad (9.52)$$

It follows, therefore, that the homogeneous Problem $\mathring{\mathbf{D}}'$ is equivalent to the homogeneous equation $v-\widetilde{\mathrm{P}}^*v = 0$.

In the general case when A is a measurable bounded function and $A_0 \in L_p(G+\Gamma)$, $p>2$, the solution of the equation (9.52) does not in general belong to $D_{2,p}$ and, consequently, it cannot be regarded as a generalized solution of Problem $\mathbf{D}'$ in the ordinary sense. In this case a special definition of the adjoint operator is required.

9.4. The method presented in this section can also be applied to a number of other boundary value problems. For instance, let us consider for the linear equation (9.36) *the second fundamental boundary value problem* which will hereafter be called *Problem* **N**:

$$L(u) = h \quad (\text{in } G)\,, \quad \frac{\partial u}{\partial \nu} = 0 \quad (\text{on } \Gamma) \qquad (9.53)$$

(ν is the outward normal to Γ). This problem was investigated by Vinogradov, [15a] whose results will be presented here.

As before, the domain G is the circle $|z|<1$. The remaining asumptions concerning the coefficients and the free term remain unchanged.

It is natural to seek the solution of the problem in the class $D_{2,p}(G+\Gamma)$, $p>2$. Then it can be represented in the form

$$u(x,y) = \iint_G \hat{g}(z,\zeta)\,\varrho(\zeta)\,d\xi\,d\eta + c \equiv \hat{\Pi}_0\varrho + c\,, \quad (9.54)$$

where c is a constant, $\varrho = \Delta u$ and $\hat{g}$ is the Neumann function for the unit circle:

$$\hat{g}(z, \zeta) = \frac{2}{\pi} \ln |(z-\zeta)(1-z\bar{\zeta})| - \frac{1}{\pi}(|z|^2 + |\zeta|^2) + \frac{3}{4}.$$

Let us note the following properties of this function:

$$(1) \quad \hat{g}(z, \zeta) = \hat{g}(\zeta, z); \qquad (2) \quad \frac{\partial \hat{g}(z, \hat{\zeta})}{\partial v_z} = 0 \ (\text{on } \Gamma, \zeta \in G)$$

$$(3) \quad \Delta \hat{g} = -\frac{4}{\pi}; \qquad (4) \quad \iint_G \hat{g}(z, \zeta) d\xi d\eta = 0.$$

If $\varrho \in L_p(G+\Gamma)$, $p > 2$, then u has continuous derivatives in the Hölder sense which are given by the formula

$$\partial_z u = \hat{\Pi}_1 \varrho \equiv \iint_G \frac{\partial g(z, \zeta)}{\partial z} \varrho(\zeta) d\xi d\eta,$$

It is also evident that

$$\hat{\Pi}_1 \varrho \in C_a(G+\Gamma), \quad a = \frac{p-2}{p}.$$

There also exist the generalized second derivatives given by the formulae

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = \varrho(z) - \frac{1}{\pi} \iint_G \varrho(\zeta) d\xi d\eta,$$

$$\frac{\partial^2 u}{\partial z^2} = \partial_z \hat{\Pi}_1 \varrho \equiv \hat{\Pi}_2 \varrho = \iint_G \frac{\partial^2 \hat{g}(z, \zeta)}{\partial z^2} \varrho(\zeta) d\xi d\eta,$$

The last integral is taken as its Cauchy principal value; $\hat{\Pi}_2 \varrho$ is a linear operator in $L_p(G+\Gamma)$, $p > 2$, (Ch. I, §9).

Hence,

$$L_p(\hat{\Pi}_2 \varrho, \bar{G}) \leqslant \hat{\Lambda}_p L_p(\varrho, \bar{G}), \quad \hat{\Lambda}_p = L_p(\hat{\Pi}_2).$$

In exactly the same way as was done above for the operator Π_2 it can be established that

$$L_2(\hat{\Pi}_2) \equiv \hat{\Lambda}_2 = 1.$$

Therefore, in view of continuity of $\hat{\Lambda}_p$ with respect to p, for any positive number $q < 1$, a number $\varepsilon > 0$ can be found such that the inequality

$$q\hat{\Lambda}_p < 1 , \quad \text{if} \quad 2 < p \leqslant 2+\varepsilon .$$

is satisfied. Introducing the above expressions for the unknown function u and its first and second derivatives into the equation $L(u) = h$ we arrive at the following singular integral equation, completely equivalent to the problem stated

$$\varrho(z) - \hat{\mathrm{P}}\varrho = -\tfrac{1}{4}cf + \tfrac{1}{4}h , \tag{9.55}$$

where

$$\hat{\mathrm{P}}\varrho \equiv -\mathrm{Re}[A(z)\hat{\Pi}_2\varrho + A_0(z)\hat{\Pi}_1\varrho] - \\ -\frac{1}{4}f(z)\hat{\Pi}_0\varrho - \frac{1}{\pi}\iint_G \varrho(\zeta)\,d\xi\,d\eta .$$

This operator is linear and acts from $L_p(G+\Gamma)$ into $L_p(G+\Gamma)$, $p > 2$. It has the form

$$\hat{\mathrm{P}}\varrho = \hat{\mathrm{P}}_2\varrho + \hat{\mathrm{P}}_1\varrho ,$$

where

$$\hat{\mathrm{P}}_2\varrho = -\mathrm{Re}[A(z)\hat{\Pi}_2\varrho] ,$$

$$\mathrm{P}_1\varrho = -\mathrm{Re}[A_0(z)\hat{\Pi}_1\varrho] - \frac{1}{4}f(z)\hat{\Pi}_0\varrho - \frac{1}{\pi}\iint_G \varrho(\zeta)\,d\xi\,d\eta .$$

In view of the relation $q\hat{\Lambda}_p < 1$ the operator $I - \hat{\mathrm{P}}_2$ has the inverse $(I - \hat{\mathrm{P}}_2)$ $(I - \hat{\mathrm{P}}_2)^{-1}$ in L_p, $p > 2$.

Applying to both sides of the equation (9.55) this operator we obtain an equation of Fredholm type

$$\varrho(z) - (I - \hat{\mathrm{P}}_2)^{-1}\hat{\mathrm{P}}_1\varrho = \tfrac{1}{4}(I - \hat{\mathrm{P}}_0)^{-1}(h - cf) .$$

Consequently, the Fredholm alternatives may be applied to the equation (9.55). The condition of solubility of this equation has the form

$$\iint_G (cf - h)\,v\,dx\,dy = 0 ,$$

where v is an arbitrary solution of the adjoint homogeneous equation

$$v-\hat{\mathbf{P}}^{*}v = 0\,. \tag{9.56}$$

This equation is also equivalent to a homogeneous Fredholm equation. It has, therefore, a finite number n of linearly independent solutions. If $n = 0$ Problem **N** has a solution for an arbitrary right-hand side h. Moreover, in this case the homogeneous Problem $\overset{\circ}{\mathbf{N}}$ has one (linearly independent) solution. This result follows from the fact that Problem $\overset{\circ}{\mathbf{N}}$ is reducible to the equation $\varrho-\hat{\mathbf{P}}\varrho = -\frac{1}{4}cf$ which has a solution for an arbitrary value of the constant c. In the case $n > 0$ let us consider the constants $f_j = \iint\limits_G fv_j\,dx\,dy\,(j = 1, \ldots, n)$ where v_j are linearly independent solutions of the equation (9.56). Obviously, there may exist two cases:

(1) $f_j = 0$ for all values of $j = 1, \ldots, n$,

(2) $f_j \neq 0$ for at least one value of j.

In the first case Problem **N** is soluble only if the conditions

$$h_j = \iint\limits_G hv_j dx\,dy = 0 \qquad (j = 1, \ldots, n)\,,$$

are satisfied. Then the homogeneous Problem $\overset{\circ}{\mathbf{N}}$ has $n+1$ solutions. In the second case Problem **N** has a solution only if the following conditions are satisfied

$$c = \frac{h_1}{f_1} \quad (f_1 \neq 0)\,, \qquad h_j - \frac{h_1}{f_1}f_j = 0 \qquad (j = 2, \ldots, n)\,.$$

Thus, in the last case there are $n-1$ integral conditions for the right-hand side of the equation. Hence, if $n = 1$ and the condition $f_1 \neq 0$ is satisfied, Problem **N** is always soluble and has a unique solution.

9.5 In this subsection we shall complete the results of the preceding one, introducing the adjoint problem.

We now make additional assumptions, namely: (1) $a, b, c \in D_{2,p}(G+\Gamma)$; (2) $d, e \in D_{1,p}(G+\Gamma)$, $p > 2$. Then it

can be proved that the solution of the integral equation (9.55) belongs to the class $D_{2,p}(G+\Gamma)$, $p>2$. Moreover, taking into account that

$$\hat{\mathrm{P}}^*v \equiv -\mathrm{Re}\iint_G \left[A(\zeta)\frac{\partial^2\hat{g}(\zeta,z)}{\partial\zeta^2}+A_0(\zeta)\frac{\partial\hat{g}(\zeta,z)}{\partial\zeta}+\right.$$
$$\left.+\frac{1}{4}f(\zeta)\hat{g}(\zeta,z)+\frac{1}{\pi}\right]v(\zeta)\,d\xi\,d\eta\,,$$

and transforming the double integrals by means of Green's identity we obtain

$$\hat{\mathrm{P}}^*v \equiv -\mathrm{Re}\iint_G\left\{\left[(Av)_{\zeta\zeta}-(A_0v)_\zeta+\frac{1}{4}f(\zeta)v\right]\hat{g}(z,\zeta)+\right.$$
$$\left.+\frac{1}{\pi}v\right\}d\xi\,d\eta-\int_\Gamma \Omega(\zeta)\hat{g}(z,\zeta)\,ds\,, \qquad (9.57)$$

where

$$\Omega(\zeta)=\mathrm{Re}\left[\frac{A(\zeta)\bar{\zeta}'^2}{4}\right]\frac{dv}{d\nu}+\mathrm{Re}\left[\frac{A(\zeta)\bar{\zeta}'^2}{2i}\right]\frac{dv}{ds}+$$
$$+\mathrm{Re}\left[\frac{1}{4}\frac{dA}{d\nu}\bar{\zeta}'^2+\frac{1}{2i}\frac{dA}{ds}\bar{\zeta}'^2+\frac{1}{2i}A\bar{\zeta}'\bar{\zeta}''-\frac{1}{2i}A_0\bar{\zeta}'\right]v\,. \qquad (9.58)$$

It should be observed that the formulae (9.57) and (9.58) are valid not only for the circle but for an arbitrary domain, provided that the function $\hat{g}(z,\zeta)$ satisfies the conditions

$$\hat{g}(z,\zeta)=\frac{2}{\pi}\ln|z-\zeta|+\hat{g}_0\,,\qquad \frac{d\hat{g}}{d\nu}=0\qquad (\text{on } \Gamma)\,,$$

where $\hat{g}_0$ is a function continuously differentiable in $G+\Gamma$. In view of (9.57) the equation (9.56) can also be written in the form

$$v(z)+\mathrm{Re}\iint_G\left\{\left[(Av)_{\zeta\zeta}-(A_0v)_\zeta+\frac{1}{4}fv\right]\hat{g}(z,\zeta)+\right.$$
$$\left.+\frac{1}{\pi}v\right\}d\xi\,d\eta+\int_\Gamma \Omega(s)\hat{g}(z,\zeta)\,ds=0\,.$$

Applying the operators Δ and $\frac{\partial}{\partial \nu}$ to this equation we obtain

$$L_* v = \iint_G f v \, d\xi \, d\eta \quad (\text{in } G), \quad \frac{dv}{d\nu} - 4\Omega = 0 \quad (\text{on } \Gamma). \quad (9.59)$$

According to (9.58) the last condition has the form

$$\alpha_* \frac{dv}{d\nu} + \beta_* \frac{dv}{ds} + \gamma_* v = 0 \qquad (\text{on } \Gamma),$$

where

$$\alpha_* = 1 - \operatorname{Re}[A(\zeta)\bar{\zeta}'^2], \qquad \beta_* = \operatorname{Re}[2iA(\zeta)\bar{\zeta}'^2],$$

$$\gamma_* = \operatorname{Re}\left[-\frac{dA}{d\nu}\bar{\zeta}'^2 + 2i\frac{dA}{ds}\bar{\zeta}'^2 + 2iA\bar{\zeta}'\bar{\zeta}'' - 2iA_0\bar{\zeta}'\right].$$

Thus, the solution of the integral equation (9.56) is a solution of the boundary value problem (9.59) which naturally should be called the problem adjoint to Problem **N**. Obviously, $\alpha_* \neq 0$ on Γ, since $|A(\zeta)| < 1$.

9.6. The method presented above can also be applied to the investigation of the Riemann–Hilbert boundary value problem for an elliptic system of equations which has not been reduced to the canonical form. This problem was investigated by Vinogradov, [15b], whose results will be given below.*

We found in Ch. III, §17, that such a system can be written in the complex form

$$\partial_{\bar{z}} w - q_1(z)\partial_z w - q_2(\partial_z \overline{w} + Aw + B\overline{w} = F, \qquad (9.60)$$

where q_1 and q_2 are measurable functions which satisfy the inequality

$$|q_1(z)| + |q_2(z)| \leqslant q_0 < 1, \qquad (9.61)$$

and the functions A, B and F belong to the class $L_p(G+\Gamma)$.

* Somewhat modifying the formulation of the problem Vinogradov recently extended these results to the case of a quasi-linear elliptic system of equations [15c, d].

The problem will be considered for the circular domain G, $|z| < 1$, and the boundary condition will be taken in the canonical form

$$\operatorname{Re}[z^{-n}w(z)] = \gamma \qquad (\text{on } \Gamma\colon |z| = 1)\,, \tag{9.62}$$

where n is an integer. The general boundary condition

$$\operatorname{Re}[\overline{\lambda(z)}\,w(z)] = \gamma\,, \qquad \lambda = \alpha + i\beta \in C_\nu(\Gamma)\,,$$

for an arbitrary simply-connected domain G of the class C^1_μ, $0 < \mu < 1$, can be reduced to the form (9.62) with the aid of a non-singular transformation of the form (9.2) and a substitution of the form

$$w(z)\,e^{\chi(z)}\,.$$

They leave invariant the form of the equation and do not violate the condition (9.61).

Besides, with no loss of generality we shall assume that $\gamma \equiv 0$.

Thus, we shall consider the following boundary value problem.

Problem $\tilde{\mathbf{A}}$. *It is required to find in the circle G, $|z| < 1$, the solution of the equation* (9.60) *satisfying the boundary condition*

$$\operatorname{Re}[z^{-n}w(z)] = 0 \qquad (\text{on } \Gamma)\,. \tag{9.63}$$

The solution of the problem will be sought in the class of functions continuous in $G+\Gamma$ and belonging to $D_{1,p}(G+\Gamma)$, $p > 2$.

We first consider the case of a non-negative index, $n \geqslant 0$. It follows from the formula (7.5) (see also [14h]) that in this case the required solution of the problem can be represented in the form

$$w(z) = P_n f + \Phi_0(z)\,, \tag{9.64}$$

where

$$P_n f = -\frac{1}{\pi}\iint_G \left(\frac{f(\zeta)}{\zeta - z} + \frac{z^{2n+1}\overline{f(\zeta)}}{1 - z\bar{\zeta}}\right) d\xi\, d\eta\,,$$

$$\Phi_0(z) = i a_n z^n + \sum_{k=0}^{n-1} a_k(z^k - z^{2n-k}) + i\beta_k(z^k + z^{2n-k})\,,$$

$\alpha_0, \beta_0, \ldots, \alpha_{n-1}, \beta_{n-1}, \alpha_n$ being arbitrary real constants. For an arbitrary function f of the class $L_p(G+\Gamma)$, $p > 2$, $P_n f \in C_\alpha\,(G+\Gamma)$, $\alpha = \dfrac{p-2}{p}$, and the right-hand side of (9.64) satisfies the boundary condition (9.63), no matter what the real constants α_k, β_k are. In addition, w has generalized derivatives with respect to $\bar{z}$ and z, which are given by the formulae

$$\partial_{\bar{z}} P_n f = f\,,$$

$$\partial_z P_n f = S_n f$$

$$= -\frac{1}{\pi}\iint\limits_G \left[\frac{f(\zeta)}{(\zeta-z)^2} + \frac{(2n+1-2n\bar{\zeta}z)z^{2n}\overline{f(\zeta)}}{(1-z\bar{\zeta})^2}\right] d\xi\, d\eta\,, \qquad (9.65)$$

According to the Zygmund–Calderon theorem $S_n f$ is a linear operator acting from L_p into L_p, $p > 1$. Introducing the expression (9.64) into the equation (9.60) and taking into account the relations (9.65) we obtain for f the following integral equation:

$$f - Sf = F_0\,, \qquad (9.66)$$

where

$$Sf \equiv q_1 S_n f + q_2 \overline{S_n f} - A P_n - B\overline{P_n f}\,,$$

$$F_0 = F + q_1 \Phi_0'(z) + q_2\overline{\Phi_0'(z)} - A\Phi_0(z) - B\overline{\Phi_0(z)}\,.$$

The operator $S_n f$ can be represented thus

$$S_n f = \hat{S}_n f + (2n+1)z^{2n}\mathrm{T}_0 f\,,$$

where

$$\hat{S}_n f \equiv \frac{\partial \mathrm{T} f}{\partial z} + z^{2n+1}\frac{\partial \mathrm{T}_0 f}{\partial z}\,,$$

and

$$\mathrm{T} f = -\frac{1}{\pi}\iint\limits_G \frac{f(\zeta)\,d\xi\,d\eta}{\zeta - z}\,, \qquad \mathrm{T}_0 f = -\frac{1}{\pi}\iint\limits_G \frac{\overline{f(\zeta)}\,d\xi\,d\eta}{1 - z\bar{\zeta}}\,.$$

Therefore equation (9.66) can be written in the form

$$f - \hat{S}f - \hat{P}_n f = F_0\,, \qquad (9.67)$$

where

$$\hat{S}f = q_1\hat{S}_n f + q_2\hat{S}_n\bar{f}\ , \tag{9.68}$$

$$\hat{P}_n f = (2n+1)z^{2n}q_1\mathrm{T}_0 f + (2n+1)\bar{z}^{2n}q_2\overline{\mathrm{T}_0 f} - AP_n f - B\overline{P_n f}\ .$$

Calculations similar to those which enabled us to prove above the inequality (9.20), make it possible to establish that

$$(\hat{S}_n f, \hat{S}_n f) \leqslant (f, f)\ , \qquad f \in L_2(G+\Gamma)\ ,$$

Here also equality can take place. Therefore

$$L_2(\hat{S}_n) \equiv \|\hat{S}_n\|_{L_2} = 1\ .$$

Denoting by Λ_p the norm of S_n in L_p, i.e. $\Lambda_p = L_p(S_n)$, we have in view of (9.61), (9.68)

$$L_p(\hat{S}) \leqslant q_0\Lambda_p\ , \qquad p > 1\ .$$

Since $\Lambda_2 = 1$, a number $p > 2$ can be found such that

$$q_0\Lambda_p < 1\ .$$

For a fixed p the operator $I - \hat{S}$ has the inverse $(I-\hat{S})^{-1}$ in L_p. Consequently, the equation (9.67) which is equivalent to the original boundary value Problem $\widetilde{\mathbf{A}}$, is reducible to the equivalent Fredholm integral equation

$$f - (I-\hat{S})^{-1}\hat{P}_n f = (I-\hat{S})^{-1}F_0\ .$$

Therefore, the Fredholm theorems can be applied to the equation (9.66). We now prove that the homogeneous equation

$$f - Sf = 0$$

has no solution. If f is a solution of this equation the function $w = \mathrm{P}_n f$ is a solution of the homogeneous Problem $\widetilde{\mathbf{A}}^\circ$ and moreover, it satisfies the conditions

$$\int_\Gamma w(z)z^{-k-1}dz = 0\ , \qquad k = 0, 1, \dots, 2n\ . \tag{9.69}$$

Consequently, we have to prove that the solution of the homogeneous Problem $\widetilde{\mathbf{A}}^\circ$ which satisfies the additional conditions (9.69) identically vanishes.

In accordance with Theorem 3.31 the function $w(z) = \mathrm{P}_n f$ constituting a solution of the homogeneous equation

$$\partial_{\bar z} w - q_1 \partial_z w - q_2 \partial_z \overline{w} + Aw + B\overline{w} = 0\,, \qquad (9.69a)$$

can be represented in the form

$$w(z) = \Phi[W(z)]e^{\varphi(z)}\,, \qquad (9.70)$$

where $W(z)$ is a homeomorphism of the plane z onto the plane W, Φ is a function holomorphic in the domain $W(G)$ and $\varphi(z)$ is a function of the class $D_{1,p}(E)$, $p > 2$, holomorphic outside G and vanishing at infinity. According to the boundary condition (9.63) we have

$$\operatorname{Re}\{z^{-n}\Phi[W(z)]e^{\varphi(z)}\} = 0 \qquad (\text{on } \Gamma)\,.$$

Performing the change of variables $\zeta = W(z)$ we obtain for the function $\Phi(\zeta)$ the boundary condition

$$\operatorname{Re}[\zeta^{-n}\lambda_0(\zeta)\Phi(\zeta)] = 0 \qquad (\text{on } \Gamma_\zeta)\,, \qquad (9.71)$$

where $\Gamma_\zeta = W(\Gamma)$

$$\lambda_0(\zeta) = e^{\varphi(z)}\left(\frac{\zeta(z)}{z}\right)^n.$$

It is readily seen that $\operatorname{ind} \lambda_0(\zeta) = 0$ with respect to the curve Γ_ζ. By virtue of Theorem 4.7 the solution $\Phi(\zeta)$ of the Riemann–Hilbert boundary value problem (9.70) can not have more than $2n$ zeros in the closed domain $G+\Gamma$. Consequently, according to (9.70), the function $w(z)$ also has not more than $2n$ zeros in $G+\Gamma$. But in view of the boundary condition (9.63) $z^{-n}w(z)$ takes on Γ purely imaginary values. Let $u_0(x, y)$ be a function harmonic in G and equal to $iz^{-n}w(z)$ on Γ, i.e.

$$u_0 = iz^{-n}w(z) \qquad (\text{on } \Gamma)\,. \qquad (9.72)$$

In view of the conditions (9.69) u_0 satisfies the relations

$$\int_0^{2\pi} u_0 e^{ik\vartheta}\,d\vartheta = 0 \qquad (k = 0, 1, \ldots, n)\,.$$

It follows, therefore, that $u_0(x, y)$ has the form

$$u_0(x, y) = \mathrm{Re}[z^{n+1}\Phi_0(z)],$$

where Φ_0 is a function holomorphic in G. Hence, there pass through the origin of coordinates not less than $2n+2$ lines $u_0 = \mathrm{const} = 0$. If u_0 does not vanish identically these lines will cut the circle Γ at $2n+2$ distinct points. This result implies in view of (9.72) that $w(z)$ has not less than $2n+2$ zeros on Γ, i.e. we have a contradiction which leads to the following result: $w = \mathrm{P}_n f \equiv 0$.

It has thus been established that the equation (9.66) has a solution for an arbitrary right-hand side belonging to L_p. Since the right-hand side of the equation (9.66) contains $2n+1$ arbitrary real constants its solution also contains these constants, the dependence being linear.

It has therefore been proved that *the non-homogeneous Problem $\widetilde{\mathbf{A}}$ is always soluble and the corresponding homogeneous Problem $\widetilde{\mathbf{A}}^\circ$ has $2n+1$ linearly independent solutions.*

We proceed to the investigation of the problem with a negative index. It follows in this case from the equation (7.32) that the required solution of Problem **A** is representable in the form

$$w(z) \equiv P_k^* f = -\frac{1}{\pi}\iint_G \left(\frac{f(\zeta)}{\zeta - z} + \frac{\bar{\zeta}^{2k-1}\overline{f(\zeta)}}{1-\bar{\zeta}z}\right) d\xi\, d\eta\,, \qquad (9.73)$$

where $k = -n > 0$ and f is the unknown function of the class $L_p(G+\Gamma)$, $p > 2$, which should satisfy the relations

$$a_j(f) \equiv -\frac{1}{\pi}\iint_G (f\zeta^{j-1} + \bar{f}\bar{\zeta}^{2k-j-1})\, d\xi\, d\eta = 0 \qquad (9.74)$$

$$(j = 1, \ldots, k)\,.$$

There are altogether $2k-1$ real relations. We shall denote the subspace of elements of the space $L_p(G+\Gamma)$ which satisfy the conditions (9.74) by $L_{p,2k-1}(G+\Gamma)$. For an arbitrary element f of the subspace $L_{p,2k-1}(G+\Gamma)$ the

function (9.73) satisfies the homogeneous boundary condition (9.63). Introducing the expression (9.73) into the differential equation (9.60) we obtain for f the following integral equation:

$$f - S^*f = F \,, \tag{9.75}$$

where

$$S^*f = q_1 \frac{\partial P_k^* f}{\partial z} + q_2 \frac{\partial \overline{P_k^* f}}{\partial z} - A P_k^* f - B \overline{P_k^* f} \,.$$

It can be shown that the singular integral operator

$$S_k^* f \equiv \frac{\partial P_k^* f}{\partial z} = -\frac{1}{\pi} \int\int \left[\frac{f(\zeta)}{(\zeta - z)^2} + \frac{\bar{\zeta}^{2k} f(\bar{\zeta})}{1 - \zeta z} \right] d\xi \, d\eta \,,$$

has the norm in L_2 equal to unity, i.e.

$$L_2(S_k^*) = 1 \,. \tag{9.76}$$

Regarding now $S_0^* f = q_1 S_k^* f + q_2 \overline{S_k^* f}$ as an operator in $L_p(G + \Gamma)$, $p > 2$, we have

$$L_p(S_0^*) \leqslant q_0 L_p(S_k^*) \,.$$

According to (9.76) a number $p > 2$ can be found such that

$$q_0 L(S_k^*) < 1 \,, \qquad q_0 < 1 \,.$$

It follows that the operator $I - S_0^*$ has an inverse in $L_p(G + \Gamma)$. Applying to both sides of the equation (9.75) the inverse operator $(\mathrm{I} - S_0^*)^{-1}$ we obtain an equivalent equation with a completely continuous operator. Hence, the Fredholm theorems may be applied to the equation (9.75). Let f satisfy the equation $f - S^*f = 0$. Let us represent f in the form $f = f_0 + f_k$ where $f_0 \in L_{p,2k-1}(\bar{G})$, $f_k = c_0 \bar{z}^{k-1} + c_1 \bar{z}^{k-2} + \dots (c_0 = \bar{c}_0)$. The polynomial f_k is uniquely determined by the relations $a_j(f - f_k) = 0$ $(j = 1, 2, \dots, k)$. If we represent $w = \mathrm{P}_k^* f$ as a solution of the equation (9.69a) in the form (9.70) and take into account that $\mathrm{P}_k^* f_0$ satisfies the condition (9.63) we obtain

$$\mathrm{Re}\,\{z^k \Phi[W(z)] e^{i\varphi(z)}\} = \mathrm{Re}\, f_k' \quad \text{on} \quad \Gamma \,,$$

where f'_k is a polynomial in z of $(k-1)$-th degree. A reasoning similar to that used above (p. 359) leads to the result $f'_k \equiv 0, \Phi \equiv 0$ and, consequently, $w \equiv 0$. The last fact proves the solubility of the equation (9.75) for an arbitrary right-hand side of the class $L_p(G+\Gamma)$, $p > 2$. Thus, the solution of the equation (9.75) is representable in the form $f = (I - S^*)^{-1}F$. This solution should be subjected to the conditions (9.74). In consequence we obtain the following $2k-1$ integral conditions of solubility of the non-homogeneous Problem $\tilde{\mathbf{A}}$:

$$\iint_G (\chi'_j F_1 + \chi''_j F_2)\,dx\,dy = 0 \qquad (j = 1, \ldots, 2k-1)\,, \qquad (9.77)$$

where F_1 and F_2 are the real and imaginary parts of the function F, and χ'_j and χ''_j are definite linearly independent functions.

Thus, *for* $n < 0$ *the homogeneous Problem* $\tilde{\mathbf{A}}^\circ$ *has no non-trivial solution and the non-homogeneous Problem* $\tilde{\mathbf{A}}$ *is soluble only if the conditions* (9.77) *are satisfied.*

§10. Remarks concerning certain papers on Problem A. Some formulations of more general problems

10.1. It was already indicated above that Problem **A** was first investigated by Hilbert, [26a], for the Cauchy–Riemann system of equations. It constitutes a particular case of a more general problem stated by Riemann in his famous thesis ([76], p. 78–80). The Riemann problem can be stated as follows: *it is required to find a function* $w(z) = u + iv$ *analytic in the domain* G, *which satisfies at every boundary point the relation*

$$F(u, v) = 0 \qquad (\text{on } \Gamma)\,.$$

Riemann, however, stated only general considerations concerning the solubility of the problem and a method of solution. Hilbert was the first to investigate the case of the linear boundary condition of the form $\alpha u + \beta v = \gamma$ (on Γ) in two papers (see [26a], Ch. X). In the paper [26b]

the problem is reduced to a singular integral equation whose investigation contains certain inaccuracies leading to untrue statements. A different method of solution of the problem for a simply-connected domain was proposed later by Hilbert in the paper [26a] (p. 82–83). The case of a multiply-connected domain was first examined by Kveselava, [39a]. He established a number of important conditions of solubility of the problem; in particular, the case $n = 0$ was investigated.

The case of discontinuous coefficients was investigated by Muskhelishvili, [60a]. More general problems were dealt with in the papers of Gakhov, [23], I. N. Vekua [14b, g], Sherman [94*], N. P. Vekua [14*a], Khvedelidse, [91], and others.

It should be observed that Hilbert's paper, notwithstanding its defects, had an important historical value. Together with a paper of Poincaré [73] devoted to the theory of tides, it greatly stimulated investigations on the theory of singular integral equations and boundary value problems for analytic functions.

It was also Hilbert who first considered the boundary value problem for the system of equations of elliptic type [26a] (Ch. XXVII)

$$u_x - v_y + au + bv = f\,, \qquad u_y + v_x + cu + dv = g\,, \tag{10.1}$$

$$u = \gamma \qquad (\text{on } \Gamma)\,. \tag{10.2}$$

He investigated this problem by means of Fredholm integral equations. His reasoning, however, contained inaccuracies which led to a number of untrue statements, ([26a], p. 219, Theorem 43). These inaccuracies were indicated in a paper of Hellwig [24].

To the general boundary value problem for the system of equations (10.1), which was called above Problem **A**, are devoted the papers of Hurwitz [29], Usmanov [89a, b], Haack and Hellwig [21], Haack [20], Hellwig [24a, c], J. and J. Nitsche [62], Joh. Nitsche [63a, b, c], Joach. Nitsche [64a], Gakhov and Khasabov [23*a], [23a].

In a more general form the problem was investigated in the author's paper [14a]. The fundamental results of the last paper completed by many new results of the same author and also the results of Bojarski (see Appendix to Ch. IV) and Vinogradov [15a, b] are treated in this chapter. An approximate method of solution of Problem **A** was elaborated in the paper of Klabukova [40*]. Boundary value problems for an elliptic system of a more general form were examined in the papers of Bers and Nirenberg [7a, b].

Bojarski investigated boundary value problems for the system of equations (10.1) with the boundary conditions of the form ([11a, a′])

$$\mathrm{Re}\left\{a_0(t)w(t)+\int_\Gamma b_0(t,t_1)w(t_1)\,dt_1\right\}=\gamma(t) \qquad (10.3)$$

and

$$\mathrm{Re}[a_0(t)\partial_t w+b_0(t)w]=\gamma(t) \qquad |a_0(t)|=1\,. \qquad (10.4)$$

Introducing the adjoint homogeneous problems he established the necessary and sufficient conditions of solubility of the problems (10.3) and (10.4), and derived the important relation

$$l-l'=2n-3(m-1)\,, \qquad (10.4\mathrm{a})$$

where l and l' are the numbers of solutions of mutually adjoint homogeneous problems, n is the index of the function $\overline{a_0(t)}$ and $m+1$ is the connectedness of the domain.

Recently Daniluk proposed a new method of investigation of the problem (10.4) ([11d, e, f]). Let us note some of his results. Taking the original equation in the form $\partial_{\bar z}w+B\overline{w}=0$ and introducing a complex vector F with components $w=u+iv$, w_z and $\overline{w}$, the problem (10.4) can be reduced to an equivalent problem of the form

$$\partial_{\bar z}F+A_1F+B_1\overline{F}=0 \qquad (\text{in } G)\,,$$

$$\mathrm{Re}[g_1(t)F]=\gamma \qquad (\text{on } \Gamma)\,, \qquad (*)$$

where A_1, B_1 and g_1 are matrices expressible explicitly in terms of the coefficients of the original problem. The

problem (*) can be tackled by the methods developed in this chapter for the solution of Problem **A**. Here as the adjoint problem we have the problem

$$\partial_z F' - A_1' F' - \bar{B}_1' \bar{F}' = 0 \quad (\text{in } G)\,,$$
$$\operatorname{Re}[g_1'^{-1}(t) F'] = 0 \quad (\text{on } \Gamma)$$

the necessary and sufficient conditions of solubility of the problem (10.4) taking an appropriate form. In particular, a new proof of the formula (10.4a) is given. It is important to make clear in what cases the values of l and l' can be determined separately. To this end we introduce a boundary value problem containing the real parameter, $B = \lambda B_0$, $b = \lambda b_0$. Restricting ourselves to the case of a simply-connected domain we can reduce this problem to a Fredholm integral equation of the form $F - T_\lambda F = \Phi$ by following the method indicated in §7.2 and §7.3. It is proved that the operator T_λ can have only a discrete spectrum Λ. We note one important result implied by these considerations. If $n \geqslant 0$ and $\lambda \in \Lambda$, then $l = 2n+3$ and $l' = 0$. If now $n < 0$ and $\lambda \notin \Lambda$, then $l \leqslant 2$ and all the cases $l = 0, 1, 2$ can occur.

10.2. We shall see in Ch. V (§8) that many geometrical problems reduce to the boundary value problem (10.4).

We now indicate some other boundary value problems whose investigation is of some interest, especially from the point of view of geometrical applications.

I. Let Γ be a simple closed curve of the class C^1_μ. Let $\nu(t)$ be a complex function of the point t of the contour Γ, which establishes a homeomorphic mapping of this contour onto itself. It is required to find in the domain G bounded by the curve Γ the solutions w_1 and w_2 of the equations

$$\partial_{\bar{z}} w_j + A_j w_j + B_j \bar{w}_j = 0 \quad (j = 1, 2)\,, \qquad (10.5)$$

satisfying the boundary conditions of the form

$$[a_1(t) w(t) + a_2(t) \overline{w_1(t)} + a_3(t) \partial_t w_1(t) + a_4(t) \partial_t \bar{w}]_{t=\nu(z)}$$
$$= a_5(z) w_2(z) + a_6(z) \overline{w_2(z)} + a_7(z) \partial_z w_2 + a_8(z) \partial_{\bar{z}} \bar{w}_2\,, \qquad (10.6)$$

where $a_j(t)$ are specified functions of the point t on the curve Γ. The coefficients of the equations (10.5) belong to the class $L_p(G+\Gamma)$, $p > 2$. This problem is a generalization of the familiar Carleman problem for analytic functions, [38b]. We arrive at such a generalized statement of the Carleman problem for instance in the investigation of the geometrical problem of contact of two surfaces of positive curvature (see Ch.V, §8).

In this respect it is of interest to establish criteria of existence or non-existence of non-trivial solutions of the problem (10.6). We should observe that it can be assumed that $\nu(t) \equiv t$ in geometrical problems.

II. Let G^+ and G^- be the inner and outer domains bounded by the curve Γ. It is required to determine in G^+ and G^- the solution of the equation

$$\partial_{\bar z} w + Aw + B\overline{w} = 0\,, \qquad A\,, B \in L_{p,\,2}(E)\,, \qquad p > 2\,, \qquad (10.7)$$

satisfying the boundary condition of the form

$$[a_1(t)\,w^+(t) + a_2(t)\,\overline{w^+(t)} + a_3(t)\,(\partial_t w)^+ + a_4(t)\,(\partial_{\bar t}\overline{w})^+]_{t=\nu(z)}$$
$$= a_5(t)\,w^-(t) + a_6(t)\,\overline{w^-(t)}\,, \qquad (10.8)$$

where $\nu(t)$ is as before the homeomorphism of the contour Γ onto itself, a_j are known functions of the point $t \in \Gamma$. The formulation of the problem, similarly to the preceding problem, is generalized to the case of a multiply-connected domain. The above indicated geometric problem can be reduced to a problem of the form (10.7)–(10.8).

III. It is of interest also to investigate the problem (10.4) in the case when the function $a_0(t)$ vanishes on a part of the boundary of the domain. There exist simple geometrical problems which lead to this case (Ch. V, §8).

APPENDIX TO CHAPTER IV

ON SPECIAL CASES OF THE RIEMANN–HILBERT PROBLEM

B. Bojarski

The special cases of the Riemann–Hilbert problem (Problem Å)

$$\operatorname{Re}\{\bar{\lambda}\Phi\} = 0 \qquad (\text{for } \Gamma)\,, \qquad \Phi \in \mathfrak{A}_0(G)\,, \tag{1}$$

are said to be the cases in which the index n of the problem satisfies the inequality

$$0 \leqslant n \leqslant m-1\,.$$

Henceforth, continuing the reasoning of Ch. IV, §7–10, we shall present a number of results concerning special cases of the Riemann–Hilbert problem. These results elucidate the nature of the solubility relations of Problem Å in the indicated cases.

In accordance with the results of §4 we shall confine ourselves to the case when the domain G is bounded by the curve Γ constituting a union of circles $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$, Γ_0 being the unit circle and all Γ_j, $j \geqslant 1$, are situated inside Γ_0. Let the point $z = 0$ belong to the domain G. The problem (1) will be reduced to the canonical form (3.14) assuming also that $\overline{\Omega_n(z)} = (z-a)^{-n}$. We shall denote by $\mathfrak{B}_n(a)$, $a \in G$ the class of functions $\Phi(z)$ analytic in G and continuous everywhere in $G+\Gamma$ except for the point $z = a$ where $\Phi(z)$ may have a pole of the order not greater than n. For $a = 0$ we shall set $\mathfrak{B}_n(0) = \mathfrak{B}_n$. Then it is readily observed that the problem (1) is equivalent to the following problem.

Problem $\mathbf{A}_n$. *To find a function* $\Phi(z) \in \mathfrak{B}_n(a)$ *satisfying the boundary condition*

$$\operatorname{Re}[e^{-\pi i\alpha(t)}\Phi(t)] = 0 \, , \tag{2}$$

where $\alpha(t)$ is a real piecewise-constant function on Γ, i.e. $\alpha(t) = \alpha_j = \text{const.}$ on Γ_j, $j = 1, \ldots, m$; $\alpha_0 = 0$. If the function $\lambda(t)$ is specified on Γ and the point a is fixed inside G, then the function $\alpha(t)$ and the sequence α_j ($j = 1, \ldots, m$) are determined uniquely in accordance with the formulae (5.68) and (5.69). If the sequence α_j is replaced by a sequence α'_j such that $\alpha_j = \alpha'_j (\operatorname{mod} 1)$ for all j, then it is evident that Problem $\mathbf{A}_n$ is unchanged. It is, therefore, natural to regard the set of numbers α_j ($j = 1, \ldots, m$) as a point of a real m-dimensional torus T_m. Then for any fixed n the torus T_m represents the variety of all problems $\mathbf{A}_n$. On the other hand, every problem $\mathring{\mathbf{A}}$ of the form (1) has, according to the formulae (5.68) and (5.69) a fully determined representative among the problems $\mathbf{A}_n$, which is completely equivalent to the given problem $\mathring{\mathbf{A}}$. The qualitative nature of the solubility relations of Problems $\mathring{\mathbf{A}}$ and $\mathbf{A}_n$ is identical. We now proceed to the investigation of the latter problem.

1. We first give two simple lemmas concerning the solutions of Problem $\mathbf{A}_n$. They are stated in a somewhat more general way than is necessary for further considerations.

LEMMA 1. *Every function* $\Phi(z)$ *meromorphic in* G, *continuous on* Γ *and satisfying the boundary condition* (2) *is analytically continued through* Γ.

PROOF. The statement follows directly from the principle of symmetry, since according to our assumption G is a circular domain and the condition (2) implies that on the plane of the variable Φ the boundary values of the functions $\Phi(z)$ on Γ_j lie on the straight line $\operatorname{Re}\{e^{-\pi i\alpha_j}\Phi\} = 0$.

Lemma 1 is still valid if we allow the presence of a finite number of points on Γ on approaching which from the

inside of G the function $\Phi(z)$ tends to ∞. For proving this assertion it is sufficient to investigate the function $\Phi_1 = \frac{1}{\Phi}$ in the vicinity of these points. According to Lemma 1 the above indicated points are the poles of a finite order of the functions $\Phi(z)$.

Suppose that the meromorphic function $\Phi(z)$ which satisfies the boundary conditions (2) takes in $G+\Gamma$ some value h. In view of Lemma 1 every h-point of the functions $\Phi(z)$ in $G+\Gamma$ has a definite finite multiplicity. Let us denote by $N_G(h)$ the sum of the multiplicities of all h-points of the function $\Phi(z)$, situated inside G. By $N_{\Gamma_j}(h)$ $(j = 0, 1, \ldots, m)$ we denote the sum of the multiplicities of the h-points of the function $\Phi(z)$ on Γ_j. Let

$$N_\Gamma(h) = \sum_{j=0}^{m} N_{\Gamma_j}(h).$$

LEMMA 2. *For all finite h-points of the function $\Phi(z)$ satisfying the boundary condition* (2), *not identically equal to a constant and continuous on Γ, we have the formula*

$$2N_G(h) + N_\Gamma(h) = 2N_G(\infty)\,, \tag{3}$$

where $N_G(\infty)$ is the sum of multiplicities of all poles of the function $\Phi(z)$ inside G. All the numbers $N_{\Gamma_j}(h)$ are even.

PROOF. Let us consider the function $\varphi(z) = \Phi(z) - h$. By virtue of Lemma 1 this function has only a finite number of zeros in $G+\Gamma$. Let $z_1, \ldots, z_p$ be the set of its zeros inside G and $z'_1, \ldots, z'_r$ the set of zeros on Γ. Let $\zeta_1, \ldots, \zeta_s$ be the poles of $\Phi(z)$ inside G. Let us choose a sufficiently small positive number ε such that the circles K_ε with centres at the poles of the function $\varphi(z)$ or at its zeros on Γ do not intersect. Removing from the domain G the sum of these circles we obtain a domain G_ε bounded by the contour Γ_ε; for a sufficiently small ε all the points $z_1, \ldots, z_p$ lie inside G_ε and the points

$\zeta_1, \ldots, \zeta_s$ and $z'_1, \ldots, z'_r$ outside G_ε. According to the principle of the argument we have

$$N_G(h) = \frac{1}{2\pi}\lim_{\varepsilon\to 0} \Delta_{\Gamma_\varepsilon} \arg\varphi(z) = \sum_{k=1}^{s} N(\zeta_k) - \sum_{k=1}^{r} \frac{N(z'_k)}{2}, \qquad (4)$$

where $N(\zeta_k)$ is the multiplicity of the pole ζ_k and $N(z'_k)$ is the multiplicity of the zero z'_k. The first sum is obvious, we shall therefore consider only the second one. It represents a sum of the increments of $\arg\varphi$ along somewhat deformed boundary curves Γ_j: in moving along the curve Γ_j, approaching the distance ε from any of the zeros z'_k we have to pass on the arc of the circle of radius ε with centre at z'_k. But according to the boundary condition (2) on the sections $\Gamma_j^{z'_k z'_i}$ of the boundary line Γ_j, between two neighbouring zeros z'_k and z'_j, $\arg\varphi(z)$ has no increment, since $\varphi(z) = -h + i\varrho(z)e^{\pi i a_j}$, $\varrho(z)$ is there real and does not vanish; in other words the values of $\varphi(z)$ cover a finite section of a straight line and the values of $\varphi(z)$ on the ends of $\Gamma_j^{z'_k z'_i}$ are equal. Therefore, if this straight line does not pass through the origin of coordinates, then $\Delta_{\Gamma_j^{z'_k z'_i}} \arg\varphi = 0$. If it does pass through the origin, then according to our condition $\varphi(z) \neq 0$ on $\Gamma_j^{z'_k z'_i}$, the values of $\varphi(z)$ on $\Gamma_j^{z'_k z'_i}$ are situated only on one semi-axis of the straight line under consideration. Hence, we have also in this case $\Delta_{\Gamma_j^{z'_k z'_i}} \arg\varphi = 0$.

Thus, $\arg\varphi$ takes an increment different from zero only in the motion along the arcs of circles of radius ε with centres at the points z'_k. But for $\varepsilon \to 0$ the arcs under consideration become semi-circles and then, according to Lemma 1 the formula (4) holds; hence, also (3) is valid. We notice also that $\frac{1}{2}N_{\Gamma_j}(h)$ is an integer, i.e. $N_{\Gamma_j}(h)$ is an even number for arbitrary j and h. This completes the proof of Lemma 2.

Lemma 2 constitutes a more precise statement of the formula (4.17) in application to Problem $\mathbf{A}_n$. The proofs of both formulae also in principle are identical.

It should be noted that Lemma 2 cannot be extended to the general case of the boundary value Problem $\mathring{\mathbf{A}}$, for the h-points are not invariant with respect to the transformation to the canonical form, if $h \neq 0$.

Lemma 2 implies immediately a new proof of insolubility of the problem (5.2) (§5.1) if the conditions $\alpha_j = 0 (\mathrm{mod}\, 1)$ are not satisfied.

In fact, then $N_G(\infty) = 0$ and according to (3) $N_G(h) = N_\Gamma(h) = 0$, i.e. the function $\Phi(z) = \mathrm{const} = c$. In view of the boundary condition (2) $c \neq 0$ only under the condition $\alpha_j = 0 (\mathrm{mod}\, 1)$, $j = 1, \ldots, m$.

For a complete investigation of Problem $\mathbf{A}_n$ in the class $\mathfrak{B}_1$ we shall need the following lemma:

LEMMA 3. *Let E_z be the domain of the variable z bounded by radial cuts $I_k (k = 0, 1, \ldots, m)$ of a finite length situated respectively along the straight lines l_k,* $\mathrm{Re}(e^{-\pi i \alpha_k} z) = 0$, $\alpha_0 = 0$. *If the function $\Phi = \Phi(z)$ maps E_z conformally and univalently onto the domain $\widetilde{E}_\Phi$ of the plane of the variable Φ, bounded by the cuts I'_k situated along the same lines l_k, respectively, and $\Phi(\infty) = \infty$, then $\Phi(z) = cz + i\tilde{c}$ where c and $\tilde{c}$ are real constants. $\tilde{c} \neq 0$ only if $\alpha_k = 0 (\mathrm{mod}\, 1)$ for all $k = 1, 2, \ldots, m$.*

PROOF. Under the conditions of the lemma the function $\Phi(z)$ has a simple pole at the point $z = \infty$, and $\lim\limits_{z \to \infty} \dfrac{\Phi(z)}{z} = \gamma \neq 0$. We note that the essence of the lemma consists in the fact that we make no assumptions about the situation of the zero of the function $\Phi(z)$ or about the number γ.

First, we examine the case in which $\Phi(0) = 0$. It can easily be verified that the function $\varphi(z) = \dfrac{\Phi(z)}{z}$ is everywhere bounded. It is obvious if the point $z = 0$ does not lie on any of the cuts I_k. If $z = 0$ lies inside the cut I_{k_0},

$0 \leqslant k_0 \leqslant m$, then the boundedness of the function $\varphi(z)$ is ensured by Lemma 1. In the case in which the point $z = 0$ is an end of one of the cuts I_k (assume that it is the cut I_0) Lemma 1 cannot be applied. Let us situate I_0 along the positive real semi-axis and let us map by means of the function $\zeta = f(z)$ the exterior of the line I_0 onto the interior of a smooth curve, say the circle K, in such a way that the point $z = 0$ remains invariant, $f(0) = 0$. Obviously, $f(z)$ has the form $f(z) = f_1(\xi)$, $\xi = \sqrt{z}$, in the vicinity of the point $z = 0$, where $f_1(\xi)$ is an analytic function in the variable ξ, $f_1(0) = 0$, $\frac{f_1(\xi)}{\xi}$ is bounded for $\xi \to 0$. The function $\Phi_1(\xi) = \Phi(z)$ satisfies a boundary condition of the form (2) on $K+K'$ where K' is the union of the images of the lines L_k, $k \neq 0$ in the mapping $\zeta = f(z)$. Besides, $\Phi_1(\zeta)$ vanishes for $\zeta = 0$ and has a pole of the first order for $\zeta = \infty$. It cannot vanish together with $\Phi(z)$ at any point $\zeta \neq 0$ of the domain bounded by the curves $K+K'$ or on these curves. Since the curve K is a circle we may apply Lemma 2. In view of the formula (3) we find that $\Phi_1(\zeta)$ has a zero exactly of order two at the point $\zeta = 0$, i.e.

$$\left|\frac{\Phi(z)}{z}\right| = \left|\frac{\Phi_1(\zeta)}{\zeta^2} \cdot \frac{f^2(z)}{z}\right| = \left|\frac{\Phi_1(z)}{\zeta^2}\left(\frac{f_1(\xi)}{\xi}\right)\frac{\xi^2}{z}\right| \leqslant \left|\frac{\Phi_1}{\xi^2}\right| \left|\frac{f_1(\xi)}{\xi}\right|^2,$$

i.e. $\frac{\Phi(z)}{z}$ is bounded.

Thus, the boundedness of $\varphi(z)$ has been proved. Now, according to the conditions of Lemma 3 $\operatorname{Im}\varphi(z) = 0$ on I_k, $k = 0, \dots, m$ whence it follows that $\varphi(z)$ can have only a real constant value.

Let now

$$\Phi(0) = ie^{-ni\theta} \neq 0 .$$

We again introduce the function $\varphi(z) = \frac{\Phi(z)-\Phi(0)}{z}$ and as before we find that $\varphi(z)$ is everywhere bounded

in E_z including the point $z=\infty$. Moreover, it satisfies the boundary condition

$$\operatorname{Im}\varphi(z) = -\operatorname{Im}\frac{\Phi(0)}{z} = \frac{\varrho_k}{t} \quad \text{on} \quad I_k\,, \quad |z| = t\,, \quad (5)$$

$$0 < t_k' \leqslant t \leqslant t_k'' < \infty \quad (k = 0, 1, \ldots, m)\,, \qquad (6)$$

where

$$\varrho_k = -\operatorname{Im} e^{2\pi i(a_k-\theta)} = -\sin 2\pi(a_k-\theta)\,. \qquad (7)$$

If $z=0$ (i.e. $t=0$) belongs to any of the intervals $[t_k', t_k'']$ defining the lines I_k, then in accordance with the conditions of the lemma the corresponding $\varrho_k = 0$. Therefore, we have written the sign of strong inequality in the first and last inequalities (6). The function $\varphi(z) = u+iv$ is a bounded and single-valued solution of the nonhomogeneous Dirichlet problem (5) for the domain E_z. We shall prove that such a solution exists only for $\varrho_k = 0$, $k = 0, 1, \ldots, m$. In this case it is a real constant.

Let us first consider the case in which all $\varrho_k \neq 0$. Then, if there are both negative and positive numbers among them we arrive at once at a contradiction. In fact, if $\varphi(z)$ is not a constant, then every boundary continuum of the domain $\varphi(E_z)$ ($\varphi(E_z)$ is the image of the domain E_z) lies in view of (5) beyond the section of the line v, $v = \dfrac{\varrho_k}{t}$, $0 < t_k' < t < t_k''$, i.e. wholly beyond the negative or positive imaginary semi-axis, and continua of both kinds occur. Such a domain, however, cannot be bounded.

Thus, all ϱ_k have the same sign. Assume that $\varrho_k > 0$, $k = 0, 1, \ldots, m$. By virtue of (5) $\dfrac{dv}{ds} = -\dfrac{\varrho_k}{t^2} < 0$ on the positive side of I_k^+ and $\dfrac{dv}{ds} = \dfrac{\varrho_k}{t^2} > 0$ on the negative side of I_k^- (by positive, we understand the side situated to the left in moving along I_k in the direction of increasing t).

Hence

$$\frac{du}{dn} = -\frac{dv}{ds} > 0 \quad \text{on } I_k^+ \text{ and } \quad \frac{du}{dn} < 0 \quad \text{on } I_k^-\,,$$

i.e. in moving in the positive direction along every curve bounding the domain $\varphi(E_z)$ we should move always in such a way that the interiors of these curves lie on the right. This is, however, impossible if $\varphi(E_z)$ is a finite domain, since then at least on one contour we should have to move in the reverse direction.

Similarly, we can consider the case in which some of the numbers ϱ_k vanish. Thus, the assumption that $\varphi(z)$ is not constant leads to a contradiction. Hence, $\varphi(z) = c = \text{const.}$ and this result implies that $\varrho_k = 0$ for $k = 0, 1, \ldots, m$ and that c is real. This result is compatible with (7) only when $a_k = 0$, $k = 1, 2, \ldots, m$, and then also we should have $\theta = 0$; otherwise $\Phi(0) = 0$.

This completes the proof of Lemma 3.

From Lemma 2 and Lemma 3 the following theorem can be obtained.

THEOREM 1. *For* $n = 1$ *any solution* $\Phi(z)$ *of Problem* $\mathbf{A}_1$ *in the class* $\mathfrak{B}_1$ *satisfying the condition* $\Phi(\infty) = 0$ *establishes a conformal mapping of the domain* G *onto the plane of the variable* Φ *with cuts along the straight lines* $\mathrm{Re}[e^{-\pi i a_k}\Phi] = 0$, $(k = 0, \ldots, m)$.

If Φ_1 *and* Φ_2 *are two such solutions, then* $\Phi_2 = c\Phi_1 + \Phi_0$ *where* c *is a real constant, and* Φ_0 *is the solution of Problem* $\mathbf{A}_0$ *in the class* $\mathfrak{B}_0$.

PROOF. According to the condition $\Phi(0) = \infty$ and the relation (3) we have $2N_G(h) + N_\Gamma(h) = 2$ for an arbitrary finite value of h acquired by the function $\Phi(z)$ in $G+\Gamma$. Hence, if $N_\Gamma(h) = 0$, then $N_G(h) = 1$, i.e. if $\Phi(z)$ does not take the value h on the boundary Γ, then $\Phi(z)$ takes it inside exactly once. If the function $\Phi(z)$ has the value h at an internal point of the domain, then $N_G(h) = 1$ and $N_\Gamma(h) = 0$, i.e. it does not belong to the boundary values of the function $\Phi(z)$ on Γ. Since in view of the boundary condition (2) and the continuity of $\Phi(z)$ on Γ the boundary values of $\Phi(z)$ belong to bounded sections I situated on the straight lines l_k, $\Phi = i\varrho_k e^{2\pi i a_k}$, ϱ_k is a real function

of the point $t \in \Gamma_k$, $\Phi(z)$ maps conformally and univalently the domain G onto the plane with the cuts I_k. This completes the proof of the first part of the theorem. It should be observed that the domains appearing in the theorem do not necessarily coincide with the canonical domains with cuts usually employed, since it is not impossible that one of the cuts I_k passes through the origin of coordinates. Obviously, two cuts I_k cannot pass through the origin; in fact, then on two contours I_k and I'_k the function $\Phi(z)$ would have the value 0. This implies that $N_{\Gamma_k}(0) > 0$ and $N_{\Gamma'_k}(0) > 0$, and since $N_{\Gamma_k}(0)$ and $N_{\Gamma'_k}(0)$ are even, $N_{\Gamma_k}(0) \geqslant 2$ and $N_{\Gamma'_k}(0) \geqslant 2$. This contradicts the formula (3).

We still have to prove the second statement of the theorem.

Let $\Phi_1(z)$ and $\Phi_2(z)$ be two linearly independent solutions of the problem under consideration. Then according to the results just proved the implicit function $\Phi_2(\Phi_1^{-1}) = f(\Phi_1)$ would establish a conformal mapping of the domain with cuts along the lines L_k onto the domain with cuts along the same straight lines. By virtue of Lemma 3 $f(\Phi_1) = c\Phi_1 + i\tilde{c}$ where c and $\tilde{c}$ are real constants. If $\tilde{c} = 0$, then $\Phi_2(z) = c\Phi_1(z)$, which contradicts the assumption. If $\tilde{c} \neq 0$, then in accordance with Lemma 3 $\alpha_k = 0 (\mathrm{mod}\, 1)$, $k = 1, \ldots, m$. In this case the constant $i\tilde{c}$ is the solution of the problem in the class $\mathfrak{B}_0$ and we obtain $\Phi_2(z) = c\Phi_1(z) + i\tilde{c}$ which was required.

2. For further considerations it is necessary to formulate the problem adjoint to Problem $\mathbf{A}_n$. It is convenient for our purpose to take the following definition.

Problem $\mathbf{A}'_n$. *To find the function* $\psi(z) \in \mathfrak{R}_{n'}$, $n' = m - n - 1$, *satisfying the boundary condition*

$$\mathrm{Re}[z^{m-1} z' e^{\pi i \alpha(z)} \psi(z)] = 0 \quad (\text{on } \Gamma) . \tag{8}$$

Transformation from Problem $\mathbf{A}'_n$ to the problem adjoint in the sense of the definitions of §2, Ch. IV, is obtained

by the simple substitution of the form $\psi = \frac{\psi_1}{z^{n'}}$. In particular, l_n and l'_n are the numbers of linearly independent solutions of the problems $\mathbf{A}_n$ and $\mathbf{A}'_n$ connected accordingly by the relation (4.24).

$$l_n - l'_n = 2n - m + 1 . \tag{9}$$

For $n \leqslant m-1$ the numbers $l_n = l_n(a)$ and $l'_n(a)$ depend on both the number n and the point $a \in T_m$ which determines Problem $\mathbf{A}_n$. Below we shall enumerate some properties of the numbers l' and l'_n in terms of n,

$$l_n \leqslant l_{n+1} \leqslant l_n + 2 , \qquad l'_{n+1} \leqslant l'_n \leqslant l'_{n+1} + 2 . \tag{10}$$

The first of the inequalities follows from the fact that $\mathfrak{B}_{n+1} \supset \mathfrak{B}_n$ and in passing from $\mathfrak{B}_n$ to $\mathfrak{B}_{n+1}$ only two linearly independent (with respect to $\mathfrak{B}_n$) functions $\frac{1}{z^{n+1}}$ and $\frac{i}{z^{n+1}}$ appear. On the contrary, in passing from n to $n+1$ in Problem $\mathbf{A}'_n$, we have $\mathfrak{B}_{m-n-1} \supset \mathfrak{B}_{m-n-2}$. This completes the proof of the second inequality (10).

We shall below make use of the following estimate for l_n when $n \leqslant m-1$:

$$\max(0, 2n - m + 1) \leqslant l_n \leqslant m . \tag{11}$$

The first part of the estimate follows from (9), for $l'_n \geqslant 0$. According to the first inequality (10), $l_n \leqslant l_{m-1} = l'_{m-1} + 2(m-1) - m + 1 = l'_{m-1} + m - 1 \leqslant m$, since l'_{m-1} is the number of linearly independent solutions of the boundary value problem with the boundary condition of the form (8) in the class $\mathfrak{B}_0$. But this problem is equivalent to Problem $\mathring{\mathbf{A}}$ in the class $\mathfrak{B}$, and in view of Lemma 2 it has not more than one solution.

This completes the proof of the estimate (11).

THEOREM 2. *If $l_n = 2n - m + 1$ for some $n < m$, then $l_{n'} = 2n' - m + 1$ for all $n' > n$.*

PROOF. If $l_n = 2n-m+1$, then according to (9) $l'_n = 0$ whence in view of (10_2) $l'_{n'} = 0$; this implies that $l_{n'} = 2n'-m+1$.

THEOREM 3. *For any* $n \leqslant m-1$ *we have the estimate*

$$l_n \leqslant n+1 \,. \tag{12}$$

This estimate is exact.

PROOF. First of all we note that, taking into account (9), we can readily prove that the estimate (12), is equivalent to the following estimate: $l_n + l'_n \leqslant m+1$. It is more convenient to prove the last inequality. Let $\Phi_1, \ldots, \Phi_{l_n}$ and $\psi_1, \ldots, \psi_{l'_n}$, be the complete systems of solutions of the homogeneous Problems $\mathbf{A}_n$ and $\mathbf{A}'_n$, respectively.

Let us consider the products $\varphi_{jk} = \Phi_j \psi_k$, $j \leqslant l_n$, $k \leqslant l'_n$. In view of the boundary conditions (2), (8) and the definition of the classes $\mathfrak{B}_n$ and $\mathfrak{B}_{n'}$ these products are solutions of the boundary value problem

$$\operatorname{Re}\{iz^{m-1}z'\varphi_{jk}\} = 0 \tag{13}$$

in the class $\mathfrak{B}_{m-1}$, for $n+n' = m-1$. Let $\tilde{l}$ be the number of linearly independent solutions of the problem (13) in the class $\mathfrak{B}_{m-1}$. Since the increment of the argument of the function $z^{m-1}z'$ along Γ is equal to zero, by a substitution of the form (5.68) we, evidently, can reduce this problem to the equivalent problem $\mathbf{A}_{n-1}$ in the class $\mathfrak{B}_{m-1}$. According to (11) we have $\tilde{l} \leqslant m$.

For the proof of the theorem it is now sufficient to prove that among the products φ_{jk} there are at least $l_n + l'_n - 1$ linearly independent. Then we shall have $l_n + l'_n - 1 \leqslant \tilde{l} \leqslant m$ which, as it was indicated above, is equivalent to (12). In order to count the number of linearly independent functions among the products φ_{jk} we may assume with no loss of generality that in the sequence $\Phi_1, \ldots, \Phi_{l_n}$ the functions are ordered according to the increase of the order of the poles at the point $z = 0$, and if Φ_j and Φ_{j+1} have equal orders of poles, say n_j, we assume that

in the vicinity of $z=0$ $\Phi_j(z) = \dfrac{1}{z^{n_j}} + , \ldots, \Phi_{j+1}(z) = \dfrac{i}{z^{n_j}} + \ldots$. In the same order we set the functions $\psi_1, \ldots, \psi_{l'_n}$. Then it is readily observed that the products $\varphi_{11}, \varphi_{21}, \ldots, \varphi_{l_n,1}$ and $\varphi_{l_n,2}, \varphi_{l_n,3}, \ldots, \varphi_{l_n,l'_n}$ are linearly independent. But there are exactly $l_n + l'_n - 1$ of them, whence $l_n + l'_n \leqslant \hat{e} + 1$. This completes the proof.

Examples illustrating the accuracy of the estimate (12) will be given below (see also §5.8, Ch.IV).

3. In consequence of (11), for $n < 0$ and $n > m-1$ we have

$$l_n = l_n(\alpha) = \max(0, 2n - m + 1)\,. \tag{14}$$

We shall denote by R_n the set of the points $\alpha = (\alpha_1, \ldots, \alpha_m)$ of the torus T_m, such that the formula (14) is valid for them. Let CR_n be the supplement of R_n to T_m. Then Theorem 4.10 can be stated in the following way, making use of the concepts of the sets R_n: *in the non-special cases of the Riemann–Hilbert problem* ($n < 0, n > m-1$) *CR_n is an empty set. Conversely, as we shall now see, in the special cases CR_n is never empty. This means that for $0 \leqslant n \leqslant m-1$ there exists for any domain Problem* Å *(or* $\mathbf{A}_n$*) whose number of linearly independent solutions is not given by the formula* (14) (it is greater).

We state the properties of the set R_n which follow immediately from the facts given in subsections 1 and 2:

(1) $CR_0 \subseteq CR_1 \subseteq \ldots \subseteq CR_n \subseteq CR_{\left[\frac{m}{2}\right]}$ where $n < \left[\dfrac{m}{2}\right]$,

(2) $CR_{\left[\frac{m}{2}\right]+1} \supseteq CR_n \supseteq \ldots \supseteq CR_{m-2} \supseteq CR_{m-1}$,

$$m-1 \geqslant n \geqslant \left[\frac{m}{2}\right].$$

The property (1) is identical with the first part of the inequality (10); the property (2) follows from Theorem 2.

Thus, the most massive are the sets $CR_{\left[\frac{m}{2}\right]}$ and $CR_{\left[\frac{m}{2}\right]+1}$.

4. Lemma 3, corollary of Lemma 2 and Theorem 1 enable us to describe completely the structure of the sets CR_0 and CR_1. Namely,

(a) CR_0 consists of the one point $\alpha = (0, \dots, 0) = 0$.

(b) CR_1 is a continuous image of the domain G completed by the point $a = 0$, i.e. CR_1, in general, is a variety of the dimension $\leqslant 2$ on the torus T_m. The statement (a) is obvious.

According to Theorem 1 the point α belongs to $CR_1 - CR_0$ if and only if the domain G can be mapped conformally, preserving the condition $\Phi(0) = \infty$, onto a domain (which is not necessarily canonical) with cuts along the straight lines $\mathrm{Re}(e^{-2\pi i\alpha_k}\Phi) = 0$ $(k = 0, 1, \dots, m)$, and $\alpha_0 = 0$. But according to well-known theorems for any $a \in G$, $a \neq 0$, there exsists a mapping of the domain G onto a domain with cuts, satisfying the following conditions: (1) $\Phi(0) = \infty$, (2) $\Phi(a) = 0$, (3) the contour Γ_0 is transformed onto a section of the real axis (i.e. $\alpha_0 = 0$). Thus, for an arbitrary point $a \neq 0$ belonging to the domain G there exists $\alpha \in CR_1$; evidently, all $\alpha \in CR_1$ are thus employed.

Similarly, if $n > 1$ it follows from Lemma 2 that any solution of Problem $\mathbf{A}_n$ having a pole at the point $z = 0$ of exactly the order n, establishes a conformal mapping of the domain G onto a Riemannian surface n times covering the domain of the plane Φ with radial cuts I_k of a finite length, situated on the straight lines $\mathrm{Re}(e^{-\pi i\alpha_k}\Phi) = 0$ $(k = 0, \dots, m)$, respectively. The edge of this surface is projected onto the cuts I_k. The surface of the above described type will be called the surace of the class F_n^α, $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_m)$. The above facts imply that a necessary and sufficient condition of solubility of Problem $\mathbf{A}_n$ can be formulated as follows:

THEOREM 4. *Problem* $\mathbf{A}_n$ *has a solution having a pole of order* n *at the point* $z = 0$ *if and only if the domain* G *can be mapped conformally onto a surface of the class* F_n^α,

in such a way that the point $z = 0$ is transformed into the point $z = \infty$.

Such a mapping is in general fully determined by the specification of n points of the domain G, the projection of which is zero. It is, therefore, geometrically evident that for $n < \frac{m}{2}$ CR_n is a set of points of the torus T_m of dimension not greater than $2n$. In this respect we note that for $n \geqslant \frac{m}{2}$ we always obtain, according to the estimate (11), $l_n > 1$, and there necessarily exists a solution having a pole of the order $1 \leqslant n' \leqslant n$ at the point $z = 0$. Thus, the first part of the inequality (11) has the following interpretation in the theory of conformal mappings.

Let l_k $(k = 0, 1, \dots, m)$ be an arbitrary prescribed set of straight lines passing through the origin of coordinates and let n be a given number, such that $n \geqslant \frac{m}{2}$. Among the lines l_k there can be identical lines. Then every domain G of connectedness $(m+1)$ can be mapped onto a Riemannian surface F, n' times covering a plane unbounded domain, where $0 < n' \leqslant n$. The edge of the surface F is projected onto some sections of a finite length I_k, $k = 0, 1, \dots, m$, lying on the straight lines l_k, respectively. A more precise statement can obviously be formulated if $n \geqslant m$. In particular, for $m = 2$ this result implies the familiar theorem asserting that an arbitrary domain of connectedness three can be mapped conformally and univalently onto a domain with cuts along three prescribed straight lines passing through the origin of coordinates.

The structure of the sets R_n and $R_{n'} (n' = m-n-1)$ is the same. Exactly these sets are connected by the transformation of the form

$$a' = a - a^0\,, \qquad a \in R_n\,, \qquad a' \in R_{n'}\,, \tag{15}$$

where α^0 is a point of the torus depending on the domain G.

It is sufficient to examine the case $n > \frac{m}{2}$. According to (9), in order to find the number l_n it is sufficient to find the number l'_n. Let us write down Problem $\mathbf{A}_n$ equivalent to Problem $\mathbf{A}'_n$. In view of the formulae of §3, Ch. IV we have for the functions $\psi_*(z) = e^{-iq(z)} \prod_{k=1}^{m} (z - z_k)\psi(z)$ where $\psi(z)$ is the solution of Problem $\mathbf{A}'_n$, the boundary condition

$$\mathrm{Re}\,\{e^{2\pi i(\alpha_k + \alpha_k^0)} \psi_*(z)\} = 0 \tag{16}$$

in the class $\mathfrak{R}_{n^*}$ where α^0 is given by the formula (5.68). Therefore, if $n \geqslant \frac{m}{2}$ and $\alpha \in R_n$, then $l_n = 2n - m + 1$, i.e. $l'_n = 0$. But the problem (16) is entirely equivalent to Problem $\mathbf{A}'_n$ under consideration. Hence, for this problem $l'_n = 0$ and according to the definition we have $\alpha - \alpha^0 \in R_{n'}$.

Thus, we have shown that if the point $\alpha \in R_n$, the point α' related to α by means of the formula (15) belongs to $R_{n'}$. In the same way we can prove the inverse statement. This completes the proof of (15).

In accordance with the formula (15) the assertions (a) and (b) of the subsection 4 are also valid for the sets CR_{m-1} and CR_{m-2}.

5. We now proceed to the proof of the important property of the sets CR_n, formulated in Theorem 5. First of all, let us, in accordance with the reasoning of §5, transform the necessary and sufficient condition of solubility of Problem $\mathbf{A}_n$ to a more convenient form. We shall solve Problem $\mathbf{A}_n$ in the class $\mathfrak{B}_n(\alpha)$.

We shall denote by $\psi_j(z; a, \alpha) = \omega_j(z)$, $j = 1, \dots, p$, the complete system of linearly independent solutions of the problem

$$\mathrm{Re}\,\{z' e^{\pi i \alpha_k} w_j\} = 0 \qquad (\text{on } \Gamma_k) \tag{17}$$

in the class ω_0.

Let $\Phi(z)$ be the solution of Problem $\mathbf{A}_n$ in $\mathfrak{B}_n(a)$. Then, representing $\Phi(z)$ in the form

$$\Phi(z) = \sum_{k=n}^{1} \frac{\gamma_k}{(z-a)^k} + \Phi_1(z) \equiv \Phi_0 + \Phi_1 ,$$

we obtain for Φ_1 the non-homogeneous problem

$$\operatorname{Re}\{e^{-\pi i a_k}\Phi_1\} = -\operatorname{Re}\{e^{-\pi i a_k}\Phi_0\} \equiv \gamma . \tag{18}$$

A necessary and sufficient condition of solubility of the problem (18) in the class $\mathfrak{B}_0$ is given by the relations

$$\int_\Gamma \gamma(t)\, w_j(t)\, e^{-\pi i a(t)}\, dt = 0 , \quad j = 1, 2, \dots, p ,$$

or, in a different form,

$$\begin{aligned} 0 &= 2 \sum_{k=0}^{m} \int_{\Gamma_k} \operatorname{Re}[\Phi_0(t)\, e^{-\pi i a(t)}]\, w_j(t)\, e^{-\pi i a(t)}\, dt \\ &= \sum_{s=1}^{n} \left[\gamma_s \int_\Gamma \frac{w_j(t)\, dt}{(t-a)^s} + \bar{\gamma}_s \sum_{k=0}^{m} \int_{\Gamma_k} \frac{w_j(t)\, e^{2\pi i a_k}}{(\bar{t}-\bar{a})^s} dt\right] \\ &= \sum_{s=1}^{n} \{\gamma'_s w_j^{(s-1)}(a) + \bar{\gamma}'_s \overline{w_j^{(s-1)}(a)}\} , \end{aligned}$$

i.e.

$$\operatorname{Re}\Big\{\sum_{s-1}^{n} \gamma'_s w_j^{(s-1)}(a)\Big\} = 0 , \quad j = 1, 2, \dots, p , \tag{19}$$

where

$$\gamma'_s = \frac{\gamma_s \cdot 2\pi i}{(s-1)!} ,$$

since

$$w_j(t)\, e^{\pi i a_k} t' = -\overline{w_j(t)}\, e^{-\pi i a_k} \bar{t}' .$$

The conditions (19) represent necessary and sufficient conditions of solubility of the problem (18) for fixed γ_s, $s = 1, \dots, n$, i.e. they express the conditions of ex-

istence of the solution of the problem (18) in $\mathfrak{B}_n(a)$, if the principal part of the expansion of the required solution in the vicinity of the pole is given. We are now, however, interested in a different problem, namely: in how many ways we can choose the vector $\{\gamma_s\}$, $s = 1, \ldots, n$, so that the problem is soluble. In other words, how many linearly independent solutions (over the field of real numbers) has the system of linear equations (19).

From (19) we obtain for the real and imaginary parts of the quantities $\gamma_s = \gamma_s' + i\gamma_s''$ the system of real equations with the matrix equivalent to the following:

$$\mathfrak{D}_n = \left\{ \begin{matrix} w_1(a) \ldots w_1^{(n-1)}(a), \overline{w_1(a)} \ldots \overline{w_1^{(n-1)}(a)} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ w_p(a) \ldots w_p^{(n-1)}(a), \overline{w_p(a)} \ldots \overline{w_p^{(n-1)}(a)} \end{matrix} \right\}$$

or

$$\mathfrak{D}_n = \{\Delta_n, \overline{\Delta}_n\},$$

where Δ_n is the matrix $\{w_j^{(s)}(a)\}$, $j = 1, \ldots, p$, $s = 0, \ldots, n-1$. The matrices Δ_n are in fact functions of the point $a \in G$ and the point $\alpha = (\alpha_1, \ldots, \alpha_m) \in T_m$, and $\Delta_n = \Delta_n(a; \alpha)$ is a function holomorphic in the point $a \in G$, continuous in $G + \Gamma$ and analytic in the real domain of the variables $\alpha_1, \ldots, \alpha_m$. The first assertion needs no explanations. The second follows from the obvious fact that the functions $w_j(z, a, \alpha)$ are solutions of an integral equation with coefficients analytic in the variables $\alpha_1, \ldots, \alpha_m$ (and also in all remaining variables). If $\alpha_1 = \alpha_2 = \ldots = \alpha_m = 0$ the functions $w_j(z)$ are connected by the formulae $\dfrac{\partial u_j}{\partial z} = w_j$ with harmonic measures u_j (see §5). In this respect we indicate an interesting property of some minors of the matrix Δ_n. For $n \leqslant m-1$ let us consider the minor $\tilde{\Delta}_n = \{w_j^{(s)}\}$, $s = 0, 1, \ldots, n-1$, $j = 1, 2, \ldots, p$. Such a minor always exists since it is readily observed that $p = m-1$ or $p = m$. It is also verified by making use of the boundary condition (17) and simple

properties of determinants that this minor is the solution of the following Riemann–Hilbert problem:

$$N = \frac{n(n-1)}{2} \text{ . Missing formula} \tag{20}$$

The last result implies in view of Lemma 2 for instance the following relationship for the numbers $N_G(\widetilde{\Delta}_n)$ and $N_\Gamma(\widetilde{\Delta}_n)$, i.e. the numbers of zeros of the determinant $\widetilde{\Delta}_n$ inside G and on Γ, respectively:

$$2N_G(\widetilde{\Delta}_n) + N_\Gamma(\widetilde{\Delta}_n) = n(n+1)(m-1)\,.$$

We note that under our assumptions about the contour Γ the functions $w_j(z)$ are analytically continuable through Γ. Hence the differentiation $(n-1)$ times of the relation (17) along Γ, necessary in the verification of (20), is fully legitimate.

Let r be the rank of the matrix $\mathfrak{D}_n$ and assume that $a \neq 0$. Then the number of linearly independent solutions of the system (19)—and therefore the number of linearly independent solutions of Problem $\mathbf{A}_n$ in the class $\mathfrak{R}_n(a)$, is equal to $l_n = 2n - r$. If $n < \frac{m}{2}$, then $r \leqslant p$. Evidently, for $a = 0$ the set R_n coincides with the set of points α of the torus T_m for which the rank r is $2n$. Let us denote by $\varphi_n(a; \alpha_1, \ldots, \alpha_n) = \varphi_n(a; \alpha)$ the sum of the squares of moduli of all minors of rank $2n$ of the matrix $\mathfrak{D}_n$. $\varphi_n(a; \alpha)$ is a function analytic with respect to the point $\alpha \in T_m$ in the real domain. Obviously the set CR_n is the set of zeros of the functions $\varphi_n(0; \alpha)$. We shall now prove that $\varphi_n(a; \alpha)$ does not vanish identically on T_m for any $a \in G$. To this end it is sufficient to indicate at least one Problem $\mathbf{A}_n$ (or $\mathring{\mathbf{A}}$) for $n < \frac{m}{2}$, having only trivial solutions, i.e. a problem for which $l_n = 0$. Then, if necessary reducing the problem thus found to the canonical form, we find a point $\widetilde{\alpha} \in T_m$ for which $\varphi_n(a; \widetilde{\alpha}) > 0$. But in view of the theorem of

Ch. IV, §5.10 such a problem is always found in the set of problems $\mathrm{Re}[(z-b)^n f(z)] = 0$, $f \in \mathfrak{A}_0$, for variable $b \in G$. Now, every set R_n for $n \geqslant \frac{m}{2}$ is associated with a set $R_{n'}$, $n' = m-1-n < \frac{m}{2}$ in accordance with the formulae (15). Thus we have proved the following theorem.

THEOREM 5. *The set CR_n is the set of zeros of a function analytic in the real domain with respect to the point $a \in T_m$, which does not identically vanish on T_m.*

Theorem 5 implies a number of statements about the sets R_n and CR_n. The set CR_n in general consists of a number of analytic varieties and points, which may intersect, of dimension smaller than m; R_n is open and everywhere dense on T_m.

Let us sum up in a general way the above results. In the non-special case of the solubility relations of Problem $\mathbf{\mathring{A}}$ or $\mathbf{A}_n$, the numbers l_n and l'_n were completely determined by the numbers m and n representing only topological conditions of the problem. Neither the structure of the domain G nor special properties of the coefficient, nor even in the more general case the properties of the class of generalized analytic functions in which Problem $\mathbf{\mathring{A}}$ was investigated, had any influence at all on the numbers l_n and l'_n. In our terminology this fact is expressed by the statement that the sets CR_n are always empty. The situation is already different in the special case of Problem $\mathbf{A}_0$. It was observed that in this case for an arbitrary domain the sets CR_n are not empty. The numbers l_n and l'_n can vary in certain range depending on the properties of the domain G and the boundary condition. The analysis carried out above, in particular Theorem 5, prove however that this influence of the domain and the boundary condition for fixed n and m occurs only in exceptional cases. In the overwhelming majority of problems, we may say in "typical" problems, when $a \in R_n$ the numbers l_n and l'_n in the special cases are to

be determined by the same formulae as in the non-special cases. Deviations from the formula (14) are encountered very seldom, "only on varieties of a smaller dimension" in the set of all problems, which in our case is the torus T_m. Geometrically these deviations are described in Theorem 4. This qualitative reasoning is in a sense valid also in the general case of Problem **Å**. Then also we may say that in the great majority of problems, in "typical" problems, the formula (14) is still valid. Similarly, other properties of the sets R_n and Problem **A** can be carried to the general case of Problem **Å**. First of all the results of Theorem 3 are valid for all problems **Å**. We now show how the property of openness of the sets R_n can be carried over to the general case. For simplicity let us assume that $|\lambda| = 1$ on Γ. Two Problems **Å** are said to be sufficiently close to each other if the functions λ and λ' determining them satisfy the inequality $|\lambda - \lambda'| \leqslant \varepsilon$ on Γ for a sufficiently small positive number ε.

THEOREM 6. *Problem* **Å** *and all problems sufficiently close to it, have the same number of solutions if and only if the formula* (14) *holds.*

The proof follows from the continuity of passage from λ and λ' to the points a and a' (formulae 5.69), and from the openness of the set R_n.

6. In some rare cases the solutions of Problem $\mathbf{A}_n$ can be constructed in an explicit form. To this end we map the domain G onto the canonical domain Δ of the plane of the complex variable ζ with radial cuts I_k, in such a way that the point $z = 0$ is carried onto the point at infinity, and the cut I_0 corresponding to the curve Γ_0 lies on the real axis. Let $\zeta = \varrho e^{\pi i a_k}$, $0 < \varrho_k < \varrho < \varrho_k'$, $k = 0, 1, \ldots, m$, be the equation of the cut I_k; by hypothesis $d_0 = 0$.

After having passed to the variable ζ Problem **Å** is stated as follows. *It is required to find in the domain Δ a holomorphic function $\Phi(\zeta)$ with a pole of*

order $\leqslant n$ *at the point* $\zeta = \infty$ *satisfying the boundary condition*

$$\operatorname{Re}\{e^{-\pi i a_k}\Phi(\zeta)\} = 0 \quad \text{on} \quad I_k . \tag{21}$$

It is assumed that the boundary condition is satisfied for the limiting values Φ^+ and Φ^- of the function Φ on the section I_k. The problem thus stated will have in some cases continuous solutions for which $\Phi^- = \Phi^+$ on I_k, $k = 0, 1, \dots, m$. It is evident that these solutions are polynomials of degree $\leqslant n$. Hence, it is very easy to find them. It is readily verified that in order that the boundary value problem (21) has a polynomial solution of the form

$$\sum_{r=0}^{n} \varrho_r e^{2\pi i \theta_r} \zeta^r ,$$

it is necessary and sufficient that the following relations are satisfied:

$$2\theta_o - 2r d_k + 2a_k = \tfrac{1}{2}(\operatorname{mod} 1) . \tag{22}$$

If $a_k = 0$, $k = 0, 1, \dots, m$, it is evident that (22) are satisfied for $d_k = 0$, $\theta_r = \frac{1}{4}$ and the solutions are the polynomials $i, i\zeta, \dots, i\zeta^n$. There are $n+1$ such solutions. Consequently, according to Theorem 3 these polynomials are all linearly independent solutions of the problem. Obviously, solutions of this kind are identical with the solutions of §5.8. We have at the same time verified the accuracy of the estimate (12). Thus, for $d_k = 0$, i.e. for domains which can be mapped conformally onto the canonical domain with radial cuts along the real axis, we have $l_n = n+1$ for any $n < m$. Obviously, Theorem 1 implies that the relation $l_n = n+1$ for any $n < m$ can hold only for domains of the above kind. More precisely: if $l_1 = 2$ the domain G can be mapped onto the domain with cuts along the imaginary axis, all $a_k = 0$ and $l_n = n+1$ for any $n < m$. Thus, for $n = 1$ the extreme case of the estimate (12) takes place only for special domains. Nevertheless, for other values of n the estimate (12) can

occur in an arbitrary domain. For instance, if $n = m-1$ the estimate (12) occurs in an arbitrary domain for the problem $\mathrm{Re}(z'f) = 0$ on Γ (see Ch. IV, §5). If a part of the cuts I_k lie on the real axis and another part on the imaginary axis then for $a = 0$ the sequence of polynomial solutions is the sequence $i, i\zeta^2, \ldots, i\zeta^{2r}$, $2r \leqslant n$. In general it is easy to construct examples of domains and problems for which the number of polynomial solutions is equal to a beforehand prescribed number s, $0 \leqslant s \leqslant n+1$.

PART TWO

SOME APPLICATIONS TO PROBLEMS OF THE THEORY OF SURFACES AND THE MEMBRANE THEORY OF SHELLS

IN THIS part of the book we shall deal with some applications of the preceding results to problems of the infinitesimal bending of surfaces (Chapter V) and the membrane theory of shells (Chapter VI). These topics of geometry and mechanics are closely related, first of all since they depend on the same partial differential equations. This fact enables us to give a mechanical interpretation to geometric results, and conversely. Besides these important but apparently purely mathematical analogues there exist deeper bonds which come to light during an analysis of the state of deformation and stress in a shell. In what follows this fact forms a basis of a joint investigation of geometric and mechanical problems in which we are interested. In the case of surfaces and shells of positive curvature these problems lead to various problems for elliptic equations. Thus, a wide field is open for the application of the results obtained in the preceding chapters to an investigation of a vast range of problems of the theory of surfaces and the theory of elasticity. These methods make it possible to overcome considerable analytic difficulties encountered in investigating many classical problems and achieve a significant progress in many respects.

CHAPTER V

FOUNDATIONS OF THE GENERAL THEORY OF INFINITESIMAL BENDINGS OF SURFACES

THE principal aim of this chapter is an investigation of some problems of infinitesimal bending of surfaces of positive curvature. Nevertheless, before starting the consideration of these problems we find it expedient to elucidate some principles of the general theory of infinitesimal bending, mainly for the sake of reader's convenience who although not a specialist on this topic will, perhaps, become interested in the problems dealt with here *. We give therefore a derivation of the basic differential equations and investigate some general properties of surfaces subject to infinitesimal bending. For instance, we derive expressions for the variation of various quantities connected with the surface. Since we shall also consider sectionally regular surfaces we derive the conjunction conditions on the contact lines of the adjacent regular parts. In the consideration of rigidity problems we present, first of all, a number of new proofs of the classical theorem on the rigidity of ovaloids;

* A considerable number of papers have been written on problems of infinitesimal bending of surfaces. Many chapters are devoted to this problem in the celebrated works of Darboux [31] and Bianchi [8]. A very complete exposition of the basic problems of this topic and the present state of investigations can be found in the papers of Cohn–Vossen [41a] (1936) and Yefimov [33a] (1948). Yefimov's paper had in 1957 a new edition in the German translation [33b], and was supplied by a number of appendices written by Rembs and Grothemeyer. These appendices contain a fairly complete survey q papers in recent years (1948-1957).

to this end in particular we employ the results of Ch. III. These devices permit us in many cases to weaken considerably the requirements of smoothness of both the surface and the displacement field. Moreover, this theorem is generalized to the case of sectionally regular convex closed surfaces. Further, employing mainly the results of Ch. IV, we investigate the rigidity conditions for convex surfaces with edges. A number of criteria is established for the rigidity of such surfaces; therefore, great attention is given to the geometrical and mechanical means of setting up rigid constraints. It is proved that in many cases the rigidity can be ensured by contacting the surfaces and applying bush constraints. The notion of optimum rigidity is introduced and conditions for the occurence of rigid constraints are investigated. We also consider some classes of rigid and non-rigid non-convex sectionally regular surfaces. In this connection we state some new boundary value problems for generalized analytic functions. We should note that this chapter contains a number of as yet unpublished results of the author, Bojarski, Sun Che-shen and others. Many of these results were presented on various occasions (1955–1957) at the seminar on geometry in the large in Moscow University, conducted jointly by M. V. Yefimov and the author.

Finally, we note that in this book problems of the bending of surfaces (in the large or in the small) which, as is known, lead to non-linear problems, are not dealt with at all.* It will be observed, however, that the methods employed here can also be used for the solution of many non-linear problems of the general theory of bending [24d], [64b, c]. In recent years many new results on this topic have been obtained by Pogorelov [68c].

*The reader can become acquainted with basic problems of bending of surfaces in the monograph quoted above of Yefimov [33a, b], and also in the monographs of Aleksandrov [2a, b] and Pogorelov [68a, b].

§1. Equations of infinitesimal bending in vectorial form

Let S be a sectionally regular surface of class C^m represented in vectorial form by the equation

$$\boldsymbol{r} = \boldsymbol{r}(x^1, x^2) .$$

Consider now the family of surfaces S_ε represented by equations of the form

$$\boldsymbol{r}_\varepsilon(x^1, x^2) = \boldsymbol{r}(x^1, x^2) + \varepsilon \boldsymbol{U}(x^1, x^2) ,$$

where ε is an arbitrary numerical parameter and $\boldsymbol{U}(x^1, x^2)$ is a continuously differentiable vector-function of a point of the surface.

The surfaces S_ε are said to be infinitesimal bendings of the surface S if the difference between the squares of their linear elements is a quantity of order ε^2, i.e.

$$ds_\varepsilon^2 - ds^2 = 0(\varepsilon^2) . \tag{1.1}$$

Since

$$ds^2 = d\boldsymbol{r}\, d\boldsymbol{r} , \qquad ds_\varepsilon^2 = d\boldsymbol{r}\, d\boldsymbol{r} + 2\varepsilon\, d\boldsymbol{r}\, d\boldsymbol{U} + \varepsilon^2 d\boldsymbol{U}\, d\boldsymbol{U} ,$$

for (1.1) to be true it is necessary and sufficient that

$$d\boldsymbol{r}\, d\boldsymbol{U} = 0 . \tag{1.2}$$

The last equation is called *the equation of infinitesimal bending* of the surface S, and the vector $\boldsymbol{U}$ satisfying this equation is called *the displacement vector of infinitesimal bendings*. Such a vector field on S will be briefly called *the displacement field.*

By virtue of (1.1) we have

$$ds_\varepsilon = ds\left[1 + \varepsilon^2\left(\frac{d\boldsymbol{U}}{ds}\right)^2\right]^{1/2} , \tag{1.3}$$

i.e.

$$ds - ds = O(\varepsilon^2) \geqslant 0 . \tag{1.3a}$$

Thus, we have the following

THEOREM 5.1. *As a result of an infinitesimal bending of the surface every element of it takes a non-negative increment of the second order of smallness.*

In other words, *infinitesimal bendings of a surface are not accompanied by compressions* (even to the second order of smallness).

In what follows we shall always consider only continuous deformations of a surface, i.e. we assume that the displacement vector $\boldsymbol{U}$ is a continuous function of a point of a surface. Moreover, we shall assume that this vector possesses all the derivatives appearing in our considerations. Usually we shall encounter derivatives of the first and second order, and sometimes of the third order of the vector $\boldsymbol{U}$. In consequence of the deformation the order of regularity of the surface in general diminishes. We shall return to this problem later (§10.3) and we shall establish precisely the order of regularity of the deformed surface in terms of the order of regularity of the original surface.

By a direct verification we find that the equation (1.2) always has solutions of the form

$$\boldsymbol{U} = \boldsymbol{\Omega} \times \boldsymbol{r} + \boldsymbol{C}\,, \tag{1.4}$$

where $\boldsymbol{\Omega}$ and $\boldsymbol{C}$ are arbitrary vectors. Since the formula (1.4) represents the displacement vector of a rigid (infinitesimal) displacement of the surface these displacements result in no intrinsic deformation of the surface. Therefore a vector field of the form (1.4) will be further called *the trivial bendings* or *the trivial displacement field.*

The basic problem of the theory of the infinitesimal bending of surfaces consists in the determination of non-trivial displacement fields satisfying the equation (1.2). If the surface is subject to some constraints, then in solving the equation (1.2) it is necessary also to take into account these constraints, i.e. *it is required to find non-trivial displacement fields which are compatible with the existing bonds.*

If the equation (1.2) with the existing constraints being taken into account has only a trivial solution of the form (1.4) the surface is said to be *rigid* with respect to infinitesimal bending, or simply rigid. Since the general

trivial bending is a linear combination of six linearly independent vectors, a rigid surface regarded as a solid body can have some number k of degrees of freedom, and $0 \leqslant k \leqslant 6$. If $k = 0$ we shall say that the surface is *kinematically rigid*. If $0 < k \leqslant 6$ the surface is said to be *geometrically rigid*.

Previously Liebmann (see [33a]) and in a simpler way Blaschke [10] established the rigidity of closed regular surfaces of positive curvature, the so-called ovaloids. At present the problem of rigidity is solved for many other classes of surfaces. In this chapter we shall consider also new classes of rigid surfaces with edges. Moreover, the very statement of the problem of rigidity will be formulated more precisely by introducing the concept of correct and incorrect rigidity.

§2. Equation of infinitesimal bending with respect to a Cartesian coordinate system. The first proof of the rigidity of ovaloids

2.1. Denoting by x, y, z and ξ, η, ζ the Cartesian coordinates of the vectors $\boldsymbol{r}$ and $\boldsymbol{U}$, respectively, we can put the equation (1.2) in the form

$$dx\,d\xi + dy\,d\eta + dz\,d\zeta = 0 . \tag{2.1}$$

Let us assume that the surface is uniquely projectable onto the plane. Then its equation has the form

$$z = f(x, y) . \tag{2.2}$$

In this case the relation (2.1) takes the form

$$(\xi_x + z_x\zeta_x)\,dx^2 + (\xi_y + \eta_x + z_x\zeta_y + z_y\zeta_x)\,dx\,dy + \\ + (\eta_y + z_y\zeta_y)\,dy^2 = 0 .$$

Hence, we have

$$\xi_x + z_x\zeta_x = 0 , \qquad \eta_y + z_y\zeta_y = 0 ,$$
$$\xi_y + \eta_x + z_x\zeta_y + z_y\zeta_x = 0 . \tag{2.3}$$

Differentiating these equations twice, the first with respect to y, the second with respect to x, and the third with

respect to x and y, adding the first and the second and subtracting the third we obtain

$$z_{xx}\zeta_{yy}-2z_{xy}\zeta_{xy}+z_{yy}\zeta_{xx}=0\ . \tag{2.4}$$

Thus for the *"vertical" component* ζ of the displacement vector $\boldsymbol{U}$ we have obtained a partial differential equation of the second order (2.4). In view of the formula for the Gaussian curvature of the surface

$$\mathrm{K}=\frac{z_{xx}z_{yy}-z_{xy}^2}{(1+z_x^2+z_y^2)^2}\,, \tag{2.5}$$

we find that the equation (2.4) is of elliptic, hyperbolic or parabolic type depending on the sign of the curvature, i.e. whether $\mathrm{K}>0$, <0 or $\mathrm{K}=0$, respectively.

If a solution $\zeta(x,y)$ of the equation (2.4) is found the other components $\xi(x,y)$ and $\eta(x,y)$ of the displacement vector $\boldsymbol{U}$ are determined by quadratures, solving the system (2.3). In particular these equations have the following trivial solutions:

$$\xi=C_1+\Omega_2 z-\Omega_3 y\ ,\qquad \eta=C_2+\Omega_3 x-\Omega_1 z\ ,$$
$$\zeta=C_3+\Omega_1 y-\Omega_2 x\ , \tag{2.5a}$$

where C_j and Ω_j are arbitrary real constants.

2.2. Introducing the notation

$$u=\xi+z_x\zeta\ ,\qquad v=\eta+z_y\zeta\ , \tag{2.6}$$

we can rewrite the system of equations (2.3) in the form

$$u_x-z_{xx}\zeta=0\ ,\qquad v_y-z_{yy}\zeta=0\ ,\qquad u_y+v_x-2z_{xy}\zeta=0\ .$$

Eliminating ζ from this system we obtain

$$z_{xx}(u_y+v_x)-2z_{xy}u_x=0\ ,\qquad z_{yy}(u_y+v_x)-2z_{xy}v_y=0\ . \tag{2.7}$$

Introducing the complex variable $a=x+iy$ and the function

$$w=u+iv\equiv\xi+i\eta+2z_{\bar a}\zeta\ ,\qquad z_{\bar a}\equiv\tfrac{1}{2}(z_x+iz_y)\ , \tag{2.8}$$

the system (2.7) can be put in the following form:

$$w_{\bar{a}} - \frac{1}{2}\frac{z_{\bar{a}\bar{a}}}{z_{a\bar{a}}}(w_a + \overline{w}_{\bar{a}}) = 0 \,, \tag{2.9}$$

where

$$z_{\bar{a}\bar{a}} = \tfrac{1}{4}(z_{xx} - z_{yy} + 2iz_{xy}) \,, \qquad z_{a\bar{a}} = \tfrac{1}{4}(z_{xx} + z_{yy}) \,.$$

Hence,

$$\left|\frac{z_{\bar{a}\bar{a}}}{z_{a\bar{a}}}\right|^2 = 1 - \frac{z_{xx}z_{yy} - z_{xy}^2}{\left(\dfrac{z_{xx}+z_{yy}}{2}\right)^2} \equiv 1 - \frac{\mathrm{K}(1+z_x^2+z_y^2)^2}{\left(\dfrac{z_{xx}+z_{yy}}{2}\right)^2} \,. \tag{2.10}$$

Let the surface under consideration belong to the class C^2 and assume that it has everywhere positive Gaussian curvature K. Such being the case there exists a positive constant $\mathrm{K}_0 < 1$ such that everywhere on the surface under consideration the inequality

$$\mathrm{K}(1+z_x^2+z_y^2)^2 \geqslant \mathrm{K}_0 \,, \qquad \frac{1}{4}(z_{xx}+z_{yy})^2 \leqslant \frac{1}{\mathrm{K}_0} \,. \tag{2.11}$$

holds. Then in view of (2.11) and (2.10) on the whole surface the following inequality is valid:

$$\left|\frac{z_{\bar{a}\bar{a}}}{z_{a\bar{a}}}\right|^2 \leqslant 1 - \mathrm{K}_0^2 < 1 \,. \tag{2.12}$$

Thus, on a surface of the class C^2 having everywhere positive Gaussian curvature, the equation (2.9) belongs to the class of elliptic equations of the first order, which were already considered in §17 of Ch. III. We may therefore apply to the equation (2.9) the theorems proved in Ch. III. We note that these theorems may also be applied to a somewhat more general case, namely to the class of surfaces satisfying the conditions (1) z_{xx}, z_{yy} and z_{xy} are bounded measurable functions, and (2) the principal curvature $\mathrm{K} \geqslant \widetilde{\mathrm{K}} > 0$ almost everywhere, where $\widetilde{\mathrm{K}}$ is a positive constant. We shall denote the class of surfaces

satisfying these conditions by $D_{2,\infty}$. For any such surface the condition (2.12) is satisfied almost everywhere and, consequently, the theorems proved in §17 of Ch. III may be employed in the investigation of the equation (2.9) for the surfaces of the class $D_{2,\infty}$. These remarks will be essentially used below, in proving the classical theorem on the rigidity of ovaloids. This method of proof makes it possible to weaken considerably the requirements with respect to the smoothness of both the displacement field and the surface.

2.3. Let us now denote the surface under consideration by S_* and the Cartesian coordinates of the position vector $\boldsymbol{r}_*$ and the displacement vector $\boldsymbol{U}_*$ by x_*, y_*, z_* and ξ_*, η_*, ζ_*, respectively. We now subject these variables to the following (projective) transformations:

$$x = \frac{x_*}{z_*}, \quad y = \frac{y_*}{z_*}, \quad z = \frac{1}{z_*}, \tag{2.13}$$

$$\xi = \frac{\xi_*}{z_*}, \quad \eta = \frac{\eta_*}{z_*}, \quad \zeta = \frac{x_*\xi_* + y_*\eta_* + z_*\zeta_*}{z_*}. \tag{2.14}$$

Then applying simple transformations we find that the condition $d\boldsymbol{r}_* d\boldsymbol{U}_* = dx_* d\xi_* + dy_* d\eta_* + dz_* d\zeta_* = 0$ implies the relation

$$d\boldsymbol{r}\, d\boldsymbol{U} \equiv dx\, d\xi + dy\, d\eta + dz\, d\zeta = 0\ , \tag{2.15}$$

where $\boldsymbol{r}$ and $\boldsymbol{U}$ are vectors with components x, y, z and ξ, η, ζ, respectively.

Thus, *in consequence of a projective transformation of the form* (2.13) *the surface* S_* *is carried into a new surface* S, *and the transformation* (2.14) *associates with every displacement field* ζ_*, η_*, ζ_* *on* S_* *a definite displacement field* ξ, η, ζ *on* S, *and conversely. It is also easily verified that a trivial displacement field is carried into another trivial field* ([33a], §25).

Let S_* be an ovaloid. If the origin of the Cartesian coordinate system is situated on S_* and the coordinate

plane $o_*x_*y_*$ on the plane tangent to S_*, then in consequence of the transformation (2.13) the ovaloid is carried into an infinite convex surface S' which is uniquely projectable on the plane oxy. Consequently, its equation can be written in the form (2.2).

Making use of Monge's notation we have the following transformation formulae:

$$p = \frac{p_*}{\varDelta}, \qquad q = \frac{q_*}{\varDelta}, \qquad \varDelta = x_* p_* + y_* q_* - z_*, \tag{2.16}$$

$$r = \frac{z_*}{\varDelta^3}[(y_* q_* - z_*)^2 r_* - 2y_* p_*(y_* q_* - z_*)s_* + y_*^2 p_*^2 t_*], \tag{2.17}$$

$$s = \frac{z_*}{\varDelta^3}[-x_* q_*(y_* q_* - z_*)r_* + \\ + (2x_* y_* p_* q_* - z_* \varDelta)s_* - y_* p_*(x_* p_* - z_*)t_*],$$

$$t = \frac{z_*}{\varDelta^3}[x_*^2 q_*^2 r_* - 2x_* q_*(x_* p_* - z_*)s_* + (x_* p_* - z_*)^2 t_*].$$

These formulae imply the relation

$$rt - s^2 = (r_* t_* - s_*^2)\frac{z_*^4}{\varDelta^4}. \tag{2.18}$$

The quantity $|\varDelta|$ is equal to the distance form the origin of coordinates to the point of intersection of the axis o_*z_* with the tangent plane to the ovaloid S_*. Hence, $\varDelta \neq 0$ everywhere except at the origin where $\varDelta = 0$. But in the vicinity of the origin the following relation is valid:

$$\varDelta \cong z_* = \frac{1}{2}(r_*^0 x_*^2 + 2s_*^0 x_* y_* + t_*^0 y_*^2) + o(x_*^2 + y_*^2). \tag{2.19}$$

For definitness let us assume that the origin of coordinates coincides with the umbilical point of the surface. Then the relation (2.19) takes the form

$$\varDelta \cong z_* = \frac{\mathrm{K}_0}{2} a_* \bar{a}_* + o(|a_*|^2), \qquad a_* = x_* + iy_*. \tag{2.20}$$

Thus, in the vicinity of the origin of coordinates

$$\frac{\Delta_*}{z_*} \cong 1 . \tag{2.21}$$

This means that if the ovaloid S_* has everywhere positive principal curvature K_*, then in consequence of the transformation (2.13) it is carried into a surface which has everywhere positive principal curvature including the point at infinity. In view of (2.21) the formulae (2.17) imply at once that r, s, t are bounded. The condition (2.12) therefore is satisfied also for the surface S obtained from the ovaloid S_* in consequence of the transformation (2.13). This result will be employed in the next subsection.

2.4. In this subsection we make use of the preceding results and some properties of solutions of an equation of elliptic type (2.9), to give a proof of the rigidity of ovaloids.

Superposing on the displacement field the trivial field (1.4) which has no influence on the state of deformation of the ovaloid (in the first approximation), and taking into account the relations (2.3) we can show that the components of the displacement field of the ovaloid satisfy the following conditions at the origin of coordinates:

$$\xi_* = 0 , \quad \eta_* = 0 , \quad \zeta_* = 0 ,$$

$$\frac{\partial \xi_*}{\partial x_*} = \frac{\partial \xi_*}{\partial y_*} = \frac{\partial \eta_*}{\partial x_*} = \frac{\partial \eta_*}{\partial y_*} = \frac{\partial \zeta_*}{\partial x_*} = \frac{\partial \zeta_*}{\partial y_*} = 0 . \tag{2.22}$$

We assume that ξ_*, η_*, ζ_* are continuous and belong to the class $D_{1,p}$, $p > 2$, on the whole ovaloid. We assume that the ovaloid belongs to the class $D_{2,\infty}$ (p. 398).

Under these conditions, taking into account the relations (2.22), (2.3) we have in the vicinity of the point $a_* = 0$ $(a_* = x_* + iy_*)$

$$\xi_*, \eta_* = O(|a_*|^2) , \quad \zeta_* = O(|a_*|^2) . \tag{2.23}$$

According to (2.9) and (2.14), and taking into account that the vicinity of the point $a_* = 0$ is carried into the vicinity of the point $a = \infty$ ($a = x + iy$), and

$$a_* = O\left(\frac{1}{|a|}\right) \quad \text{near} \quad a = \infty\,, \tag{2.24}$$

we have

$$\xi, \eta = O(1)\,, \quad \zeta = O\left(\frac{1}{|a|}\right) \quad (\text{near} \quad a = \infty)\,. \tag{2.25}$$

Moreover, it follows from (2.16) and (2.20) that

$$z_a = O(|a|) \quad (\text{near} \quad a = \infty)\,. \tag{2.26}$$

Under these conditions the function

$$w(a) = \xi + i\eta + 2z_a\zeta \tag{2.27}$$

is continuous on the entire plane, belongs to the class $D_{1,p}$ on every bounded part of the plane a, satisfies the equation (2.29) and also is bounded at infinity, i.e.

$$w(a) = O(1) \quad (\text{near } a = \infty)\,. \tag{2.28}$$

Writing the equations (2.9) in the form

$$w_{\bar{a}} - qw_a = 0\,, \quad q = \frac{1}{2}\frac{z_{\bar{a}\bar{a}}}{z_{a\bar{a}}}\left(1 + \frac{\overline{w}_a}{w_a}\right) \tag{2.28a}$$

and taking into account the inequality $|q| \leqslant q_0 < 1$ we may write

$$w(a) = \Phi[W(a)]\,,$$

where W is the basic homeomorphism of the equation (2.28), which exists on account of Theorem 2.15, and Φ is an entire function. But this function, according to the condition (2.28) is, evidently, equal to a constant. Consequently, $w = \text{const}$. Now making use of the formulae (2.3) we easily infer that $\xi = \text{const}$, $\eta = \text{const}$ and $\zeta \equiv 0$. Finally, taking into account the conditions (2.22) we

obtain $\xi \equiv \eta \equiv 0$. Thus, the rigidity of an ovaloid is proved.

The proof given in this article was obtained by the author and is published here for the first time.

2.5. Now, making use of the equation of second order (2.4) we shall give a proof of an important property of maximum and minimum for the displacement vector, which will be employed later in the proof of the rigidity of sectionally regular convex surfaces. This proof was first given by Bojarski and Yefimow [12*].

THEOREM 5.2. *Let S be a sectionally regular convex surface of positive curvature, which is uniquely projectable on a plane E. Let $\boldsymbol{n}$ be the normal to the plane E. Then under an arbitrary non-trivial infinitesimal bending of the surface S the quantity $\zeta = \boldsymbol{U}\cdot\boldsymbol{n}$ where $\boldsymbol{U}$ is the displacement field, attains its minimum and maximum value on the boundary of the surface S.*

PROOF. We shall give the proof for the case when the surface S has no vertices, i.e. when its ribs do not intersect. In this proof it is sufficient to assume that every section of the surface is twice differentiable, since these assumptions are sufficient for the derivation of the equation (2.4) [12*]. Taking E as the plane xy of the Cartesian coordinate system we obtain for ζ the equation (2.4). This equation implies immediately that the maximum (minimum) of ζ cannot take place at internal points of a section S_i of which the surface is composed. In fact, at these points the equation (2.4) is of elliptic type and its solution ζ has continuous derivatives. Hence, in this case the statement of the theorem follows from the familiar principle of maximum (minimum) for such equations.

Let us now assume that ζ has a maximum at a point P of a rib L along which two regular sections S^+ and S^- of the surface S are in contact. In what follows the projections of the surfaces S^+ and S^- and the rib L on E will be denoted

by the same letters supplying them with a dash "′". For definitness we assume that the surface S is convex in the direction of the plane E. Then we have

$$z^+_{xx} > 0\,, \quad z^+_{yy} > 0 \quad \text{on} \quad S'^+\,, \quad z^-_{xx} > 0\,, \quad z^-_{yy} > 0 \quad \text{on} \quad S'^-$$

and

$$z^+ = z^- \quad \text{on} \quad L'\,. \tag{*}$$

For convenience the origin of coordinates is situated at a point P', the $x-$ axis is directed along L' and the $y-$ axis inside S'^-. Let $x = x(s)$ and $y = y(s)$ be the equations of L' in the vicinity of P'. Then at the point P' we have $\dot{x} = 1\,, \dot{y} = 0\,, \ddot{x} = 0\,, \ddot{y} = k_0 \geqslant 0$ where k_0 is the curvature of L' at P'. Differentiating $(*)$ along L' we obtain at P'

$$z^+_y - z^-_y = -\lambda\,, \qquad z^+_{xx} - z^-_{xx} = \lambda k_0\,, \tag{2.29}$$

and in view of the convexity it is easily seen that

$$\lambda \geqslant 0\,.$$

Since P' is the point of the maximum lying on the boundary of the domains S'^+ and S'^-, according to a familiar result (see e.g. [66a]) the normal derivative of the function ζ at the point P' does not vanish, i.e.

$$\zeta^-_y < 0 \quad \text{and} \quad \zeta^+_y > 0 \quad \text{at the point} \quad P'\,. \tag{2.29a}$$

Differentiating twice the relation $\eta^+ = \eta^-$ on L' along L' we obtain at the point P' the equation

$$\eta^+_{xx} - \eta^-_{xx} + k_0(\eta^+_y - \eta^-_y) = 0\,. \tag{2.30}$$

Differentiating the third equation (2.3) with respect to x, the first with respect to y and subtracting the results we obtain

$$\eta^+_{xx} + z^+_y \zeta^+_{xx} + z^+_{xx} \zeta^+_y = 0 \tag{2.31}$$

on S'^+, and a similar equation on S'^-. Substituting (2.31) into (2.30) and taking into account the second equa-

tion (2.3) we have

$$z_{xx}^{+}\zeta_{y}^{+}+z_{y}^{+}\zeta_{xx}^{+}+k_0 z_{y}^{+}\zeta_{y}^{+}=z_{xx}^{-}\zeta_{y}^{-}+z_{y}^{-}\zeta_{xx}^{-}+k_0 z_{y}^{-}\zeta_{y}^{-}\,. \tag{2.32}$$

Differentiating the relation $\zeta^{+}=\zeta^{-}$ twice with respect to s along L' we obtain

$$\zeta_{xx}^{+}+k_0\zeta_{y}^{+}=\zeta_{xx}^{-}+k_0\zeta_{y}^{-}=\zeta_{ss}^{-}=\mu\,, \tag{2.33}$$

and since ζ has the maximum at the point P'

$$\mu\leqslant 0\,. \tag{2.34}$$

Taking into account (2.29) we obtain from (2.32) and (2.33)

$$z_{xx}^{+}\zeta_{xx}^{+}=z_{xx}^{-}\zeta_{xx}^{-}\,,$$

whence, in view of (2.29)

$$\lambda\zeta_{xx}^{-}=z_{xx}^{-}(\zeta_{y}^{+}-\zeta_{y}^{-})\,. \tag{2.35}$$

However, this is impossible. In fact, it follows from (2.35) and (2.29a) that $\zeta_{xx}^{+}>0$, i.e. then we have $\mu=\zeta_{xx}^{+}+$ $+k_0\zeta_{y}^{+}>0$ at the point P', which contradicts (2.34). In deriving (2.35) we have to divide throughout once by k_0. Therefore, we have to assume that $k_0>0$. But for $k_0=0$ the contradiction follows readily from the formula (2.32).

This completes the proof of the theorem.

Theorem 5.2 can be used for the proof of rigidity of surfaces. For instance it implies

THEOREM 5.3. *The surface S satisfying the conditions of Theorem 5.2 has no non-trivial sliding bending with respect to the plane E.*

By sliding bendings of the surface S with respect to the plane E we understand such infinitesimal bendings in which the edges of the surface S have no displacement orthogonal to E (§8.11). In particular, if the edge L of the surface S lies in the plane E or in a plane parallel to E, then under sliding bendings L should be deformed parallely to E, i.e. $\zeta\equiv 0$ along L. Con-

sequently, according to Theorem 5.2 $\zeta \equiv 0$ everywhere, thus completing the proof of Theorem 5.3.

It is necessary to observe that for both Theorem 5.2. and its implication the requirement of single-valued projectivity of the surface S on the plane E is essential. Examples of sliding bendings of spherical sections greater than the semi-sphere (§11.8) indicate that without this requirement Theorem 5.3 is not valid.

Theorem 5.2. for the class of regular surfaces was used by Pogorelov, [68a], for a proof of rigidity of regular ovaloids. Making use in this proof of the above indicated extension of the theorem to the case of sectionally smooth surfaces we obtain a proof of the rigidity of sectionally regular ovaloids (see also §9.3).

§3. The system of equations for the components of the displacement field in an arbitrary coordinate system on the surface. Some criteria of rigidity

The equation of infinitesimal bendings can be reduced to a system of equations of the first order (2.3) or to the equation of the second order (2.4) only when the surface is uniquely projectable on the plane. For a sufficiently small section of the surface this property always holds, but globally this is possible only for a very special class of surfaces. Hence, the equation (2.4) or the system (2.7) can be used always if a local property of infinitesimal bendings is being investigated. But in investigating global properties we cannot, in general, use these equations. Bearing in mind this fact, we shall now derive the system of equations of infinitesimal bendings in an arbitrary coordinate system on the surface.

3.1. Let the surface be referred to a coordinate system x^1, x^2. Then the vectors

$$\boldsymbol{r}_\alpha = \frac{\partial \boldsymbol{r}}{\partial x^\alpha} \quad (\alpha = 1, 2) \tag{3.1}$$

constitute *the base of the coordinate system* x^1, x^2. It is convenient to introduce also *the reciprocal base* *

$$\boldsymbol{r}^\alpha = a^{\alpha\beta}\boldsymbol{r}_\beta \qquad (\alpha = 1, 2), \tag{3.2}$$

where

$$a^{11} = \frac{a_{22}}{a}; \qquad a^{12} = a^{21} = -\frac{a_{12}}{a}; \qquad a^{22} = \frac{a_{11}}{a}, \tag{3.3}$$

and

$$a_{\beta\alpha} = a_{\alpha\beta} = \boldsymbol{r}_\alpha \boldsymbol{r}_\beta, \qquad a = a_{11}a_{22} - a_{12}^2 > 0. \tag{3.4}$$

The square of the linear element, i.e. *the first fundamental quadratic form* of the surface has the form

$$ds^2 = a_{\alpha\beta}dx^\alpha dx^\beta \qquad (a_{\alpha\beta} = a_{\beta\alpha}). \tag{3.5}$$

It is easily seen that

$$a^{\alpha\lambda}a_{\lambda\beta} = \delta^\alpha_\beta, \qquad \boldsymbol{r}^\alpha \boldsymbol{r}_\beta = \delta^\alpha_\beta. \tag{3.6}$$

The quantities $a_{\alpha\beta}$ and $a^{\alpha\beta}$ are *the covariant and contravariant components of the metric tensor* of the surface. We shall hereafter with the help of these tensors perform *the operations of raising and lowering the indices*; for instance if $C_{\alpha\beta\gamma}^{\cdot\cdot\cdot}$ and $C^{\alpha\cdot\cdot}_{\cdot\beta\gamma}$ are two tensors, they are connected by the relations

$$C^{\alpha\cdot\cdot}_{\cdot\beta\gamma} = a^{\alpha\lambda}C_{\lambda\beta\gamma}; \qquad C_{\alpha\beta\gamma} = a_{\alpha\lambda}C^{\lambda\cdot\cdot}_{\cdot\beta\gamma}.$$

Let $\boldsymbol{n}$ be the unit vector of the normal to the surface. Obviously,

$$\boldsymbol{n} = \frac{\boldsymbol{r}_1 \times \boldsymbol{r}_2}{\sqrt{a}}, \qquad a = a_{11}a_{22} - a_{12}^2 > 0. \tag{3.7}$$

The rule for the vector product will be such that the triplet of vectors $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{n}$ constitutes the right-hand system at every point of the surface. The same orientation then has the triplet $\boldsymbol{r}^1, \boldsymbol{r}^2, \boldsymbol{n}$.

If on the surface S a vector field $\boldsymbol{U}(x^1, x^2)$ is given we can at every point perform the decomposition

$$\boldsymbol{U} = u_\alpha \boldsymbol{r}^\alpha + u_0\boldsymbol{n} \quad \text{or} \quad \boldsymbol{U} = u^\alpha \boldsymbol{r}_\alpha + u_0\boldsymbol{n}, \tag{3.8}$$

* We employ the familiar Einstein summation rule, the dummy Greek indices $(\alpha, \beta, \gamma, \ldots)$ taking the values $1, 2$ and the Latin indices $1, 2, 3$.

where

$$u_\alpha = \boldsymbol{U}\boldsymbol{r}_\alpha , \qquad u^\alpha = \boldsymbol{U}\boldsymbol{r}^\alpha = a^{\alpha\beta}u_\beta , \qquad u_0 = \boldsymbol{U}\boldsymbol{n} . \tag{3.9}$$

The quantities u^α and u_α constitute *the contravariant and covariant components of the tangent displacement vector*, and u_0 is *the normal displacement.* The formulae

$$u_{(\alpha)} = \frac{u_\alpha}{\sqrt{a_{\alpha\alpha}}} \qquad \text{(no summation with respect to } \alpha) \tag{3.10}$$

yield *the physical components* of the tangent displacement vector, i.e. the projections of the vector U on the base vectors $\boldsymbol{r}_\alpha$ of the coordinate system.

Differentiating throughout the relation (3.8) with respect to x^β we obtain

$$\frac{\partial \boldsymbol{U}}{\partial x^\beta} = \frac{\partial u^\alpha}{\partial x^\beta}\boldsymbol{r}_\alpha + u^\alpha \boldsymbol{r}_{\alpha\beta} + \frac{\partial u_0}{\partial x^\beta}\boldsymbol{n} + u_0 \boldsymbol{n}_\beta . \tag{3.11}$$

Let us now write the Gauss equations

$$\boldsymbol{r}_{\alpha\beta} \equiv \frac{\partial^2 \boldsymbol{r}}{\partial x^\alpha \partial x^\beta} = \Gamma^\lambda_{\alpha\beta}\boldsymbol{r}_\lambda + b_{\alpha\beta}\boldsymbol{n} , \tag{3.12}$$

and also the formulae for the first derivatives of the normal

$$\boldsymbol{n}_\alpha \equiv \frac{\partial \boldsymbol{n}}{\partial x^\alpha} = -b_{\alpha\beta}\boldsymbol{r}^\beta = -b^\beta_\alpha \boldsymbol{r}_\beta , \tag{3.13}$$

where

$$\Gamma^\lambda_{\alpha\beta} = \boldsymbol{r}^\lambda \boldsymbol{r}_{\alpha\beta} = a^{\lambda\gamma}\boldsymbol{r}_\gamma \boldsymbol{r}_{\alpha\beta} = a^{\lambda\gamma}\Gamma_{\alpha\beta,\lambda} , \tag{3.14}$$

$$b_{\alpha\beta} = -\boldsymbol{n}_\alpha \boldsymbol{r}_\beta = \boldsymbol{n}\boldsymbol{r}_{\alpha\beta} , \qquad b^\beta_\alpha = a^{\beta\lambda}b_{\alpha\lambda} = -\boldsymbol{n}_\alpha \boldsymbol{r}^\beta \tag{3.15}$$

$$(b_{\alpha\beta} = b_{\beta\alpha}) .$$

The quantities $\Gamma_{\alpha\beta,\lambda}$ and $\Gamma^\lambda_{\alpha\beta}$ are called the Christoffel symbols of the first and the second kind; $b_{\alpha\beta}$ are the coefficients of the second fundamental quadratic form of the surface.

Substituting (3.12) and (3.13) into (3.11) we obtain

$$\frac{\partial U}{\partial x^\alpha} = \left(\nabla_\alpha u^\beta - b^\beta_\alpha u_0\right) r_\beta + \left(\nabla_\alpha u_0 + b_{\alpha\lambda} u^\lambda\right) n\,, \tag{3.16}$$

where ∇_α is the symbol of covariant differentiation. We recall that if φ is a scalar

$$\nabla_\alpha \varphi = \frac{\partial \varphi}{\partial x^\alpha} \quad (\alpha = 1, 2)\,. \tag{3.17}$$

If u_β and u^β are covariant and contravariant vectors, then

$$\nabla_\alpha u_\beta = \frac{\partial u_\beta}{\partial x^\alpha} - \Gamma^\lambda_{\alpha\beta} u_\lambda\,, \qquad \nabla_\alpha u^\beta = \frac{\partial u^\beta}{\partial x^\alpha} + \Gamma^\beta_{\alpha\lambda} u^\lambda\,. \tag{3.18}$$

In general, for a tensor of an arbitrary rank

$$\nabla_\alpha a^{\cdot\gamma\cdot\cdot}_{\beta\cdot\cdot\cdot} = \frac{\partial a^{\cdot\gamma\cdot\cdot}_{\beta\cdot\cdot\cdot}}{\partial x^\alpha} - \Gamma^\lambda_{\alpha\beta} a^{\cdot\gamma\cdot\cdot}_{\lambda\cdot\cdot\cdot} + \Gamma^\gamma_{\alpha\lambda} a^{\cdot\lambda\cdot\cdot}_{\beta\cdot\cdot\cdot} + \ldots \tag{3.19}$$

On the basis of these formulae, making use of the relations

$$\Gamma_{\alpha\beta,\lambda} = \frac{1}{2}\left(\frac{\partial a_{\lambda\alpha}}{\partial x^\beta} + \frac{\partial a_{\lambda\beta}}{\partial x^\alpha} - \frac{\partial a_{\alpha\beta}}{\partial x^\lambda}\right) \equiv r_\lambda r_{\alpha\beta}\,, \tag{3.20}$$

it is readily proved that

$$\nabla_\alpha a_{\beta\lambda} = 0\,, \qquad \nabla_\alpha a^{\beta\lambda} = 0 \qquad (\alpha, \beta, \lambda = 1, 2)\,. \tag{3.21}$$

According to these relations the formulae (3.16) can be written in the form

$$\frac{\partial U}{\partial x^\alpha} = (\nabla_\alpha u_\beta - b_{\alpha\beta} u_0) r^\beta + (\nabla_\alpha u_0 + b_{\alpha\beta} u^\beta) n\,. \tag{3.22}$$

Substituting (3.22) into the equation $dr\,dU = 0$ and taking into account that

$$dU = \frac{\partial U}{\partial x^\alpha} dx^\alpha\,, \qquad dr = r_\beta dx^\beta\,, \qquad n r_\alpha = 0\,, \tag{3.23}$$

we obtain

$$\tfrac{1}{2}(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) - b_{\alpha\beta} u_0 = 0 \qquad (\alpha, \beta = 1, 2)\,. \tag{3.24}$$

The last system of equations which is evidently the index form of the equation (1.2), will be called *the kinematic system of equations* or *the system of equations of the displacement field* of infinitesimal bending of surface.

This system contains three equations with three unknown functions u_1, u_2, u_0 of two variables x^1, x^2. The function u_0 however can easily be eliminated from the system.

In fact, putting the system (3.24) in the form

$$\tfrac{1}{2}(\nabla_\alpha u^\beta + \nabla^\beta u_\alpha) - b_\alpha^\beta u_0 = 0 ,$$

and contracting the left-hand side with respect to the indices α, β, i.e. putting $\alpha = \beta$ and summing, we obtain

$$u_0 = \frac{1}{2\mathrm{H}} \nabla_\alpha u^\alpha \equiv \frac{1}{2\mathrm{H}} \frac{1}{\sqrt{a}} \frac{\partial \sqrt{a}\, u^\alpha}{\partial x^\alpha} = \frac{1}{2\mathrm{H}} \frac{1}{\sqrt{a}} \frac{\partial \sqrt{a}\, a^{\alpha\beta} u_\beta}{\partial x^\alpha} . \tag{3.25}$$

In deriving this relation we have taken into account the familiar formulae

$$2\mathrm{H} = b_\alpha^\alpha , \quad \Gamma_{\alpha\beta}^\beta = \frac{1}{\sqrt{a}} \frac{\partial \sqrt{a}}{\partial x^\alpha} \quad (\alpha = 1, 2) , \tag{3.26}$$

where H is the mean curvature of the surface. Introducing (3.25) into (3.24) we obtain a system of two equations with two unknown functions u_1, u_2

$$b_{\alpha\beta} \nabla_\lambda u^\lambda - \mathrm{H}(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) = 0 \quad (\alpha, \beta = 1, 2) . \tag{3.27}$$

The formula (1.4) indicates that the system (3.24) has six linearly independent trivial solutions

$$\begin{aligned} u_\alpha^{(k)} &= \boldsymbol{e}_k \boldsymbol{r} \boldsymbol{r}_\alpha , \quad u_0^{(k)} = \boldsymbol{e}_k \boldsymbol{r} \boldsymbol{n} \quad (k = 1, 2, 3,\ \alpha = 1, 2) , \\ u_\alpha^{(k)} &= \boldsymbol{e}_k \boldsymbol{r}_\alpha , \quad u_0^{(k)} = \boldsymbol{e}_k \boldsymbol{n} \quad (k = 4, 5, 6,\ \alpha = 1, 2) , \end{aligned} \tag{3.28}$$

where $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are the unit vectors of a spatial Cartesian coordinate system.

3.2. In many cases it is expedient to employ as the coordinate system the lines of curvature. Then

$$a_{12} = b_{12} = 0 \tag{3.29}$$

and, consequently, the first and second fundamental quadratic forms of the surface take the form

$$\begin{aligned} \mathrm{I} &\equiv ds^2 = \mathrm{A}^2 d\xi^2 + \mathrm{B}^2 d\eta^2 , \\ \mathrm{II} &\equiv k_s ds^2 = \mathrm{A}^2 k_1 d\xi^2 + \mathrm{B}^2 k_2 d\eta^2 , \end{aligned} \tag{3.30}$$

where

$$\xi = x^1 , \quad \eta = x^2 , \quad \mathrm{A}^2 = a_{11} , \quad \mathrm{B}^2 = a_{22} , \tag{3.31}$$

$$b_{11} = \mathrm{A}^2 k_1 , \quad b_{22} = \mathrm{B}^2 k_2 , \tag{3.32}$$

and k_1 and k_2 are *the principal curvatures* of the surface. Besides, we have

$$a^{11} = \frac{1}{\mathrm{A}^2} , \quad a^{22} = \frac{1}{\mathrm{B}^2} , \quad a^{12} = a^{21} = 0 , \quad a = \mathrm{A}^2\mathrm{B}^2 , \tag{3.33}$$

$$\begin{aligned} &\Gamma^1_{11} = \frac{1}{\mathrm{A}}\frac{\partial A}{\partial \xi} , \quad \Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{\mathrm{A}}\frac{\partial \mathrm{A}}{\partial \eta} , \quad \Gamma^1_{22} = -\frac{\mathrm{B}}{\mathrm{A}^2}\frac{\partial \mathrm{B}}{\partial \xi} , \\ &\Gamma^2_{11} = -\frac{\mathrm{A}}{\mathrm{B}^2}\frac{\partial \mathrm{A}}{\partial \eta} , \quad \Gamma^1_{12} = \Gamma^2_{21} = \frac{1}{\mathrm{B}}\frac{\partial \mathrm{B}}{\partial \xi} ; \quad \Gamma^1_{22} = \frac{1}{\mathrm{B}}\frac{\partial \mathrm{B}}{\partial \eta} . \end{aligned} \tag{3.34}$$

We also write down the Codazzi and Gauss equations (for their derivation see §7.3)

$$\frac{\partial}{\partial \xi}(k_2 \mathrm{B}) = k_1 \frac{\partial \mathrm{B}}{\partial \xi} , \quad \frac{\partial}{\partial \eta}(k_1 \mathrm{A}) = k_2 \frac{\partial \mathrm{A}}{\partial \eta} , \tag{3.35}$$

$$\mathrm{K} = -\frac{1}{\mathrm{AB}}\left[\frac{\partial}{\partial \xi}\left(\frac{1}{\mathrm{A}}\frac{\partial \mathrm{B}}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{1}{\mathrm{B}}\frac{\partial \mathrm{A}}{\partial \eta}\right)\right] . \tag{3.36}$$

In this coordinate system the system of equations (3.24) takes the following form:

$$\begin{aligned} &\frac{1}{\mathrm{A}}\frac{\partial}{\partial \xi}\left(\frac{u_1}{\mathrm{A}}\right) + \frac{1}{\mathrm{AB}}\frac{\partial \mathrm{A}}{\partial \eta}\frac{u_2}{\mathrm{B}} - k_1 u_0 = 0 , \\ &\frac{1}{\mathrm{B}}\frac{\partial}{\partial \eta}\left(\frac{u_2}{\mathrm{B}}\right) + \frac{1}{\mathrm{BA}}\frac{\partial \mathrm{B}}{\partial \xi}\frac{u_1}{\mathrm{A}} - k_2 u_0 = 0 , \\ &\mathrm{A}^2\frac{\partial}{\partial \eta}\left(\frac{u_1}{\mathrm{A}^2}\right) + \mathrm{B}^2\frac{\partial}{\partial \xi}\left(\frac{u_2}{\mathrm{B}^2}\right) = 0 . \end{aligned} \tag{3.37}$$

If we introduce the so-called physical components of the displacement vector

$$u = \frac{u_1}{A}, \quad v = \frac{u_2}{B}, \tag{3.38}$$

which are the projections of the vector $\boldsymbol{U}$ on the principal directions of the surface at the point under consideration, then taking into account the Codazzi equations the system of equations (3.37) after having eliminated u_0 can be written thus:

$$-\frac{k_2^2 B}{A}\frac{\partial}{\partial\xi}\left(\frac{u}{k_2 B}\right)+\frac{k_1^2 A}{B}\frac{\partial}{\partial\eta}\left(\frac{v}{k_1 A}\right)=0, \tag{3.39}$$

$$\frac{A}{B}\frac{\partial}{\partial\eta}\left(\frac{u}{A}\right)+\frac{B}{A}\frac{\partial}{\partial\xi}\left(\frac{v}{B}\right)=0,$$

$$u_0=\frac{1}{k_1 A}\frac{\partial u}{\partial\xi}+\frac{v}{k_1 AB}\frac{\partial A}{\partial\eta}\equiv\frac{1}{k_2 B}\frac{\partial v}{\partial\eta}+\frac{u}{k_2 BA}\frac{\partial B}{\partial\xi}. \tag{3.40}$$

The last formula can also be written in the form

$$u_0=\frac{1}{2HAB}\left(\frac{\partial Bu}{\partial\xi}+\frac{\partial Av}{\partial\eta}\right). \tag{3.41}$$

3.3. Suppose that a surface has zero principal curvature, $K=0$. First, we assume that one of the principal curvatures does not vanish, for instance $k_1\equiv 0$, $k_2\neq 0$. Then according to the equations of Codazzi and Gauss we have

$$A=\alpha(\xi), \quad k_2 B=\gamma(\eta), \tag{3.42}$$

where α and γ are functions of ξ and η, respectively.

Let $\dfrac{\partial B}{\partial\xi}\equiv 0$, i.e. B is a function of the variable η only, i.e. $B=\beta(\eta)$. Then it follows immediately from the system of equations (3.39) and (3.40) that the components of the displacement vector are given by

$$u=\varphi(\eta), \quad v=-\frac{\varphi'(\eta)}{\beta(\eta)}\int\alpha(\xi)d\xi+\psi(\eta), \tag{3.43}$$

$$u_0=-\frac{1}{\gamma(\eta)}\left(\frac{\varphi'(\eta)}{\beta(\eta)}\right)'\int\alpha(\xi)d\xi+\psi'(\eta), \tag{3.44}$$

where $\varphi(\eta)$ and $\psi(\eta)$ are arbitrary functions of the variable η. If, now, $\dfrac{\partial B}{\partial \xi} \neq 0$, we have

$$u = \varphi(\eta)\,, \qquad v = \frac{\mathrm{A}}{\dfrac{\partial \mathrm{B}}{\partial \xi}}\varphi'(\eta) + \mathrm{B}\psi(\eta)\,, \tag{3.45}$$

$$u_0 = \frac{1}{k_2 \mathrm{B}}\frac{\partial v}{\partial \eta} + \frac{u}{k_2 \mathrm{BA}}\frac{\partial \mathrm{B}}{\partial \xi}\,. \tag{3.46}$$

In the case of a cylindrical surface we may put $\mathrm{A} = \mathrm{B} = 1$ and, consequently, we have the formulae

$$u = \varphi(\eta)\,, \qquad v = -\varphi'(\eta)\xi + \psi(\eta)\,, \tag{3.47}$$

$$u_0 = \frac{1}{k_2}[-\varphi'(\eta)\xi + \psi'(\eta)]\,. \tag{3.48}$$

We note that the lines $\eta = \text{const}$ are the generators of the cylinder and the lines $\xi = \text{const.}$ are the directrices perpendicular to the generators.

Let L be an arc belonging to the surface of zero principal curvature ($k_1 = 0$, $k_2 \neq 0$) such that it is nowhere tangent to the asymptotic directions of the surface. Let S_L denote the strip of the surface generated by the (asymptotic) lines $\eta = \text{const.}$ intersecting the curve L at least in one point. In particular it may occur that S_L covers the whole surface S.

Assume that the surface S is subject to constraints which do not allow an extension along the line L. Then it follows from (1.3) that

$$\frac{d\boldsymbol{U}}{ds} = 0 \quad (\text{on } L)\,, \quad \text{i.e.} \quad \boldsymbol{U} = \boldsymbol{C} \quad (\text{on } L)\,, \tag{3.49}$$

where $\boldsymbol{C}$ is a constant vector. Thus, the vector $\hat{\boldsymbol{U}} = \boldsymbol{U} - \boldsymbol{C}$ is the displacement vector taking on the arc L the zero value

$$\hat{\boldsymbol{U}} \equiv 0 \quad (\text{on } L)\,, \quad \text{i.e.} \quad \hat{u} = \hat{v} = \hat{u}_0 = 0 \quad (\text{on } L)\,.$$

Then according to the formulae (3.44) or (3.46) we find that $\hat{u} = \hat{v} = u_0 = 0$ everywhere on S_L, i.e. the displacement vector $\boldsymbol{U} = \boldsymbol{C}$ on S_L.

Thus, we have proved the following:

THEOREM 5.4. *If on a surface of zero Gaussian curvature* $(k_1 = 0, k_2 \neq 0)$ *constraints are present such that they ensure inextensibility of an arc L of the surface, then under these conditions the strip S_L is (geometrically) rigid.*

We note that constraints of the above kind can be set up by *a rigid contact of the surface with an inextensible but perfectly flexible cord.* By a rigid contact of a cord with a surface we understand a conjunction of the points of the cord with the points of the surface, such that in the process of deformation the corresponding points undergo identical displacements. Other cases of rigidity of surfaces of zero curvature are investigated in papers of Posnyak [69a, b, c].

The case of the infinitesimal bending of a section of plane $(k_1 = k_2 = 0)$ should be considered separately. In this case the equation $d\boldsymbol{r}\, d\boldsymbol{U} = 0$ implies that

$$d\boldsymbol{U} = \boldsymbol{n}\, d\Phi\,, \quad \text{i.e.} \quad \boldsymbol{U} = \boldsymbol{n}\Phi + \Omega \times \boldsymbol{r} + \boldsymbol{C}\,, \qquad (3.50)$$

where $\boldsymbol{n}$ is the normal to the plane and Φ is an arbitrary function of a point of the plane, $\boldsymbol{\Omega}$ and $\boldsymbol{C}$ are constant vectors. In view of the complete arbitrariness of the function Φ the investigation of the infinitesimal bending of a plane is a completely indefinite problem. Nevertheless it is not expedient to exclude this case completely from our considerations. For many reasons it is meaningful to consider infinitesimal bendings of a plane confining ourselves only to trivial bendings (rigid plates)

$$\boldsymbol{U} = \boldsymbol{\Omega} \times \boldsymbol{r} + \boldsymbol{C}\,. \qquad (3.51)$$

Therefore, in what follows when we speak of infinitesimal bendings of plane sections of a surface (for instance polyhedra) we have in mind diplacements of the form (3.51).

3.4. If the principal curvature of the surface is everywhere negative it is convenient to employ the asymptotic lines as the coordinate system. In this system

$$b_{11} = b_{22} = 0\,, \qquad b_{12} = \sqrt{-a\mathrm{K}} \tag{3.52}$$

and the system of equations (3.24) takes the form

$$\frac{\partial u_1}{\partial x} - \Gamma^{\lambda}_{11} u_\lambda = 0\,, \qquad \frac{\partial u_2}{\partial y} - \Gamma^{\lambda}_{22} u_\lambda = 0 \tag{3.53}$$

$$(x^1 = x\,,\ x^2 = y)\,,$$

$$u_0 = \frac{1}{2\sqrt{-a\mathrm{K}}}\left(\frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y}\right) - \frac{1}{\sqrt{-a\mathrm{K}}}\Gamma^{\lambda}_{12} u_\lambda\,. \tag{3.54}$$

Let L be an arc lying on a surface S of negative curvature and assume that it is nowhere tangent to the asymptotic direction of the surface S. Let S'_L and S''_L be parts of the surface S generated by asymptotic lines intersecting the curve L. Let S_L be the common part (the intersection) of S'_L and S''_L, $S_L = S'_L S''_L$. Let us assume that the surface is subject to constraints which exclude extensions along the line L. Then, as before (p. 412) the displacement vector $\boldsymbol{U}$ a constant value $\boldsymbol{C}$ has along L. Consequently, the components of the vector $\hat{\boldsymbol{U}} = \boldsymbol{U} - \boldsymbol{C}$ vanish on the arc L. Since, moreover, the tangent components $\hat{u}_1$ and $\hat{u}_2$ of the vector $\hat{\boldsymbol{U}}$ satisfy the system of equations (3.53), according to the uniqueness of the solution of the Cauchy problem we have $\hat{u}_1 = \hat{u}_2 = 0$ on S_L. In other words S_L is rigid.

Thus, we have proved the following:

THEOREM 5.5. *In presence of constraints excluding extensions along an arc L of a surface S of negative curvature, the part S_L of the surface is (geometrically) rigid.*

It was already indicated above that such constraints can be set up by joining an inextensible perfectly flexible cord to the surface.

3.5. If the principal curvature K is everywhere positive it is convenient to employ the isometric—conjugate coordinate system. As we found in §6, Ch. II, in this case

$$b_{11} = b_{22} = \Lambda = \sqrt{a\mathrm{K}}\,, \qquad b_{12} = 0 \tag{3.55}$$

and the system of equations (3.24) after elimination of u_0, can be written in the form

$$\begin{gathered} \frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial y} - \left(\Gamma_{11}^{\lambda} - \Gamma_{22}^{\lambda}\right) u_\lambda = 0\,, \\ \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} - 2\Gamma_{12}^{\lambda} u_\lambda = 0\,, \end{gathered} \tag{3.56}$$

$$u_0 = \frac{1}{2\sqrt{a\mathrm{K}}}\left[\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} - \left(\Gamma_{11}^{\lambda} + \Gamma_{22}^{\lambda}\right) u_\lambda\right]. \tag{3.57}$$

Introducing *the complex displacement function*

$$w = u_1 + iu_2 \equiv 2\boldsymbol{U}\boldsymbol{r}_{\bar z} \qquad \left(\boldsymbol{r}_{\bar z} = \tfrac{1}{2}(\boldsymbol{r}_x + i\boldsymbol{r}_y)\right), \tag{3.58}$$

we see that the system of equations (3.56) can be written in the form

$$\partial_{\bar z} w + Aw + B\overline{w} = 0\,, \tag{3.59}$$

where

$$\begin{aligned} A &= \frac{1}{4}\left(\Gamma_{22}^{1} - \Gamma_{11}^{1} - 2\Gamma_{12}^{2}\right) + \frac{i}{4}\left(\Gamma_{11}^{2} - \Gamma_{22}^{2} - 2\Gamma_{12}^{1}\right), \\ B &= \frac{1}{4}\left(\Gamma_{22}^{1} - \Gamma_{11}^{1} + 2\Gamma_{12}^{2}\right) - \frac{i}{4}\left(\Gamma_{11}^{2} - \Gamma_{22}^{2} + 2\Gamma_{12}^{1}\right). \end{aligned} \tag{3.60}$$

Thus, we observe that the complex displacement function in the case of a surface of positive curvature is a generalized analytic function

$$w(z) \in \mathfrak{A}(A, B, G)\,.$$

We recall that (see Ch. II, §6.6)

$$\left.\begin{aligned} A &= -\partial_{\bar z} \ln \sqrt{a\sqrt{\mathrm{K}}}\,, \\ B &= -\left(\operatorname{Arch}\frac{\mathrm{H}}{\sqrt{\mathrm{K}}}\right)_{\bar z} e^{i\varphi} - \frac{\mathrm{K}_z}{4\mathrm{K}} e^{2i\varphi}\,. \end{aligned}\right\} \tag{3.61}$$

For the normal component u_0 of the displacement vector we have

$$u_0 = \frac{1}{\mathrm{K}\sqrt{a}} \operatorname{Re}\left[\frac{\partial \sqrt{\mathrm{K}}w}{\partial z}\right]. \tag{3.62}$$

This formula can easily be derived from (3.57) if we make use the relations (6.63) and (6.54) of Ch. II.

Taking for $\boldsymbol{U}$ the trivial bending (3.51) we easily find that the equation (3.59) has the following linearly independent trivial solutions:

$$X_{\bar{z}},\ Y_{\bar{z}},\ Z_{\bar{z}},\ XY_{\bar{z}}-YX_{\bar{z}},\ YZ_{\bar{z}}-ZY_{\bar{z}},\ ZX_{\bar{z}}-XZ_{\bar{z}}, \tag{3.63}$$

where X, Y, Z are the Cartesian coordinates of a point of the surface.

We shall now derive a formula which makes it possible to express the displacement vector by the complex displacement function w. We have

$$\boldsymbol{U} = u_\alpha \boldsymbol{r}^\alpha + u_0 \boldsymbol{n} \equiv \tfrac{1}{2} w(\boldsymbol{r}^1 - i\boldsymbol{r}^2) + \tfrac{1}{2}\overline{w}(\boldsymbol{r}^1 + i\boldsymbol{r}^2) + u_0 \boldsymbol{n}. \tag{3.64}$$

Taking into account that

$$\frac{1}{2}(\boldsymbol{r}^1 - i\boldsymbol{r}^2) = -\frac{1}{\sqrt{a\mathrm{K}}}\boldsymbol{n}_z, \tag{3.64a}$$

we can, using (3.62), put the relation (3.64) in the form

$$\boldsymbol{U} = -\frac{1}{\sqrt{a\mathrm{K}}}(w\boldsymbol{n}_z + \overline{w}\boldsymbol{n}_{\bar{z}}) + \frac{1}{\mathrm{K}\sqrt{a}} \operatorname{Re}(\sqrt{\mathrm{K}}\,w)_z \boldsymbol{n}. \tag{3.65}$$

or, else,

$$\boldsymbol{U} = \operatorname{Re}\left\{-2\mathrm{K}^{-1/4}\,U(z)\,\boldsymbol{n}_z + \frac{1}{\mathrm{K}\sqrt{a}}\partial_z\left(\sqrt{a}\,\mathrm{K}^{3/4}U\right)\boldsymbol{n}\right\}, \tag{3.66}$$

where

$$U = \frac{w(z)}{\sqrt{a\sqrt{\mathrm{K}}}}. \tag{3.66a}$$

Taking into account (3.61) we readily see that the function U satisfies the equation

$$\partial_{\bar{z}}U + B\overline{U} = 0. \tag{3.67}$$

It is evident that for $B \equiv 0$, U is an analytic function in z. We shall discover in the following section that this case occurs for convex surfaces of the second order (and only for them).

THEOREM 5.6. *Let a surface S of positive curvature be subject to constraints which exclude extensions along a (internal or boundary) arc L of the surface. In this case the surface S is rigid.*

PROOF. In fact, we have already seen above (p. 412) that the displacement vector $\boldsymbol{U}$ takes a constant value $\boldsymbol{C}$ along L. Consequently, the displacement vector $\hat{\boldsymbol{U}} = \boldsymbol{U} - \boldsymbol{C}$ vanishes along L. Such being the case, the appropriate complex displacement function U vanishes on L. Since this function satisfies the equation (3.67), according to the uniqueness theorem (3.5) we have $\hat{w} \equiv 0$ everywhere. This proves the (geometrical) rigidity of the surface S.

We recall once more that constraints of this kind are set up by joining the surface to an arbitrarily small, inextensible and perfectly flexible cord.

With the help of the equation (3.67) one more proof of the rigidity of an ovaloid can be given.

Let the ovaloid belong to the class $D_{3,p}$, $p > 2$. Then, as was established in Ch. III (§6.9) $B \in L_{p,2}(E)$, $p > 2$. Moreover, near the point $z = \infty$ we have

$$\sqrt{a} = O\left(|z|^{-4}\right), \qquad r_{\bar{z}} = O\left(|z|^{-2}\right). \tag{3.68}$$

Assume that the displacement field $\boldsymbol{U}$ belongs to the class $D_{1,p}$, $p > 2$. Then in view of (3.68) the function U is continuous and satisfies the equation (3.67) everywhere on the plane. Moreover, in view of (3.58), (3.66a) and (3.68) near the point $z = \infty$ we have

$$U = O(|z|^2). \tag{3.69}$$

It follows in accordance with Theorem 3.12 that U is a generalized polynomial of the second degree. But every such polynomial is a linear combination, with real coef-

ficients, of six linearly independent generalized polynomials, which obviously correspond to the six linearly independent trivial solutions (3.63). This means that all solutions of the equation (3.67), continuous on the plane and satisfying at infinity the condition (3.69), are given by the solutions to which correspond trivial displacement fields. This completes the proof of the rigidity of the ovaloid.

The formula (3.65) indicates that the tangent displacement field is given by the formula

$$\boldsymbol{U}_s = -\frac{1}{\sqrt{a\mathrm{K}}}(w\boldsymbol{n}_z + \overline{w}\boldsymbol{n}_{\bar{z}}) \equiv \frac{1}{2}(\boldsymbol{r}^1 - i\boldsymbol{r}^2)w + \frac{1}{2}(\boldsymbol{r}^1 + i\boldsymbol{r}^2)\overline{w}\ .$$

Hence, we have (see Ch. II, §6.4)

$$\begin{aligned}\boldsymbol{U}_s^2 &= \frac{1}{4}[(a^{11} - a^{22} - 2ia^{12})w^2 + (a^{11} - a^{22} + 2ia^{12})\overline{w}^2 + \\ &\quad + 2(a^{11} + a^{22})w\overline{w}] = \frac{1}{4a}[2(a_{11} + a_{22})|w|^2 - \\ &\quad - (a_{11} - a_{22} - 2ia_{12})w^2 - (a_{11} - a_{22} + 2ia_{12})\overline{w}^2] \\ &= \frac{|w|^2}{\sqrt{a\mathrm{K}}}\left(\mathrm{H} - \sqrt{\mathrm{E}}\cos 2\left(\frac{\psi}{2} - \theta\right)\right) \equiv k_s|w_0|^2\ , \qquad (3.69\mathrm{a})\end{aligned}$$

where

$$k_s = \mathrm{H} - \sqrt{\mathrm{E}}\cos 2\left(\frac{\psi}{2} - \theta\right), \qquad \theta = \arg w(z)\ ,$$
$$w_0(z) = w(z)(a\mathrm{K})^{-1/4}\ .$$

Consequently, k_s is the normal curvature of the surface in a direction which depends on the displacement field under consideration. The function w_0 is a generalized analytic function. Therefore in view of the principle of the maximum (Ch. III, §4.8) we have

$$|w_0(z)| \leqslant M_0 \max_L |w_0(t)|\ ,$$

where M_0 is a positive constant depending only on the surface under consideration. Accordingly, we obtain from (3.69a)

$$|\boldsymbol{U}_s(M)| \leqslant \sqrt{k_s(M)}\, M_0 \max_{N \in L} \frac{U_s(N)}{\sqrt{k_s(N)}}\ .$$

Since $k_1 \geqslant k_s \geqslant k_2$, denoting the maximum and minimum of k_1 and k_2 by k_1^0 and k_2^0, respectively, we have

$$|\boldsymbol{U}_s(M)| \leqslant M' \max_{N \in L} |\boldsymbol{U}_s(M)| , \qquad M' = \sqrt{\frac{k_1^0}{k_2^0}} M_0 . \quad (3.69\text{b})$$

Thus for a tangent displacement field on a surface of positive curvature the principle of the maximum is valid in the form indicated above. It should be borne in mind that the positive constant M' appearing in inequality (3.69b) depends only on the surface and is independent of the displacement field.

3.6. We shall now indicate one more simple way of deriving the equation (3.59). As we found in §6.5 of Ch. II, the position vector $\boldsymbol{r}$ satisfies the equation (for $\mathrm{K} > 0$ of course)

$$\boldsymbol{r}_{\bar{z}\bar{z}} + A\boldsymbol{r}_{\bar{z}} + B\boldsymbol{r}_z = 0 . \qquad (3.70)$$

Multiplying this equation throughout scalarly by $2\boldsymbol{U}$ we obtain

$$(2\boldsymbol{U}\boldsymbol{r}_{\bar{z}})_{\bar{z}} - 2\boldsymbol{U}_{\bar{z}}\boldsymbol{r}_{\bar{z}} + A(2\boldsymbol{U}\boldsymbol{r}_{\bar{z}}) + B(2\boldsymbol{U}\boldsymbol{r}_{\bar{z}}) = 0 . \qquad (3.71)$$

But U satisfies the equation $d\boldsymbol{r}\, d\boldsymbol{U} = 0$ which is equivalent to two following relations

$$\boldsymbol{U}_{\bar{z}}\boldsymbol{r}_{\bar{z}} = 0 , \qquad \boldsymbol{U}_z\boldsymbol{r}_{\bar{z}} + \boldsymbol{U}_{\bar{z}}\boldsymbol{r}_z = 0 .$$

Hence, taking into account the notation (3.58) we obtain from (3.71) the equation (3.59).

§4. A property of surfaces of the second order

Introducing a new complex displacement function

$$U = \frac{w}{\sqrt{a\sqrt{\mathrm{K}}}} , \qquad (4.1)$$

we can write the equation (3.59) in the form

$$\partial_{\bar{z}} U + B\overline{U} = 0 . \qquad (4.2)$$

If

$$B \equiv 0 , \qquad (4.3)$$

we have the Cauchy–Riemann equation

$$\frac{\partial U}{\partial \bar{z}} = 0 , \tag{4.4}$$

i.e. U is a function analytic in z. Consequently, for surfaces satisfying the condition (4.3) the problem is reduced to the Cauchy–Riemann equation and the complex displacement function has the form

$$w(z) = \sqrt{a\sqrt{\mathrm{K}}}\,\Phi(z) , \tag{4.5}$$

where $\Phi(z)$ is a function analytic in z.

The condition (4.3), as follows immediately from the formula (3.61), is satisfied for spherical surfaces; in this case, taking into account that (for the unit sphere)

$$a_{11} = a_{22} = \sqrt{a} = \frac{4}{(1+z\bar{z})^2} , \qquad a_{12} = 0 , \qquad \mathrm{K} = \mathrm{const} \tag{4.6}$$

the formula (4.5) takes the form

$$w(z) = (1+z\bar{z})^{-2}\Phi(z) . \tag{4.7}$$

Now naturally the question arises—for which other surfaces the relation (4.3) is valid. The answer to this question is given by the following

THEOREM 5.7. *The condition* (4.3) *is satisfied for all surfaces of the second order of positive curvature, and only for these surfaces.*

PROOF. If $B \equiv 0$ the equation (3.70) can be written in the form

$$\frac{\partial}{\partial \bar{z}}\left(\frac{1}{\sqrt{a\sqrt{\mathrm{K}}}}\boldsymbol{r}_z\right) = 0 .$$

Hence

$$X_{\bar{z}} = \sqrt{a\sqrt{\mathrm{K}}}\,f(z) , \qquad Y_{\bar{z}} = \sqrt{a\sqrt{\mathrm{K}}}\,\varphi(z) , \qquad Z_{\bar{z}} = \sqrt{a\sqrt{\mathrm{K}}}\,\psi(z) , \tag{4.8}$$

where X, Y, Z are Cartesian coordinates of a point of the surface and f, φ, ψ are arbitrary functions analytic

in z. At least one of these functions does not identically vanish, otherwise we would have $X = \text{const}$, $Y = \text{const}$, $Z = \text{const}$. Let, for instance, $f(z) \neq 0$. Then by the change of the variable

$$z_* = \int_{z_0}^{z} \frac{dz}{f(z)}, \qquad f(z_0) \neq 0 \qquad (z_0\text{—fixed point})$$

we can obtain the result $f \equiv 1$ in the vicinity of the point z_0.

In fact, on one hand

$$X_{\bar z} = X_{\bar z_*} \frac{d\bar z_*}{d\bar z} = X_{\bar z_*} \frac{1}{\overline{f(z)}} .$$

On the other hand (we make use of the formulae (6.77) of Ch. 2)

$$X_{\bar z} = \sqrt{a\sqrt{\mathrm{K}}}\, f(z) = \sqrt{a_* \sqrt{\mathrm{K}}\frac{dz_*}{dz}\frac{d\bar z_*}{d\bar z}}\, f(z) = \sqrt{a_* \sqrt{\mathrm{K}}}\, \frac{1}{\overline{f(z)}} .$$

Thus, these two relations imply that

$$X_{\bar z_*} = \sqrt{a_* \sqrt{\mathrm{K}}}\,, \qquad \text{i.e.} \qquad f_* = 1 .$$

Dropping hereafter the sign $*$ we have

$$X_y = 0\,, \qquad \text{i.e.} \qquad X = X(x)\,, \qquad \sqrt{a\sqrt{\mathrm{K}}} = \frac{1}{2} X'(x) .$$

Thus, X is a function only of x. Hence, the last two relations (4.8) can now be written in the form

$$\big(Y - X(x)\varphi(z)\big)_{\bar z} = 0\,, \qquad \big(Z - X(x)\psi(z)\big)_{\bar z} = 0 .$$

This means that

$$Y = X(x)\varphi(z) + \varphi_1(z)\,, \qquad Z = X(x)\psi(z) + \psi_1(z)\,,$$

where φ_1 and ψ_1 are functions analytic in z. Since Y and Z are real functions we have

$$Xu + u_1 = 0\,, \qquad Xv + v_1 = 0\,,$$

where u, u_1, v, v_1 are the imaginary parts of the functions $\varphi, \varphi_1, \psi, \psi_1$, respectively. It follows however from

these relations that Xu and Xv are harmonic functions of the variables x and y. Now, the conditions of harmonicity of u and v imply at once that this fact is possible only if

$$\begin{aligned} X(x) &= \frac{2}{\lambda}\tan h\lambda(x-\xi)\,, \\ u &= \alpha\cos h\lambda(x-\xi)\cos\lambda(y-\eta)\,, \\ v &= \beta\cos h\lambda(x-\xi)\cos\lambda(y-\eta')\,, \\ u_1 &= -\frac{2}{\lambda}\alpha\sin h\lambda(x-\xi)\cos\lambda(y-\eta)\,, \\ v_1 &= -\frac{2}{\lambda}\beta\sin h\lambda(x-\xi)\cos\lambda(y-\eta')\,, \end{aligned}$$

where $\alpha, \beta, \lambda, \xi, \eta$ and η' are arbitrary real constants. The harmonic functions conjugate to u, v, u_1 and v_1 are given by the formulae

$$\begin{aligned} u_* &= -\alpha\sin h\lambda(x-\xi)\sin\lambda(y-\eta)+\gamma\,, \\ v_* &= -\beta\sin h\lambda(x-\xi)\sin\lambda(y-\eta')+\gamma'\,, \\ u_{1*} &= \frac{2\alpha}{\lambda}\cos h\lambda(x-\xi)\sin\lambda(y-\eta)+\gamma_1\,, \\ v_{1*} &= \frac{2\beta}{\lambda}\cos h\lambda(x-\xi)\sin\lambda(y-\eta')+\gamma_1'\,, \end{aligned}$$

where $\gamma, \gamma', \gamma_1, \gamma_1'$ are real constants.

Thus, the equation of the surfaces for which $B \equiv 0$ can be written thus:

$$X = X(x)\,, \quad Y = X(x)u_* + u_{1*}\,, \quad Z = X(x)v_* + v_{1*}\,. \tag{4.9}$$

It is easily found that with no loss of generality we may put $\xi = \eta = 0$, $\gamma_1 = \gamma_1' = 0$ in these equations. Then the equations (4.9) can also be written in the form

$$\begin{aligned} X = X(x)\,, \qquad Y-\gamma X &= \frac{2\alpha}{\lambda}\frac{\sin\lambda y}{\cos h\lambda x}\,, \\ Z-\gamma' X &= \frac{2\beta}{\lambda}\frac{\sin\lambda y\cos\lambda\eta' - \cos\lambda y\sin\lambda\eta'}{\cos h\lambda x}\,. \end{aligned} \tag{4.10}$$

Eliminating the variables x and y from these equations we obtain

$$\left(\frac{\gamma^2}{\alpha^2}+\frac{\gamma'^2}{\beta^2}-\frac{2\gamma\gamma'}{\alpha\beta}\cos\lambda\eta'+\sin^2\lambda\eta'\right)X^2+\frac{Y^2}{\alpha^2}+\frac{Z^2}{\beta^2}+$$
$$+\left(-\frac{2\gamma}{\alpha^2}+\frac{2\gamma'\cos\lambda\eta'}{\alpha\beta}\right)XY-\frac{2\cos\lambda\eta'}{\alpha\beta}ZY+$$
$$+\left(\frac{2\gamma\cos\lambda\eta'}{\alpha\beta}-\frac{2\gamma'}{\beta^2}\right)XZ=\frac{4\sin^2\lambda\eta'}{\lambda^2}. \qquad (4.11)$$

Thus, it has been proved that *the class of surfaces for which $B\equiv 0$ contains only algebraic surfaces of the second order.* Varying the constants $\alpha,\beta,\gamma,\gamma',\lambda$ and η' entering the equation (4.11) we can obtain an arbitrary surface of the second order of positive Gaussian curvature, i.e. ellipsoids (in particular spheres), hyperboloids of two sheets and paraboloids.*

We note that these surfaces possess the following property: *the net of curvature lines is the isometric-conjugate net of lines.*

We shall prove this assertion. As is seen from equation (6.66) of Ch. II, the relation $B\equiv 0$ means that

$$\frac{\sqrt{\mathrm{E}}}{\mathrm{K}\sqrt{a}}e^{i\varphi}=f(z)\,, \qquad (4.12)$$

where f is a function analytic in z. For a spherical surface $E\equiv 0$ and the theorem is trivial. Hence, we shall assume that $E>0$ in the vicinity of a point z_0. Then $f(z)\neq 0$ and taking for the new independent variable

$$z_*=\int_{z_0}^{z}\frac{dz}{\sqrt{f(z)}}\,, \qquad f(z_0)\neq 0\,, \qquad (4.13)$$

* It should be observed that it was known earlier that surfaces of the second order with $\mathrm{K}>0$ lead to the Cauchy–Riemann system (see e. g. [18]).

we find that

$$\frac{\sqrt{\mathrm{E}}}{\mathrm{K}\sqrt{a_*}} e^{i\psi_*} = 1\,, \qquad \text{i.e.} \qquad \psi_* = 0\,. \tag{4.14}$$

Hence, according to (6.30) of Ch. II we have for this coordinate system

$$a_{12}^* = 0\,. \tag{4.15}$$

This means that the coordinate lines of the chosen isometric–conjugate coordinate system are the curvature lines of the surface. Moreover, for this coordinate system in view of (4.14) and (4.15) the first and the second fundamental quadratic forms are given by the relations

$$\mathrm{I} = \frac{\sqrt{\mathrm{E}}}{\mathrm{K}^{3/2}}(k_1 dx^2 + k_2 dy^2)\,, \qquad \mathrm{II} = \sqrt{\frac{\mathrm{E}}{\mathrm{K}}}(dx^2 + dy^2)\,. \tag{4.16}$$

It should be noted that these formulae are valid only in the vicinity of every non-umbilical point, since the change of variables in accordance with the formula (4.13) is legitimate only in the vicinity of the point for which $f(z) \neq 0$; at an umbilical point $f(z_0) = 0$ and the transformation (4.13) is meaningless.

§5. The rotation field. The characteristic equation of infinitesimal bending

5.1. The equation (1.2) implies that

$$d\boldsymbol{U} = \boldsymbol{V} \times d\boldsymbol{r}\,, \tag{5.1}$$

where $\boldsymbol{V}$ is a vector-function. This relation indicates that as a consequence of an infinitesimal bending of the surface all linear elements outcoming from the point (x^1x^2) undergo a rotation with the angular velocity $\boldsymbol{V}(x^1, x^2)$. In other words, *under an infinitesimal bending of a surface every elementary area is displaced as a rigid body.*

The vector $\boldsymbol{V}$ is called *the rotation vector*. Such a vector field on a surface will hereafter be called *the rotation field.*

We shall now prove that the rotation field is uniquely determined by the displacement field.

Representing $\boldsymbol{V}$ in the form

$$\boldsymbol{V} = v^\lambda \boldsymbol{r}_\lambda + v\boldsymbol{n} , \tag{5.2}$$

we have

$$d\boldsymbol{U} = v^\lambda \boldsymbol{r}_\lambda \times \boldsymbol{r}_\alpha dx^\alpha + v\boldsymbol{n} \times \boldsymbol{r}_\alpha dx^\alpha . \tag{5.3}$$

It can easily be shown that

$$\boldsymbol{r}_\alpha \times \boldsymbol{r}_\beta = c_{\alpha\beta}\boldsymbol{n} , \qquad \boldsymbol{n} \times \boldsymbol{r}_\alpha = c_{\alpha\beta}\boldsymbol{r}^\beta , \tag{5.4}$$

where

$$c_{11} = c_{22} = 0 , \qquad c_{12} = -c_{21} = \sqrt{a} , \tag{5.5}$$

$c_{\alpha\beta}$ is an antisymmetric covariant tensor of rank two. In view of (5.3) and (5.2), the relation (5.1) takes the form

$$(\nabla_\alpha u_\beta - b_{\alpha\beta}u_0)\, dx^\alpha \boldsymbol{r}^\beta + (\nabla_\alpha u_0 + b_{\alpha\lambda}u^\lambda)\, dx^\alpha \boldsymbol{n} = c_{\lambda\alpha}v^\lambda dx^\alpha \boldsymbol{n} + v c_{\alpha\beta} dx^\alpha \boldsymbol{r}^\beta .$$

Hence, we have

$$v c_{\alpha\beta} = \nabla_\alpha u_\beta - b_{\alpha\beta} u_0 , \tag{5.6}$$

$$v^\lambda c_{\lambda\alpha} = \nabla_\alpha u_0 + b_{\alpha\beta} u^\lambda . \tag{5.7}$$

From (5.6), taking into account (5.5), we again obtain the relations (3.24). Besides, we have

$$v = \frac{1}{2\sqrt{a}} (\nabla_1 u_2 - \nabla_2 u_1) = \frac{1}{2\sqrt{a}} \left(\frac{\partial u_2}{\partial x^1} - \frac{\partial u_1}{\partial x^2}\right) , \tag{5.8}$$

$$v^\alpha = c^{\alpha\beta}(\nabla_\beta u_0 + b_{\beta\lambda} u^\lambda) \qquad (\alpha = 1, 2) , \tag{5.9}$$

where $c^{\alpha\beta} = a^{\alpha\lambda} a^{\beta\gamma} c_{\lambda\gamma}$ is an antisymmetric contravariant tensor of rank two, and

$$c^{11} = c^{22} = 0 , \qquad c^{12} = -c^{21} = \frac{1}{\sqrt{a}} , \qquad c^{\alpha\lambda} c_{\beta\lambda} = \delta^\alpha_\beta . \tag{5.10}$$

Thus, the components of the rotation vector $\boldsymbol{V}$ are uniquely determined by the components of the displacement vector $\boldsymbol{U}$. Conversely, if a rotation field is prescribed the corresponding displacement field $\boldsymbol{U}$ can be de-

termined to within a translation from the relation (5.1) by quadratures.

Evidently, $\boldsymbol{U}$ is a trivial displacement field if and only if $\boldsymbol{V}$ is a constant vector field.

5.2. Let L be a curve lying on the surface S. Let $\boldsymbol{s}, \boldsymbol{m}, \boldsymbol{b}$ be the natural triplet of the curve L where $\boldsymbol{s}$ is the unit tangent vector, $\boldsymbol{m}$ the unit vector of the principal normal and $\boldsymbol{b}$ is the unit binormal vector. We assume that the orientation of this triplet is the same as that of the triplet $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{n}$. Hence

$$\boldsymbol{s}\times\boldsymbol{m} = \boldsymbol{b}\,, \qquad \boldsymbol{m}\times\boldsymbol{b} = \boldsymbol{s}\,, \qquad \boldsymbol{b}\times\boldsymbol{s} = \boldsymbol{m}\,. \tag{5.11}$$

In addition we assume that $\boldsymbol{m}$ is directed always towards the concavity of the curve.

Decomposing the displacement vector $\boldsymbol{U}$ with respect to the natural triplet $\boldsymbol{s}, \boldsymbol{m}, \boldsymbol{b}$ we obtain

$$\boldsymbol{U} = u_s\boldsymbol{s} + u_m\boldsymbol{m} + u_b\boldsymbol{b}\,. \tag{5.12}$$

Differentiating throughout this equation with respect to the arc s of the curve L and making use of Serret–Frenet formulae

$$\frac{d\boldsymbol{s}}{ds} = k\boldsymbol{m}\,, \qquad \frac{d\boldsymbol{m}}{ds} = -k\boldsymbol{s} + \varkappa\boldsymbol{b}\,, \qquad \frac{d\boldsymbol{b}}{ds} = -\varkappa\boldsymbol{m}\,, \tag{5.13}$$

where k is the curvature and $\varkappa$ the torsion of the curve L, we obtain

$$\frac{d\boldsymbol{U}}{ds} = \left(\frac{du_s}{ds} - ku_m\right)\boldsymbol{s} + \left(\frac{du_m}{ds} + ku_s - \varkappa u_b\right)\boldsymbol{m} + \left(\frac{du_b}{ds} + \varkappa u_m\right)\boldsymbol{b}\,. \tag{5.14}$$

On the other hand

$$\frac{d\boldsymbol{U}}{ds} = \boldsymbol{V}\times\boldsymbol{s} = (v_s\boldsymbol{s} + v_m\boldsymbol{m} + v_b\boldsymbol{b})\times\boldsymbol{s} = v_b\boldsymbol{m} - v_m\boldsymbol{b}\,, \tag{5.15}$$

where v_s, v_m, v_b are the projections of the rotation vector on the unit vectors of the natural triplet.

Comparing (5.14) and (5.15) we have

$$\frac{du_s}{ds}-ku_m=0\,, \quad \text{i.e.} \quad u_m=\frac{1}{k}\frac{du_s}{ds}\,, \tag{5.16}$$

$$v_b=\frac{du_m}{ds}+ku_s-\varkappa u_b\equiv\frac{d}{ds}\left(\frac{1}{k}\frac{du_s}{ds}\right)+ku_s-\varkappa u_b\,,$$

$$v_m=-\frac{du_b}{ds}-\varkappa u_m=-\frac{du_b}{ds}-\frac{\varkappa}{k}\frac{du_s}{ds}\,. \tag{5.17}$$

We have therefore obtained formulae representing the projections v_b and v_m of the rotation vector by the projections u_s and u_b of the displacement vector on the unit vectors of the natural triplet of the curve L.

If L is a straight line $k\equiv 0$ and it follows from (5.16) that

$$u_s=\text{const.} \quad (\text{along } L)\,. \tag{5.18}$$

Thus, we have the following

THEOREM 5.8. *If the surface contains a straight line the component of the displacement vector along this straight line is a constant quantity.*

It follows that in an infinitesimal bending of a ruled surface the projection of the displacement vector on the generator is a function depending only on the corresponding point of the directrix, i.e.

$$u_s=f(s)\,, \tag{5.19}$$

where $f(s)$ is a function of the length of arc s of the directrix of the ruled surface.

Since (see below §7.5)

$$u_m=u_0\cos\theta-u_l\sin\theta \qquad (\boldsymbol{l}=\boldsymbol{n}\times\boldsymbol{s})\,, \tag{5.20}$$

where θ is the angle between the normal of the surface and the principal normal of the curve L, in view of the formula (5.16) we have

$$u_0=\frac{1}{k_s}\frac{du_s}{ds}+\tan\theta u_l\,, \qquad k_s=k\cos\theta\,. \tag{5.21}$$

If L is a geodesic curve then $\operatorname{tg}\theta = 0$ and it follows from (5.21) that

$$k_s u_0 = \frac{du_s}{ds} \quad \text{(along } L\text{)} . \tag{5.22}$$

If now L is an asymptotic line of the surface, then $k_s = 0$ and

$$\frac{du_s}{ds} + k u_l = 0 , \quad \text{i.e.} \quad u_l = -\frac{1}{k}\frac{du_s}{ds} , \quad k \neq 0 . \tag{5.23}$$

5.3. From (5.1) we have

$$\frac{\partial \boldsymbol{U}}{\partial x^1} = \boldsymbol{V} \times \boldsymbol{r}_1 , \quad \frac{\partial \boldsymbol{U}}{\partial x^2} = \boldsymbol{V} \times \boldsymbol{r}_2 .$$

Differentiating the first of the above equations with respect to x^2, the second with respect to x^1 and subtracting the second from the first we obtain

$$\frac{\partial \boldsymbol{V}}{\partial x^2} \times r_1 - \frac{\partial \boldsymbol{V}}{\partial x^1} \times \boldsymbol{r}_2 = 0 . \tag{5.24}$$

Since according to the formula (3.16)

$$\frac{\partial \boldsymbol{V}}{\partial x^\alpha} = (\nabla_\alpha v^\lambda - b_\alpha^\lambda v)\boldsymbol{r}_\lambda + (\nabla_\alpha v + b_{\alpha\lambda} v^\lambda)\boldsymbol{n} , \tag{5.25}$$

the relation (5.24) takes the form (we make also use of the formulae (5.4))

$$(\nabla_\alpha v^\alpha - b_\alpha^\alpha v)\boldsymbol{n} - (\nabla_\alpha v + b_{\alpha\lambda} v^\lambda)\boldsymbol{r}^\alpha = 0 ,$$

i.e.

$$\nabla_\alpha v^\alpha - 2\mathrm{H} v = 0 \quad (2\mathrm{H} = b_\alpha^\alpha) , \tag{5.26}$$

$$\nabla_\alpha v + b_{\alpha\lambda} v^\lambda = 0 \quad (\alpha = 1, 2) . \tag{5.27}$$

Let us now assume that the principal curvature of the surface does not vanish, i.e.

$$\mathrm{K} = \frac{b_{11} b_{22} - b_{12}^2}{a_{11} a_{22} - a_{12}^2} \neq 0 .$$

Then, solving the system (5.26) with respect to v^1 and v^2 we obtain

$$v^{\alpha} = -d^{\alpha\beta}\frac{\partial v}{\partial x^{\beta}} \quad (\alpha = 1, 2), \tag{5.28}$$

where

$$d^{\alpha\beta} = d^{\beta\alpha} = \frac{1}{\mathrm{K}} c^{\alpha\lambda} c^{\beta\lambda} b_{\lambda\gamma}, \quad \mathrm{K} \neq 0. \tag{5.29}$$

Inserting the expressions (5.28) into (5.25) we arrive at the equation

$$\nabla_{\alpha}(d^{\alpha\beta}\Delta_{\beta}v) + 2\mathrm{H}v = 0. \tag{5.30}$$

This equation was first derived by Weingarten. It is called *the characteristic equation of the rotation field* under infinitesimal bendings of the surface, its solutions being termed *the characteristic functions.*

The equation (5.30) has three linearly independent trivial solutions

$$\lambda = \boldsymbol{e}_1\boldsymbol{n}, \quad \mu = \boldsymbol{e}_2\boldsymbol{n}, \quad \nu = \boldsymbol{e}_3\boldsymbol{n}, \tag{5.31}$$

representing the direction cosines of the normal of the surface.

If v is a solution of the equation (5.30) then

$$\boldsymbol{V} = -d^{\alpha\beta}\frac{\partial v}{\partial x^{\beta}}\boldsymbol{r}_{\alpha} + v\boldsymbol{n} \tag{5.32}$$

represents the rotation vector corresponding to an infinitesimal bending of the surface.

It will be proved in the next chapter (§4) that the equation (5.30) is also obtained in the investigation of the problem of membrane state of stress of shells.

5.4. Let the surface be referred to the curvature lines as the coordinate system. Then

$$d^{11} = \frac{1}{k_1\mathrm{A}^2}, \quad d^{22} = \frac{1}{k_2\mathrm{B}^2}, \quad d^{12} = d^{21} = 0 \tag{5.33}$$

and the equation (5.30) and the formula (5.32) take the form

$$\frac{1}{AB}\left[\frac{\partial}{\partial \xi}\left(\frac{B}{k_1 A}\frac{\partial v}{\partial \xi}\right)+\frac{\partial}{\partial \eta}\left(\frac{A}{k_2 B}\frac{\partial v}{\partial \eta}\right)\right]+2Hv=0\,, \tag{5.34}$$

$$\boldsymbol{V}=-\frac{1}{k_1 A^2}\frac{\partial v}{\partial \xi}\boldsymbol{r}_\xi-\frac{1}{k_2 B^2}\frac{\partial v}{\partial \eta}\boldsymbol{r}_\eta+v\boldsymbol{n}\,. \tag{5.35}$$

If the principal curvature of the surface is of a variable sign the characteristic equation is of mixed type. It is elliptic in these parts of the surface in which $K > 0$ and hyperbolic where $K < 0$.

On a surface of negative curvature we can consider the coordinate system to consist of the asymptotic lines. Then the characteristic equation (5.30) and the formula (5.32) take the form

$$\frac{1}{\sqrt{a}}\left[\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{-K}}\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{-K}}\frac{\partial v}{\partial x}\right)\right]+2Hv=0\,, \tag{5.36}$$

$$\boldsymbol{V}=-\frac{1}{\sqrt{-aK}}\left(\frac{\partial v}{\partial \xi}\boldsymbol{r}_\eta+\frac{\partial v}{\partial \eta}\boldsymbol{r}_\xi\right)+v\boldsymbol{n}\,. \tag{5.37}$$

We have here taken into account that

$$d^{11}=d^{22}=0\,,\qquad d^{12}=d^{21}=\frac{1}{\sqrt{-aK}}\,. \tag{5.38}$$

Introducing the new function

$$\omega=(-K)^{-\frac{1}{4}}v\,, \tag{5.39}$$

we arrive at the equation

$$\frac{\partial^2 \omega}{\partial x \partial y}+M\omega=0\,, \tag{5.40}$$

where

$$M=H\sqrt{-aK}-(-K)^{\frac{1}{4}}\frac{\partial^2}{\partial x \partial y}[(-K)^{-\frac{1}{4}}]\,. \tag{5.41}$$

On a surface of positive curvature we can introduce the isometric-conjugate coordinate system. Then

$$d^{11}=d^{22}=\frac{1}{\sqrt{aK}}\,,\qquad d^{12}=d^{21}=0 \tag{5.42}$$

and the equation (5.30) and the formula (5.32) take the form

$$\frac{1}{\sqrt{a}}\left[\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{\mathrm{K}}}\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{\mathrm{K}}}\frac{\partial v}{\partial y}\right)\right]+2\mathrm{H}v=0\,, \tag{5.43}$$

$$\boldsymbol{V}=-\frac{1}{\sqrt{a\mathrm{K}}}\left(\frac{\partial v}{\partial x}\boldsymbol{r}_x+\frac{\partial v}{\partial y}\boldsymbol{r}_y\right)+v\boldsymbol{n}\,. \tag{5.44}$$

If we introduce the new function

$$\omega=\mathrm{K}^{-\frac{1}{4}}v\,, \tag{5.45}$$

we have for this function the following equation:

$$\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}+\mathrm{M}\omega=0\,, \tag{5.46}$$

where

$$\mathrm{M}=2\sqrt{a\mathrm{K}}\mathrm{H}-\mathrm{K}^{\frac{1}{4}}\{(\mathrm{K}^{-\frac{1}{4}})_{xx}+(\mathrm{K}^{-\frac{1}{4}})_{yy}\}\,. \tag{5.47}$$

According to the formulae (5.8) the function v can be expressed by means of the complex displacement function $w=u_1+iu_2$ in the following way:

$$v=\frac{1}{2\sqrt{a}}\left(\frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial y}\right)\equiv\frac{1}{\sqrt{a}}\mathrm{Im}\left(\frac{\partial w}{\partial z}\right)\,. \tag{5.48}$$

The equation (5.46) has three trivial solutions

$$w_i=\mathrm{K}^{-\frac{1}{4}}\boldsymbol{e}_i\boldsymbol{n}\qquad(i=1,2,3)\,, \tag{5.49}$$

where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ are the unit vectors of spatial Cartesian coordinate system. To these solutions there correspond trivial displacement fields. Linear combinations of these solutions constitute all trivial solutions of the equation (5.46). In the case of an ovaloid all solutions bounded on the entire plane are linear combinations of the functions (5.49). This fact follows from the rigidity of ovaloids; otherwise the ovaloid would be non-rigid. Let us observe that near infinity the function M satisfies the condition

$$\mathrm{M}=O(|z|^{-4})\,. \tag{5.50}$$

§6. Bending fields. Static field

6.1. According to (5.27) the relation (5.25) has the form

$$\frac{\partial \boldsymbol{V}}{\partial x^\alpha} = \overset{*}{T}{}^\beta_\alpha \boldsymbol{r}_\beta , \qquad \text{where} \qquad \overset{*}{T}{}^\beta_\alpha = \boldsymbol{r}^\beta \frac{\partial \boldsymbol{V}}{\partial x^\alpha} \equiv \nabla_\alpha v^\beta - b^\beta_\alpha v . \quad (6.1)$$

The quantities $\overset{*}{T}{}^\beta_\alpha$ constitute a mixed tensor of rank two. The formulae (5.8) and (5.9) indicate that the tensor $\overset{*}{T}{}^\beta_\alpha$ is uniquely determined by the vector $\boldsymbol{U}$.

We have from (6.1)

$$\frac{\partial^2 \boldsymbol{V}}{\partial x^1 \partial x^2} - \frac{\partial^2 \boldsymbol{V}}{\partial x^2 \partial x^1} = \frac{\partial \overset{*}{T}{}^\lambda_1 \boldsymbol{r}_\lambda}{\partial x^2} - \frac{\partial \overset{*}{T}{}^\lambda_2 \boldsymbol{r}_\lambda}{\partial x^1} = 0 ,$$

or

$$c^{\alpha\beta} \overset{*}{T}{}^\lambda_\beta \boldsymbol{r}_{\lambda\alpha} + c^{\alpha\beta} \frac{\partial \overset{*}{T}{}^\lambda_\beta}{\partial x^\alpha} \boldsymbol{r}_\lambda = 0 .$$

According to (3.12) the last relation can also be written in the form

$$c^{\alpha\beta} \left(\frac{\partial \overset{*}{T}{}^\lambda_\beta}{\partial x^\alpha} + \overset{*}{T}{}^\gamma_\beta \Gamma^\lambda_{\gamma\alpha} \right) \boldsymbol{r}_\lambda + c^{\alpha\beta} \overset{*}{T}{}^\lambda_\beta b_{\lambda\alpha} \boldsymbol{n} = 0 ,$$

i.e.

$$\left.\begin{aligned} c^{\alpha\beta} \left(\frac{\partial \overset{*}{T}{}^\lambda_\beta}{\partial x^\alpha} + \overset{*}{T}{}^\gamma_\beta \Gamma^\lambda_{\gamma\alpha} \right) = 0 \qquad (\lambda = 1, 2) , \\ c^{\alpha\beta} \overset{*}{T}{}^\lambda_\beta b_{\lambda\alpha} = 0 . \end{aligned}\right\} \quad (6.2)$$

Let us introduce a new contravariant tensor

$$\hat{T}^{\alpha\beta} = c^{\alpha\lambda} \overset{*}{T}{}^\beta_\lambda , \qquad \text{i.e.} \qquad \overset{*}{T}{}^\beta_\alpha = c_{\lambda\alpha} \hat{T}^{\lambda\beta} . \quad (6.3)$$

According to (6.3) and (5.26)

$$\overset{*}{T}{}^\alpha_\alpha = \overset{*}{T}{}^1_1 + \overset{*}{T}{}^2_2 = \sqrt{a}(\hat{T}^{12} - \hat{T}^{21}) = 0, \quad \text{i.e.} \quad \hat{T}^{12} = \hat{T}^{21}. \quad (6.4)$$

Consequently, $\hat{T}^{\alpha\beta}$ is a symmetric contravariant tensor of rank two. If we take into account the relations (5.5) and (5.10) we obtain according to the formula (3.19)

$$\nabla_\lambda c^{\alpha\beta} = 0 , \qquad \nabla_\lambda c_{\alpha\beta} = 0 . \quad (6.5)$$

Hence, the equations (6.3) can be written in the form

$$\nabla_\alpha \hat{T}^{\alpha\beta} = 0 \quad (\beta = 1, 2), \qquad b_{\alpha\beta} \hat{T}^{\alpha\beta} = 0 \quad (\hat{T}^{\alpha\beta} = \hat{T}^{\beta\alpha}). \tag{6.6}$$

Thus, if on the surface a rotation field V is given, then the formulae

$$\hat{T}^{\alpha\beta} = c^{\alpha\lambda} r^\beta \frac{\partial V}{\partial x^\lambda} \tag{6.7}$$

determine a contravariant tensor of rank two $\hat{T}^{\alpha\beta}$ which satisfies the system of equations (6.6). We shall see below that this tensor completely describes the deformation of the surface under an infinitesimal bending. We shall therefore call this tensor *the contravariant bending tensor*.

We shall find in the next chapter that the system of equations (6.6) is satisfied by the contravariant components of the stress tensor of a membrane state of equilibrium of a shell, and every solution of the system is associated with a fully definite membrane state of stress. Therefore every tensor field $T^{\alpha\beta}$ satisfying the system of equations (6.6) will be called also *the static field*.

The following question now naturally arises. Is it possible to represent every static field in the form (6.7)? In other words, may a given static field be always interpreted as a (contravariant) bending field?

We shall see below that for a simply-connected surface this is always possible, and that in the case of a multiply-connected surface such an interpretation leads in general to multi-valued displacement fields.

6.2. If we now introduce the covariant tensor (covariant bending field)

$$t_{\alpha\beta} = c_{\alpha\lambda} c_{\beta\gamma} \hat{T}^{\lambda\gamma}, \qquad \text{i.e.} \qquad \hat{T}^{\alpha\beta} = c^{\alpha\lambda} c^{\beta\gamma} t_{\lambda\gamma}, \tag{6.8}$$

in view of (6.3) and (6.5) we obtain for this tensor the system of equations

$$\begin{gathered} \nabla_1 t_{22} - \nabla_2 t_{12} = 0, \qquad \nabla_2 t_{11} - \nabla_1 t_{12} = 0, \\ b_{11} t_{22} + b_{22} t_{11} - 2 b_{12} t_{12} = 0 \qquad (t_{12} = t_{21}). \end{gathered} \tag{6.9}$$

We shall examine below the geometrical meaning of this tensor field (§7.4).

It readily follows from the formulae (6.1) (6.3) and (6.8) that the following relation between the rotation field and the bending field is valid:

$$dV = c^{\beta\lambda} t_{\alpha\lambda} r_\beta dx^\alpha \equiv \hat{T}^{\alpha\beta} r_\alpha c_{\beta\lambda} dx^\lambda . \tag{6.10}$$

Thus, if the rotation field is given the corresponding bending field can be found by differentiation. If the bending field is given the corresponding rotation field and then the displacement field can be found by means of quadratures.

Let $\boldsymbol{s}$ and $\boldsymbol{l}$ be two mutually perpendicular tangent unit vectors the orientation of which is such that

$$\boldsymbol{l} \times \boldsymbol{s} = \boldsymbol{n} , \qquad \boldsymbol{s} \times \boldsymbol{n} = \boldsymbol{l} , \qquad \boldsymbol{n} \times \boldsymbol{l} = \boldsymbol{s} . \tag{6.11}$$

Obviously, two of these relations are a consequence of the third.

Introducing into these relations the expressions

$$\boldsymbol{s} = s_\alpha \boldsymbol{r}^\alpha \equiv s^\alpha \boldsymbol{r}_\alpha , \qquad \boldsymbol{l} = l_\beta \boldsymbol{r}^\beta = l^\beta \boldsymbol{r}_\beta \tag{6.12}$$

and taking into account the formulae (5.4) we obtain

$$s_\alpha = c_{\beta\alpha} l^\beta , \qquad s^\alpha = c^{\beta\alpha} l_\beta , \qquad l_\alpha = c_{\alpha\beta} s^\beta , \qquad l^\alpha = c^{\alpha\beta} s_\beta . \tag{6.13}$$

According to these formulae the relation (6.10) can be written as follows:

$$\frac{dV}{ds} = \hat{T}^{\alpha\beta} \boldsymbol{r}_\alpha l_\beta . \tag{6.14}$$

If V is a rotation field, then we obviously have the relations

$$\int_L dV = 0 , \qquad \int_L \boldsymbol{r} \times dV = 0 , \tag{6.15}$$

where L is an arbitrary piecewise smooth simple closed curve, lying on the surface. The first of the above relations is obvious and the second follows from the formula

$$\boldsymbol{r} \times dV \equiv d(\boldsymbol{r} \times V) + dU . \tag{6.16}$$

Thus, *in order that a static field (i.e. a solution of the system of equations* (6.6)) *be also a bending field it is necessary that*

$$\int_L T^{\alpha\beta}\boldsymbol{r}_\alpha l_\beta ds = 0\,, \qquad \int_L \boldsymbol{r}\times T^{\alpha\beta}\boldsymbol{r}_\alpha l_\beta ds = 0\,, \qquad (6.17)$$

where L is an arbitrary piecewise smooth closed curve belonging to the surface.

We now prove that these conditions are sufficient. In fact, it is seen from the formulae (6.14) and (6.16) that

$$\boldsymbol{V}(M) = \int_{M_0M} T^{\alpha\beta}\boldsymbol{r}_\alpha l_\beta ds + \boldsymbol{V}_0\,, \qquad (6.18)$$

$$\boldsymbol{U}(M) = \boldsymbol{V}\times\boldsymbol{r} + \int_{M_0M} \boldsymbol{r}\times d\boldsymbol{V} + \boldsymbol{U}_0\,, \qquad (6.19)$$

where M_0 is a fixed point and M is a variable point of the surface; $\boldsymbol{U}_0$ and $\boldsymbol{V}_0$ are constant vectors. The integrals are taken along an arbitrary rectifiable curve lying on the surface S and connecting the points M_0 and M. In view of the relations (6.17) these integrals are independent of the path of integration. It is seen from the last formulae that to the static field $T^{\alpha\beta}$ satisfying the system of equations (6.6) there correspond single-valued fields of displacement and rotation if and only if the relations (6.17) are valid. In this case every static field can be interpreted as an infinitesimal bending of the surface. We now prove that the condition (6.17) is always satisfied in the case of simply-connected surfaces. To this end we shall prove an integral identity.

Let $\boldsymbol{U}$ be a displacement field and $T^{\alpha\beta}$ a static field, i.e. $T^{\alpha\beta}$ satisfies the system of equations (6.6). Then we have the identity

$$\int_L \boldsymbol{U}\boldsymbol{T}_{(l)} ds \equiv \int_L T^{\alpha\beta}u_\alpha l_\beta ds = 0\,, \qquad (6.20)$$

where L is the boundary of the surface S and $\boldsymbol{T}_{(l)}$ the vector

$$\boldsymbol{T}_{(l)} = T^{\alpha\beta}l_\alpha \boldsymbol{r}_\beta\,. \qquad (6.21)$$

This vector will be called *the stress vector* (see Ch. VI, §1). We assume that L consists of a finite number of piecewise smooth simple closed contours $L_0, L_1, \ldots, L_m$. Moreover we assume that the fields $\boldsymbol{U}$ and $T^{\alpha\beta}$ are continuous in $S+L$.

Applying Green's identity and making use of (3.24) and (6.6) we have

$$\int_L UT_{(l)}ds \equiv \int_L T^{\alpha\beta}u_\beta l_\alpha ds = \iint_S \nabla_\alpha(T^{\alpha\beta}u_\beta)dS$$

$$= \iint_S u_\beta \nabla_\alpha T^{\alpha\beta}dS + \iint_S T^{\alpha\beta}\nabla_\beta u_\alpha dS$$

$$= \tfrac{1}{2}\iint_S T^{\alpha\beta}(\nabla_\beta u_\alpha + \nabla_\alpha u_\beta)ds = \tfrac{1}{2}\iint_S T^{\alpha\beta}b_{\alpha\beta}dS = 0\ .$$

We have taken into account above the symmetry of the tensor $T^{\alpha\beta}$. The integral identity (6.20) expresses the property of mutual adjointness of the systems of equations (3.27) and (6.6).

In particular if $\boldsymbol{U}$ and $\boldsymbol{V}$ are two arbitrary fields of displacement and rotation on S we have the identity

$$\int_L \boldsymbol{U}d\boldsymbol{V} = 0\ . \tag{6.22}$$

It should be borne in mind that $\boldsymbol{U}$ and $\boldsymbol{V}$ in general are not connected by the relation $d\boldsymbol{U} = \boldsymbol{V}\times d\boldsymbol{r}$. We shall prove that if the last relation is satisfied the relation (6.22) holds for an arbitrary closed piecewise smooth simple curve belonging to the surface.

In fact, if $d\boldsymbol{U} = \boldsymbol{V}\times d\boldsymbol{r}$ and L is a closed curve on S then

$$\int_L \boldsymbol{U}d\boldsymbol{V} = -\int_L \boldsymbol{V}d\boldsymbol{U} = \int_L \boldsymbol{V}(\boldsymbol{V}\times d\boldsymbol{r}) = 0\ .$$

Thus, if the vector fields $\boldsymbol{U}$ and $\boldsymbol{V}$ are continuous on S and are connected by the relation $d\boldsymbol{U} = \boldsymbol{V}\times d\boldsymbol{r}$, the following relations are valid

$$\int_{L_j} \boldsymbol{U}d\boldsymbol{V} = 0 \qquad (j = 0, 1, \ldots, m)\ . \tag{6.23}$$

It should be observed that in view of the formula (6.22) one of these relations is a consequence of the remaining ones.

If the surface is simply-connected, then taking in (6.20) for $\boldsymbol{U}$ trivial deformations (3.51) we obtain the relations (6.17). Thus, the relations (6.17) are always satisfied in the case of a simply-connected surface, for an arbitrary static field. Consequently, in this case *the formulae* (6.18) *and* (6.19) *associate with every static field* (to within a trivial bending) *single-valued fields of displacement and rotation.* In other words, in the case of a simply-connected surface any static field $T^{\alpha\beta}$ can be interpreted as a bending field. In the case of a multiply-connected surface it is not so, since the conditions (6.17) are not satisfied for every static field (Ch. VI, §3.4). Let us observe that the relations (6.17), in view of the notation (6.21) can be written thus:

$$\int_{L_j} \boldsymbol{T}_{(l)} ds = 0 , \qquad \int_{L_j} \boldsymbol{r} \times \boldsymbol{T}_{(l)} ds = 0 \qquad (6.24)$$

$$(j = 1, 2, \ldots, m) .$$

In the next chapter we shall elucidate the mechanical meaning of the formula (6.20) and the relations (6.24). Moreover, we shall elucidate the mechanical meaning of the vector $\boldsymbol{T}_{(l)}$.

6.3. With the help of the formulae (6.3), (6.1), (5.8), (5.9) it is easy to establish that the bending field corresponding to a displacement field can be determined according to the formulae

$$\hat{T}^{\alpha\beta} = -c^{\alpha\gamma} \nabla_\gamma (d^{\lambda\beta} \nabla_\lambda u_0) - c^{\alpha\gamma} b^\beta_\gamma v \qquad (\alpha, \beta = 1, 2) , \qquad (6.25)$$

where

$$u_0 = \frac{1}{2\mathrm{H}} \nabla_\alpha u^\alpha , \qquad v = \frac{1}{2} c^{\alpha\beta} \nabla_\alpha u_\beta , \qquad (6.26)$$

or, since

$$\hat{T}^{\alpha\beta} = c^{\alpha\lambda} \overset{*}{T}{}^\beta_\lambda \equiv c^{\alpha\lambda} \nabla_\lambda v^\beta - c^{\alpha\lambda} b^\beta_\lambda v , \qquad (6.27a)$$

we have in view of (5.28)

$$\hat{T}^{\alpha\beta} = -c^{\alpha\gamma}\nabla_\gamma(d^{\beta\lambda}\nabla_\lambda v) - c^{\alpha\gamma}b_\gamma^\beta v\,, \tag{6.27}$$

where v is the characteristic function.

It follows from the relation $\hat{T}^{12} = \hat{T}^{21}$ that

$$v = -\frac{1}{2\mathrm{H}}\nabla_\alpha(d^{\alpha\beta}\nabla_\beta u_0)\,. \tag{6.28}$$

Thus, the characteristic function is obtained as a consequence of twice differentiating the normal component of the displacement vector.

We observe that the formulae (6.25) or (6.27) yield an arbitrary single-valued bending field (i.e. a solution of the system of equations (6.6)) if the surface is simply-connected. In the case of multiply-connected surface these formulae cannot yield all (single-valued) solutions of the system of equations (6.6) if we remain in the class of single-valued fields of displacement and rotation. We shall return below (§6.8) to this problem.

6.4. If we refer the surface to the coordinate system consisting of the curvature lines, introducing the quantities

$$N_1 = \mathrm{A}^2T^{11}\,,\qquad N_2 = \mathrm{B}^2T^{22}\,,$$
$$H = H_1 = -H_2 = \mathrm{AB}T^{12}\,, \tag{6.29}$$

the geometrical and mechanical meaning of which will be elucidated in Chapter VI, we reduce the system of equations (6.6) to the form

$$\frac{\partial \mathrm{B}N_1}{\partial\xi} + \frac{\partial \mathrm{A}H_1}{\partial\eta} + \frac{\partial \mathrm{A}}{\partial\eta}H_1 - \frac{\partial \mathrm{B}}{\partial\xi}N_2 = 0\,, \tag{6.30}$$

$$\frac{\partial \mathrm{A}N_2}{\partial\eta} + \frac{\partial \mathrm{B}H_1}{\partial\xi} + \frac{\partial \mathrm{B}}{\partial\xi}H_1 - \frac{\partial \mathrm{A}}{\partial\eta}N_1 = 0\,, \tag{6.31}$$

$$k_1N_1 + k_2N_2 = 0\,.$$

Introducing the new notation

$$p = -\frac{N_1}{k_2} \equiv \frac{N_2}{k_1}\,,\qquad q = H_1 = -H_2\,, \tag{6.32}$$

we see that the system of equations (6.30) can also be written as follows:

$$
\begin{aligned}
&-\frac{1}{k_2 \mathrm{B}^2 \mathrm{A}} \frac{\partial}{\partial \xi}(k_2^2 \mathrm{B}^2 p)+\frac{1}{\mathrm{A}^2 \mathrm{B}} \frac{\partial \mathrm{A}^2 q}{\partial \eta}=0\,, \\
&\frac{1}{k_1 \mathrm{A}^2 \mathrm{B}} \frac{\partial}{\partial \eta}(k_1^2 \mathrm{A}^2 p)+\frac{1}{\mathrm{B}^2 \mathrm{A}} \frac{\partial \mathrm{B}^2 q}{\partial \xi}=0\,.
\end{aligned}
\tag{6.33}
$$

Below, in §7.7 we shall elucidate the geometrical meaning of the quantities p and q. Let us observe that the systems of equations (3.39) and (6.33) are mutually adjoint.

6.5. Suppose that the principal curvature is negative, i.e. $\mathrm{K} < 0$. Then, considering on the surface the coordinate system consisting of the asymptotic lines we easily find that

$$T^{12} = T^{21} = 0\,. \tag{6.34}$$

Let us introduce the notation

$$u' = \sqrt{a}T^{11}\,, \qquad v' = \sqrt{a}T^{22}\,, \tag{6.35}$$

Then the system of equations (6.6) takes the form

$$\frac{\partial u'}{\partial x}+\Gamma_{11}^1 u'+\Gamma_{22}^1 v'=0\,, \qquad \frac{\partial v'}{\partial y}+\Gamma_{22}^2 v'+\Gamma_{11}^2 u'=0\,. \tag{6.36}$$

6.6. We now consider the case of a surface of positive curvature. Referring the surface to the isometric—conjugate coordinate system and taking into account the symmetry of the tensor $T^{\alpha\beta}$ we can rewrite the system of equations (6.6) in the form

$$T^{11} = -T^{22}\,, \tag{6.37}$$

$$
\begin{aligned}
&\frac{\partial T^{11}}{\partial x}+\frac{\partial T^{12}}{\partial y}+(2\Gamma_{11}^1+\Gamma_{12}^2-\Gamma_{22}^1)\,T^{11}+ \\
&\qquad\qquad +(3\Gamma_{12}^1+\Gamma_{22}^2)\,T^{12}=0\,, \\
&\frac{\partial T^{11}}{\partial y}-\frac{\partial T^{12}}{\partial x}+(2\Gamma_{22}^2+\Gamma_{12}^1-\Gamma_{11}^2)\,T^{11}- \\
&\qquad\qquad -(3\Gamma_{12}^2+\Gamma_{11}^1)\,T^{12}=0\,.
\end{aligned}
\tag{6.38}
$$

If we introduce *the complex stress function*

$$w' = \sqrt{a}(T^{11} - iT^{12}) , \tag{6.39}$$

the system of equations (6.38) takes the complex form

$$\frac{\partial w'}{\partial \bar{z}} - Aw' - \bar{B}\bar{w}' = 0 , \tag{6.40}$$

where A and B are functions represented by the relations (3.60) or (3.61).

Thus, *the complex stress function w' is a generalized analytic function satisfying the equation adjoint to the equation* (3.59) *for the complex displacement function.*

As we found in Ch. III, §9.1 the conditions of adjointness of the equations (3.59) and (6.40) are given by the relations

$$\operatorname{Re}\left[\frac{1}{i}\int_{\Gamma'} ww'\,dz\right] = 0 \tag{6.41}$$

where w and w' are arbitrary solutions continuous in the domain G of these two equations, respectively, and Γ' is an arbitrary piecewise smooth closed simple curve belonging to the domain G. If w and w' are continuous in $G+\Gamma$ where Γ is the boundary of the domain G consisting of a finite number of piecewise smooth simple curves, then for Γ' in the relation (6.41) we can also take Γ.

The complex stress function w' which uniquely determines the static field $T^{\alpha\beta}$ will be called *the complex bending function* if the tensor $T^{\alpha\beta}$ satisfies the conditions (6.17). These conditions can be written in the complex form

$$\operatorname{Im}\int_{\Gamma_j} w'\boldsymbol{r}_{\bar{z}}\,dz = 0 , \quad \operatorname{Im}\int_{\Gamma_j} w'\boldsymbol{r}\times\boldsymbol{r}_{\bar{z}}\,dz = 0 \tag{6.42}$$

$$(j = 1, 2, \dots, m) ,$$

where $\Gamma_0, \Gamma_1, \dots, \Gamma_m$ are the boundary contours of the domain G onto which the surface S is homeomorphically

mapped. The curves $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$ are the homeomorphic images of the boundary curves $L_0, L_1, \ldots, L_m$, respectively.

If the surface is simply-connected ($m = 0$) the conditions (6.42) are always satisfied in view of (6.20). Consequently, in this case the complex stress function can always be interpreted as the complex bending function. In the case of a multiply-connected surface however this is not always possible. There exist complex stress functions which do not satisfy the conditions (6.42) (Ch. VI, §3.4).

Hereafter we denote the complex bending function by $\hat{w}'$. It is a solution of the equation (6.40) satisfying the conditions (6.42), i.e.

$$\hat{w}' = \sqrt{a}(\hat{T}^{11} - i\hat{T}^{12}) . \tag{6.43}$$

Making use of the formulae (6.37) and (6.39) we can write the formula (6.14) for the derivative of the rotation vector in the form

$$\frac{d\boldsymbol{V}}{ds} = 2\,\mathrm{Im}\left(\hat{w}'\frac{dz}{ds}\boldsymbol{r}_{\bar{z}}\right), \quad \frac{dz}{ds} = \frac{dx}{ds} + i\frac{dy}{ds} . \tag{6.44}$$

Of course $\hat{w}'$ is here the complex bending function. If now w' is the complex stress function, then in view of (6.21) and (6.43) we have the formula

$$\boldsymbol{T}_{(l)} = 2\,\mathrm{Im}\left(w'\frac{dz}{ds}\boldsymbol{r}_{\bar{z}}\right), \tag{6.45}$$

expressing the stress vector of the membrane state which cannot already in general be interpreted as a bending field (see also Ch. VI, §3.4).

Making use of the formulae (6.39) and (6.25) we can now express the complex bending function $\hat{w}'$ by the complex displacement function. The formulae (6.39) and (6.27a) yield

$$\hat{w}' = \frac{1}{2i}(\nabla_1 - i\nabla_2)(v^1 - iv^2) + \frac{1}{2i}(b_1^1 - b_2^2 - 2ib_1^2)v , \tag{6.46}$$

where ∇_1 and ∇_2 are the operators of covariant differentiation, v is the normal component of the rotation vector, i.e. a solution of the characteristic equation (6.30) and v^α are the contravariant components of the rotation vector given (in the isometric-conjugate coordinate system) by

$$v^1 = -\frac{1}{\sqrt{a\mathrm{K}}}\frac{\partial v}{\partial x}, \qquad v^2 = -\frac{1}{\sqrt{a\mathrm{K}}}\frac{\partial v}{\partial y}. \tag{6.47}$$

According to the formulae (6.72), (6.63), (6.24) and (6.26) of Ch. II and (5.48) we have

$$\hat{w}' = i\sqrt{\mathrm{E}}e^{-i\varphi}v - 2i\mathrm{K}^{\frac{1}{4}}(\partial_z w_0 - \bar{B}\bar{w}_0), \tag{6.48}$$

where

$$w_0 = a^{-\frac{1}{2}}\mathrm{K}^{-\frac{3}{4}}\frac{\partial v}{\partial z} \equiv a^{-\frac{1}{2}}\mathrm{K}^{-\frac{3}{4}}\left(\frac{1}{\sqrt{a}}\operatorname{Im}\hat{w}_z\right)_z. \tag{6.49}$$

Thus, the formula (6.48) expresses the solutions of the differential equation (6.40) by the solutions of its adjoint equation (3.59), or by the solutions of the characteristic equation (5.30). In other words the complex bending function can be obtained by applying to the complex displacement function a definite linear differential operation of the third order, or as a result of applying to the characteristic function a definite differential operation of the second order.

It should be observed that the formula (6.48) gives a general representation of all single-valued solutions of the equation (6.40) in the case of a simply-connected domain. In the case of a multiply-connected domain this is not so. If we apply the formula (6.48) to single-valued solutions of the equation (3.59) it does not yield the whole class of single-valued solutions of the adjoint equation (6.40). Nevertheless in the case of a multiply-connected surface the formula (6.48) gives the whole class of the complex bending functions. Consequently, the class of single-valued solutions of the equation (6.40) which are not representable in the form (6.48) cannot

correspond to the class of single-valued displacement fields. This class of single-valued solutions of the equation (6.40) corresponds to a class of multi-valued solutions of the equation (3.59), i.e. to a class of multi-valued displacement fields.

§7. Variations of various geometrical quantities under infinitesimal bending of a surface. Some criteria of rigidity

7.1. Infinitesimal bending of a surface form a special case of the general deformation of the surface, in which the coefficients of the first quadratic form have no variations of the first order (the precise definition of the concept of variation will be given later). Therefore all quantities which are expressible by the coefficients of the first quadratic form and their derivatives (for instance the principal curvature, the Christoffel symbols, etc.) will also have no variation. The coefficients of the second quadratic form will, however, in general, have non-zero variations, since otherwise the deformation would be a rigid-body motion, i.e. a trivial bending.

Thus, the geometrical quantities, which are expressible by the coefficients of the second fundamental quadratic form acquire, under infinitesimal bending of the surface, definite increments which in general are expressed by the displacement vector $\boldsymbol{U}$ in a non-linear way. If the displacement vector has the form $\varepsilon\boldsymbol{U}$ where ε is a small parameter, then the increment $\Delta\mathcal{A}$ of a quantity $\mathcal{A}$ describing a property of the surface has the form

$$\Delta\mathcal{A} = \varepsilon\delta A + \varepsilon^2\delta^2\mathcal{A} + \dots$$

The coefficients of this expansion $\delta\mathcal{A}, \delta^2\mathcal{A}, \dots$ will be called *the first, second,* etc. *variations of the quantity* $\mathcal{A}$, *and it is evident that* $\delta\mathcal{A}$ *is a homogeneous additive function in* $\boldsymbol{U}$.

Since we consider infinitesimal deformations of a surface it is a sufficient approximation to take $\Delta\mathcal{A} = \varepsilon\delta\mathcal{A}$.

In what follows we shall consider only the first variations of the quantities describing some properties of the surface under infinitesimal bending. We shall therefore call them simply the variations of the corresponding quantities omitting the words "first" and "under infinitesimal bending of the surface".

7.2. The relations (1.1) and (1.2) indicate that the variation of the first fundamental quadratic form

$$I \equiv ds^2 \equiv a_{\alpha\beta} dx^\alpha dx^\beta \tag{7.1}$$

vanishes, i.e.

$$\delta I = 0\,, \quad \text{i.e.} \quad \delta a_{\alpha\beta} = 0 \quad (\alpha, \beta = 1, 2)\,. \tag{7.2}$$

Hence, we have also

$$\delta a = 0\,, \quad a = a_{11}a_{22} - a_{12}^2\,, \tag{7.3}$$

$$\delta a^{\alpha\beta} = 0\,, \quad (\alpha, \beta = 1, 2)\,, \tag{7.4}$$

$$\delta c_{\alpha\beta} = 0\,, \quad \delta c^{\alpha\beta} = 0 \quad (\alpha, \beta = 1, 2)\,, \tag{7.5}$$

$$\delta \Gamma_{\alpha\beta,\lambda} = 0\,, \quad \delta \Gamma^{\lambda}_{\alpha\beta} = 0 \quad (\alpha, \beta, \lambda = 1, 2)\,. \tag{7.6}$$

The last relations follow immediately from the formulae

$$\Gamma_{\alpha\beta,\lambda} = \frac{1}{2}\left(\frac{\partial a_{\lambda\alpha}}{\partial x^\beta} + \frac{\partial a_{\lambda\beta}}{\partial x^\alpha} - \frac{\partial a_{\alpha\beta}}{\partial x^\lambda}\right), \quad \Gamma^{\lambda}_{\alpha\beta} = a^{\lambda\gamma}\Gamma_{\alpha\beta,\gamma}\,, \tag{7.7}$$

if the relations (7.2) and (7.4) are taken into account.

We observe that *the variation of a tensor field given on a surface is also a tensor field of the same rank, and the relations connecting the components of a tensor of various types (covariant, contravariant and mixed) are preserved also for the variations.* For instance if f_α and f^β are covariant and contravariant components of a vector belonging to the surface, then $f_\alpha = a_{\alpha\beta} f^\beta$. Hence, taking into account the relations (7.2) we obtain $\delta f_\alpha = a_{\alpha\beta}\delta f^\beta$ which proves our assertion. In a similar way this assertion can be proved for a tensor of an arbitrary rank.

7.3. Making use of the Gauss equations (3.12) and the formulae (3.13) we obtain

$$r_{\alpha\beta\mu} \equiv \frac{\partial r_{\alpha\beta}}{\partial x^\mu} = \left(\frac{\partial \Gamma_{\alpha\beta,\lambda}}{\partial x^\mu} + \Gamma^{\nu}_{\alpha\beta}\Gamma_{\nu\mu,\lambda} - b_{\alpha\beta}b_{\mu\lambda}\right)\boldsymbol{r}^\lambda + \left(\frac{\partial b_{\alpha\beta}}{\partial x^\mu} + \Gamma^{\nu}_{\alpha\beta}b_{\nu\mu}\right)\boldsymbol{n}\,.$$

Changing the places of β and μ in this relation and subtracting one relation from the other we have

$$b_{\alpha\beta}b_{\lambda\mu}-b_{\alpha\mu}b_{\beta\lambda}=\frac{\partial\Gamma_{\alpha\beta,\lambda}}{\partial x^{\mu}}-\frac{\partial\Gamma_{\alpha\mu,\lambda}}{\partial x^{\beta}}+\Gamma^{\nu}_{\alpha\beta}\Gamma_{\nu\mu,\lambda}-\Gamma^{\nu}_{\alpha\mu}\Gamma_{\nu\beta,\lambda}\,, \tag{7.8}$$

$$\frac{\partial b_{\alpha\beta}}{\partial x^{\mu}}-\frac{\partial b_{\alpha\mu}}{\partial x^{\beta}}+\Gamma^{\nu}_{\alpha\beta}b_{\nu\mu}-\Gamma^{\nu}_{\alpha\mu}b_{\nu\beta}=0\,. \tag{7.9}$$

The system of equations (7.8) is essentially equivalent to one equation

$$a\mathrm{K}\equiv b_{11}b_{22}-b_{12}^{2}=\frac{\partial\Gamma_{11,2}}{\partial x^{2}}-\frac{\partial\Gamma_{12,2}}{\partial x^{1}}+\Gamma^{\nu}_{11}\Gamma_{\nu2,2}-\Gamma^{\nu}_{12}\Gamma_{\nu1,2}\,, \tag{7.10}$$

which is called *the Gauss equation.* The system (7.9) is called *the Codazzi system of equations.*

We observe that with respect to the coordinate system consisting of the curvature lines, the Codazzi system of equations and the Gauss equation have the forms (3.35) and (3.36), respectively.

Thus, the principal curvature K of the surface is expressed by the coefficients of the first fundamental quadratic form and their first and second derivatives. Therefore, according to the relations (7.3) and (7.6) we have

$$\delta\mathrm{K}\equiv 0\,. \tag{7.11}$$

Hence, in view of the formulae (7.3) and (7.10) it follows that the variations of the coefficients of the second fundamental quadratic form satisfy the following algebraic relation:

$$b_{11}\delta b_{22}+b_{22}\delta b_{11}-2b_{12}\delta b_{12}=0\,. \tag{7.12}$$

The Codazzi system of equations contains only two independent equations which are satisfied by the three coefficients (b_{11}, $b_{12}=b_{21}$, b_{22}) of the second fundamental quadratic form

$$\mathrm{II}=b_{\alpha\beta}dx^{\alpha}dx^{\beta}\,. \tag{7.13}$$

We find from (7.9), taking into account the relations (7.6), that the variations of the coefficients of the form II satisfy the following system of differential equations:

$$\begin{aligned} &\frac{\partial \delta b_{11}}{\partial x^2} - \frac{\partial \delta b_{12}}{\partial x^1} + \Gamma^{\lambda}_{11} \delta b_{\lambda 2} - \Gamma^{\lambda}_{12} \delta b_{\lambda 1} = 0 , \\ &\frac{\partial \delta b_{22}}{\partial x^1} - \frac{\partial \delta b_{12}}{\partial x^2} + \Gamma^{\lambda}_{22} \delta b_{\lambda 1} - \Gamma^{\lambda}_{12} \delta b_{\lambda 2} = 0 . \end{aligned} \tag{7.14}$$

This system can be briefly written thus:

$$\nabla_2 \delta b_{11} - \nabla_1 \delta b_{12} = 0 , \qquad \nabla_1 \delta b_{22} - \nabla_2 \delta b_{12} = 0 . \tag{7.15}$$

7.4. The variations of the coefficients of the form II can readily be expressed in terms of the components of the bending field.

Considering the two mutually perpendicular unit vectors tangent to the surface

$$\boldsymbol{l} = \frac{d\boldsymbol{r}}{dl} , \qquad \boldsymbol{s} = \frac{d\boldsymbol{r}}{ds} \tag{7.16}$$

and bearing in mind the relations

$$\boldsymbol{l} \times \boldsymbol{s} = \boldsymbol{n} , \qquad d\boldsymbol{U} = \boldsymbol{V} \times d\boldsymbol{r} , \tag{7.17}$$

we have

$$\begin{aligned} &\delta \boldsymbol{s} = \frac{d\boldsymbol{U}}{ds} = \boldsymbol{V} \times \boldsymbol{s} , \qquad \delta \boldsymbol{l} = \frac{d\boldsymbol{U}}{dl} = \boldsymbol{V} \times \boldsymbol{l} , \\ &\delta \boldsymbol{n} = \delta \boldsymbol{l} \times \boldsymbol{s} + \boldsymbol{l} \times \delta \boldsymbol{s} = \boldsymbol{V} \times \boldsymbol{n} . \end{aligned} \tag{7.18}$$

In deriving the last relations we used the formula for the double vector product

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A}\boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A}\boldsymbol{B})\boldsymbol{C} . \tag{7.19}$$

Further, the relations

$$b_{\alpha\beta} = \boldsymbol{n}\boldsymbol{r}_{\alpha\lambda} = -\boldsymbol{n}_{\alpha}\boldsymbol{r}_{\beta} \tag{7.20}$$

imply

$$\begin{aligned} \delta b_{\alpha\beta} &= \delta \boldsymbol{n} \boldsymbol{r}_{\alpha\beta} + \boldsymbol{n} \boldsymbol{U}_{\alpha\beta} = (\boldsymbol{V} \times \boldsymbol{n}) \boldsymbol{r}_{\alpha\beta} + \boldsymbol{n} (\boldsymbol{V} \times \boldsymbol{r}_{\alpha})_{\beta} \\ &= \boldsymbol{V}\boldsymbol{n}\boldsymbol{r}_{\alpha\beta} + \boldsymbol{n}\boldsymbol{V}\boldsymbol{r}_{\alpha\beta} + \boldsymbol{n}\boldsymbol{V}_{\beta}\boldsymbol{r}_{\alpha} = \boldsymbol{n}\boldsymbol{V}_{\beta}\boldsymbol{r}_{\alpha} . \end{aligned} \tag{7.21}$$

We have taken into account here that the variation of the position vector $\boldsymbol{r}$ is the displacement vector $\boldsymbol{U}$.

Now, according to (6.1) and (6.3)

$$\boldsymbol{V}_\beta = \overset{*}{T}{}^{\lambda}_{\beta}\boldsymbol{r}_\lambda = c_{\alpha\lambda}\hat{T}^{\alpha\lambda}\boldsymbol{r}_\lambda \,. \tag{7.22}$$

Hence, taking into account the relations $\boldsymbol{r}_\alpha \times \boldsymbol{r}_\beta = c_{\alpha\beta}\boldsymbol{n}$ and bearing in mind the formulae (6.8) we have

$$\delta b_{\alpha\beta} = c_{\alpha\lambda}c_{\beta\gamma}\hat{T}^{\lambda\gamma} \equiv t_{\alpha\beta} \,. \tag{7.23}$$

Thus, the components of the covariant tensor $t_{\alpha\beta}$ considered above (§6.2) represent the variations of the coefficients of the second fundamental quadratic form under an infinitesimal bending of the surface.

Substituting the expressions (7.23) into (7.12) and (7.15) we obtain the system of equations

$$\nabla_\alpha T^{\alpha\beta} = 0 \quad (\beta = 1, 2)\,, \qquad b_{\alpha\beta}T^{\alpha\beta} = 0\,, \tag{7.24}$$

which was already derived above (§6.1) in a different way.

Let the surface S have everywhere positive Gaussian curvature. Then, referring the surface to the isometric–conjugate net of lines we have

$$\delta b_{11} = -\delta b_{22}\,, \qquad \delta b_{12} = \delta b_{21} \,. \tag{7.25}$$

Making use of the formulae (7.23) and (6.43) we obtain the formula

$$\hat{w}' = \frac{1}{\sqrt{a}}(\delta b_{22} + i\delta b_{12})\,, \tag{7.26}$$

expressing the variations of the coefficients of the second fundamental quadratic form in terms of the complex bending function. It should be observed that here $\hat{w}'$ is the complex bending function which satisfies the equation (6.40) and the condition (6.42). In the case of a simply-connected domain the conditions (6.42) are absent and $\hat{w}' \equiv w'$.

7.5. Considering a smooth curve L on the surface and differentiating the position vector of the surface with

respect to the arc of this curve we obtain the unit vector of the tangent

$$\boldsymbol{s} = \frac{d\boldsymbol{r}}{ds} = \boldsymbol{s}_\alpha \boldsymbol{r}^\alpha \equiv s^\alpha \boldsymbol{r}_\alpha , \qquad s^\alpha = \frac{dx^\alpha}{ds} . \tag{7.27}$$

Differentiating this relation once more with respect to s and making use of the equations of Gauss (3.12) we obtain the so-called *vector of the principal normal* of L:

$$\frac{d^2\boldsymbol{r}}{ds^2} \equiv k\boldsymbol{m} \equiv \left(\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\right)\boldsymbol{r}_\lambda + b_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\cdot\boldsymbol{n} , \tag{7.28}$$

where $\boldsymbol{m}$ is the unit vector of the principal normal and k is the curvature of the curve L. It should be borne in mind that the vector $\dfrac{d^2\boldsymbol{r}}{ds^2}$ and, consequently, its unit vector $\boldsymbol{m}$ are directed towards the concavity of the curve L. It is evident that the curvature k of the curve L is a non-negative quantity, i.e. $k \geqslant 0$. If $k = 0$ on an arc of the curve L, then this arc is a section of a straight line. The points at which $k > 0$ will be called *the ordinary points* of the curve L. In the vicinity of an ordinary point the curve is situated on one side of its rectifying plane.

It is known that the curve L is uniquely determined to within a rigid-body motion, by prescribing the values of the curvature k and the torsion $\varkappa$. It should be borne in mind that k and $\varkappa$ are functions of the point of the curve and are independent of the choice of the direction on the curve L. This fact follows at once from the formulae

$$k = \boldsymbol{m}\frac{d\boldsymbol{s}}{ds} = \boldsymbol{m}\frac{d^2\boldsymbol{r}}{ds^2} , \qquad \varkappa = \boldsymbol{b}\frac{d\boldsymbol{m}}{ds} = -\boldsymbol{m}\frac{d\boldsymbol{b}}{ds} . \tag{7.29}$$

It is sufficient to take into account that as a result of a change of the direction on the curve, the directions of $\boldsymbol{s}$ and $\boldsymbol{b}$ change into opposite and $\boldsymbol{m}$ remains unaltered (it is always directed towards the concavity of the curve).

Let us now consider at every point of the curve L also the triplet $\boldsymbol{s}, \boldsymbol{n}, \boldsymbol{l}$ having the same orientation as the triplet $\boldsymbol{s}, \boldsymbol{m}, \boldsymbol{b}$, $\boldsymbol{l}$ being the tangential normal of the curve L.

Let us denote by θ the angle between the principal normal of the curve L and the normal of the surface; we assume that $-\pi \leqslant \theta \leqslant \pi$. The angle θ takes a positive value if in consequence of a rotation of the triplet $\boldsymbol{s}, \boldsymbol{m}, \boldsymbol{b}$ about $\boldsymbol{s}$

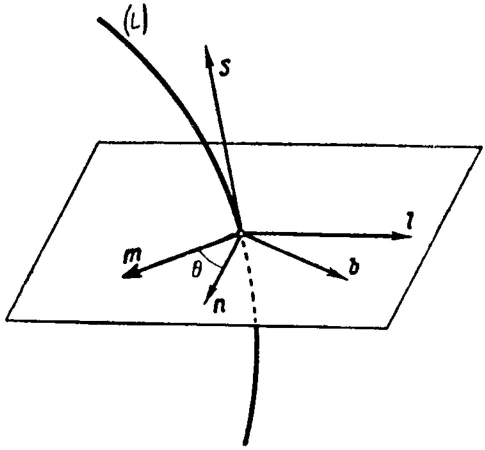

FIG. 1.

through the angle θ counter-clockwise, we obtain the triplet $\boldsymbol{s}, \boldsymbol{n}, \boldsymbol{l}$ (Fig. 1). In the converse case θ is negative.

It is readily observed that the following formulae hold:

$$\boldsymbol{m} = \boldsymbol{n}\cos\theta - \boldsymbol{l}\sin\theta\,, \qquad \boldsymbol{b} = \boldsymbol{n}\sin\theta + \boldsymbol{l}\cos\theta\,, \tag{7.30}$$

$$\boldsymbol{n} = \boldsymbol{m}\cos\theta + \boldsymbol{b}\sin\theta\,, \qquad \boldsymbol{l} = -\boldsymbol{m}\sin\theta + \boldsymbol{b}\cos\theta\,. \tag{7.31}$$

We obtain from the relations (7.30) in view of (7.18) [14j]

$$\delta\boldsymbol{m} = \boldsymbol{V}\times\boldsymbol{m} - \delta\theta\boldsymbol{b}\,, \tag{7.32}$$

$$\delta\boldsymbol{b} = \boldsymbol{V}\times\boldsymbol{b} + \delta\theta\boldsymbol{m}\,. \tag{7.33}$$

Differentiating the relations (7.31) and taking into account (5.13) we easily derive the relations:

$$\begin{gathered} \frac{d\boldsymbol{s}}{ds} = k_s\boldsymbol{n} - \sigma_s\boldsymbol{l}\,, \qquad \frac{d\boldsymbol{n}}{ds} = -k_s\boldsymbol{s} + \tau_s\boldsymbol{l}\,, \\ \frac{d\boldsymbol{l}}{ds} = \sigma_s\boldsymbol{s} - \tau_s\boldsymbol{n}\,, \end{gathered} \tag{7.34}$$

where

$$k_s = k\cos\theta\,, \qquad \sigma_s = k\sin\theta\,, \qquad \tau_s = \varkappa + \frac{d\theta}{ds}\,. \quad (7.35)$$

These quantities have the following names: k_s—*the normal curvature of the surface in direction* s; σ_s—*the geodesic curvature of the curve* L and τ_s—*the geodesic torsion of the curve* L.

Taking the scalar product and then the vector product of the equation (7.28) with $\boldsymbol{n}$ and bearing in mind the formulae

$$\boldsymbol{mn} = \cos\theta\,, \quad \boldsymbol{m}\times\boldsymbol{n} = \sin\theta\cdot\boldsymbol{s}\,, \quad \boldsymbol{n}\times\boldsymbol{r}_\lambda = c_{\lambda\gamma}\boldsymbol{r}^\gamma\,, \quad (7.36)$$

we obtain

$$k_s \equiv k\cos\theta = b_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} \equiv \frac{\mathrm{II}}{\mathrm{I}}\,, \quad (7.37)$$

$$k\sin\theta\cdot\boldsymbol{s} \equiv \sigma_s\cdot\boldsymbol{s} \equiv -\left(\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\right)c_{\lambda\gamma}\boldsymbol{r}^\gamma\,. \quad (7.38)$$

It follows that k_s depends only on the direction of the tangent of the curve L and is entirely independent of its other properties. Multiplying the last relation scalarly by $\boldsymbol{s}$ and $\boldsymbol{l}$ and taking into account the formulae (6.13) we obtain

$$\begin{aligned} \sigma_s \equiv k\sin\theta &= -\left(\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\right)c_{\lambda\gamma}s^\gamma \\ &\equiv -\left(\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\right)l_\lambda\,, \end{aligned} \quad (7.39)$$

$$\left(\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\right)s_\lambda = 0\,.$$

The curve for which $\sigma_s = 0$ is called *the geodesic curve of the surface*. Thus, the coordinates of the geodesic curve of the surface satisfy the equations

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 \qquad (\lambda = 1, 2)\,. \quad (7.40)$$

If there is a straight line on the surface ($k \equiv 0$) it is obviously a geodesic line. If $k \neq 0$ we have along the geodesic line $\sin\theta = 0$. This means that *the principal*

normal and the normal of the surface are collinear along the geodesic line (and only in this case).

7.6. THEOREM 5.9. *Under an infinitesimal bending of a surface the geodesic lines are carried into geodesic lines of the deformed surface.*

This result follows at once from the equations (7.40) if the relations (7.6) be taken into account.

We infer on the basis of (7.6) and (7.39) that the variation of the geodesic curvature of a curve vanishes i.e.

$$\delta\sigma_s \equiv \delta k \sin\theta + k\cos\theta\,\delta\theta = 0\,. \tag{7.41}$$

This relation can also be obtained in the following way: Making use of the formulae (7.34) we have

$$\begin{aligned}\delta\sigma_s &= \delta\left(\boldsymbol{s}\frac{d\boldsymbol{l}}{ds}\right) = \delta\boldsymbol{s}\frac{d\boldsymbol{l}}{ds} + \boldsymbol{s}\frac{d\,\delta\boldsymbol{l}}{ds}\\ &= (\boldsymbol{V}\times\boldsymbol{s})\frac{d\boldsymbol{l}}{ds} + \boldsymbol{s}\left(\boldsymbol{V}\times\frac{d\boldsymbol{l}}{ds}\right) + \boldsymbol{s}\left(\frac{d\boldsymbol{V}}{ds}\times\boldsymbol{l}\right) = \frac{d\boldsymbol{V}}{ds}\boldsymbol{n} = 0\,.\end{aligned}$$

It follows immediately from (7.37) that the variation of the normal curvature is given by the formula

$$\delta k_s \equiv \delta k\cos\theta - k\sin\theta\,\delta\theta = \delta b_{\alpha\beta}s^\alpha s^\beta\,. \tag{7.42}$$

According to the formulae (7.41) and (7.42) we have

$$(1)\quad \delta k = \delta k_s\cos\theta\,,\qquad (2)\quad k\,\delta\theta = -\delta k_s\sin\theta\,. \tag{7.43}$$

If L is an asymptotic line of the surface and $k \neq 0$, then $\cos\theta = 0$ along this line and (7.43) implies.

THEOREM 5.10. *Under an infinitesimal bending of a surface the variation of the curvature of an asymptotic curve vanishes.*

Substituting into (7.42) the expressions (7.23) and taking into account the formulae (6.13) we obtain

$$\delta k_s = \hat{T}^{\alpha\beta}l_\alpha l_\beta\,. \tag{7.44}$$

This formula can also be derived as follows:

$$\begin{aligned}\delta k_s &= \delta\left(\boldsymbol{n}\frac{d\boldsymbol{s}}{ds}\right) = \delta\boldsymbol{n}\frac{d\boldsymbol{s}}{ds} + \boldsymbol{n}\frac{d\,\delta\boldsymbol{s}}{ds}\\ &= (\boldsymbol{V}\times\boldsymbol{n})\frac{d\boldsymbol{s}}{dn} + \boldsymbol{n}\frac{d}{ds}(\boldsymbol{V}\times\boldsymbol{s}) = \boldsymbol{n}\left(\frac{d\boldsymbol{V}}{d\boldsymbol{s}}\times\boldsymbol{s}\right)\\ &= \boldsymbol{n}(\hat{T}^{\lambda\beta}l_\lambda\boldsymbol{r}_\beta\times s^\alpha\boldsymbol{r}_\lambda) = \boldsymbol{n}(\hat{T}^{\lambda\beta}l_\lambda s^\alpha c_{\beta\alpha}\boldsymbol{n}) \equiv \hat{T}^{\alpha\beta}l_\alpha l_\beta\,.\end{aligned}$$

We have employed here the formulae (7.18), (5.4) and (6.13).

Taking into account the formulae (3.13) we have

$$\tau_s = \boldsymbol{l}\frac{d\boldsymbol{n}}{ds} = \boldsymbol{l}\boldsymbol{n}_\alpha \frac{dx^\alpha}{ds} = -b_{\alpha\beta}l^\alpha s^\beta \,. \tag{7.45}$$

It follows therefore that τ_s depends only on the direction of the tangent of the curve L and is entirely independent of its other properties. Therefore, both τ_s and k_s are quantities characterizing a property of the surface. Let us note the following relations which follow at once from the formula (7.45):

$$\tau_s = \tau_{(-s)} \,, \qquad \tau_s = -\tau_l \,. \tag{7.46}$$

It follows from (7.45) that $\tau_s = 0$ for the principal directions of the surface (this very condition determines the principal directions of a surface). Consequently, $\tau_s = 0$ along the lines of curvature of the surface. In particular at the umbilical points $(k_1 = k_2 \neq 0)$ $\tau_s = 0$ for any direction. Therefore, on a spherical surface (and only on such a surface) $\tau_s = 0$ everywhere.

If the formulae (6.13) be taken into account, (7.45) implies that

$$\delta\tau_s = -\delta b_{\alpha\beta}l^\alpha s^\beta = T^{\alpha\beta}l_\alpha s_\beta \,. \tag{7.47}$$

In view of (7.43) we have from (7.35) [14j]

$$\delta\varkappa = \delta\tau_s - \frac{d\delta\theta}{ds} \equiv \delta\tau_s + \frac{d}{ds}\left(\frac{\sin\theta}{k}\,\delta k_s\right). \tag{7.48}$$

This formula can also be derived as follows:

$$\delta\varkappa = \delta\left(\boldsymbol{b}\frac{d\boldsymbol{m}}{ds}\right) = \delta\boldsymbol{b}\,\frac{d\boldsymbol{m}}{ds} + \boldsymbol{b}\,\frac{d\delta\boldsymbol{m}}{ds}$$

$$= (\boldsymbol{V}\times\boldsymbol{b})\,\frac{d\boldsymbol{m}}{ds} + \delta\,\theta\boldsymbol{m}\,\frac{d\boldsymbol{m}}{ds} + \boldsymbol{b}\left(\frac{d\boldsymbol{V}}{ds}\times\boldsymbol{m}\right) +$$

$$+\boldsymbol{b}\left(\boldsymbol{V}\times\frac{d\boldsymbol{m}}{ds}\right) - \boldsymbol{b}\,\frac{d\delta\,\theta\boldsymbol{b}}{ds} = \frac{d\boldsymbol{V}}{ds}(\boldsymbol{m}\times\boldsymbol{b}) - \frac{d\delta\theta}{ds} = \delta\tau_s - \frac{d\delta\theta}{ds}.$$

The formula (6.14) indicates that the vector $\dfrac{d\boldsymbol{V}}{ds}$ lies in the tangent plane of the surface. Therefore it can be written in the form

$$\frac{d\boldsymbol{V}}{ds}=\left(\boldsymbol{l}\,\frac{d\boldsymbol{V}}{ds}\right)\boldsymbol{l}+\left(\boldsymbol{s}\,\frac{d\boldsymbol{V}}{ds}\right)\boldsymbol{s}\,.$$

Now, in view of (6.14), (7.44) and (7.47) we have

$$\boldsymbol{l}\,\frac{d\boldsymbol{V}}{ds}=T^{\alpha\beta}l_\alpha l_\beta=\delta k_s\,,\qquad \boldsymbol{s}\,\frac{d\boldsymbol{V}}{ds}=T^{\alpha\beta}l_\alpha s_\beta=\delta\tau_s\,.$$

Consequently,

$$\frac{d\boldsymbol{V}}{ds}=\delta k_s\boldsymbol{l}+\delta\tau_s\boldsymbol{s}\,. \tag{7.49}$$

In particular if we consider a surface of positive curvature, by means of the formulae (6.44), (6.13) and (7.49) we can express the variations of the curvature and the torsion in terms of the complex bending function, namely we have

$$\begin{aligned}\delta k_s&=-\sqrt{a}\,\mathrm{Re}\left[\hat{w}'(z)\left(\frac{dz}{ds}\right)^2\right],\\ \delta\tau_s&=\sqrt{a}\,\mathrm{Re}\left[\hat{w}'(z)\,\frac{dz}{dl}\,\frac{dz}{ds}\right].\end{aligned} \tag{7.50}$$

Let $\boldsymbol{U}$ and $\boldsymbol{V}$ be two arbitrary fields of displacement and rotation and assume that they are continuous in $S+L$. Then in accordance with the formulae (6.22) and (7.49) we obtain the identity

$$\int_L (u_l\delta k_s+u_s\delta\tau_s)\,ds=0\,, \tag{7.51}$$

where u_l and u_s are the projections of the displacement vector $\boldsymbol{U}$ on the tangent and the tangent normal to L, and δk_s and $\delta\tau_s$ are the variations of the normal curvature and the geodesic torsion of the surface, corresponding to the rotation field $\boldsymbol{V}$. Taking into account the formulae

(7.48), (7.43) and (7.30) we can easily prove that the relation (7.51) is equivalent to the following:

$$\int_L (u_b \delta k + u_s \delta \varkappa)\, ds = 0 \,. \tag{7.52}$$

If $\boldsymbol{U}$ and $\boldsymbol{V}$ correspond to an infinitesimal bending of the surface, i.e. if $d\boldsymbol{U} = \boldsymbol{V} \times d\boldsymbol{r}$, then we have

$$\int_{L_j} (u_l \delta k_s + u_s \delta \tau_s)\, ds = 0 \qquad \text{or} \qquad \int_{L_j} (u_b \delta k + u_s \delta \varkappa)\, ds = 0 \tag{7.53}$$

$$(j = 0, 1, \ldots, m) \,.$$

We observe that one of these relations is a consequence of the other m relations.

7.7. In this article we shall prove that a bending field of a surface may be completely described by two scalar functions which have a definite geometric meaning.

If $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are principal directions of the surface at a point we have

$$\boldsymbol{s}_2 \frac{d\boldsymbol{n}}{ds_1} = 0 \,, \qquad \boldsymbol{s}_1 \frac{d\boldsymbol{n}}{ds_2} = 0 \qquad (\boldsymbol{s}_1 \times \boldsymbol{s}_2 = \boldsymbol{n}) \,. \tag{7.54}$$

Denoting by χ the angle of inclination of the unit vector $\boldsymbol{s}$ to $\boldsymbol{s}_1$ we obtain

$$\boldsymbol{s} = \boldsymbol{s}_1 \cos\chi + \boldsymbol{s}_2 \sin\chi \,, \qquad \boldsymbol{l} = \boldsymbol{s}_1 \sin\chi - \boldsymbol{s}_2 \cos\chi \,. \tag{7.55}$$

In view of (7.54) and (7.55)

$$\tau_s \equiv \boldsymbol{l} \frac{d\boldsymbol{n}}{ds} = (\boldsymbol{s}_1 \sin\chi - \boldsymbol{s}_2 \cos\chi) \left(\frac{d\boldsymbol{n}}{ds_1} \cos\chi + \frac{d\boldsymbol{n}}{ds_2} \sin\chi \right)$$
$$= \boldsymbol{s}_1 \frac{d\boldsymbol{n}}{ds_1} \sin\chi\cos\chi - \boldsymbol{s}_2 \frac{d\boldsymbol{n}}{ds_2} \sin\chi\cos\chi \,.$$

Since according to (7.54)

$$\frac{d\boldsymbol{n}}{ds_1} = -k_1 \boldsymbol{s}_1 \,, \qquad \frac{d\boldsymbol{n}}{ds_2} = -k_2 \boldsymbol{s}_2 \,, \tag{7.56}$$

where k_1 and k_2 are the principal curvatures of the surface, we have the following new expression for the geodesic torsion:

$$\tau_s = (k_2 - k_1)\sin\chi\cos\chi = \frac{k_2 - k_1}{2}\sin 2\chi\,. \qquad (7.57)$$

Thus, as was already indicated above, at an umbilical point ($k_1 = k_2 \neq 0$) the geodesic torsion vanishes for any direction. At a non-umbilical point ($k_1 \neq k_2$) for $\chi = \frac{\pi}{4}, \frac{3\pi}{4}$ it takes the extremum values equal to $\mp\,\tau_0$, respectively, where

$$\tau_0 = \frac{k_1 - k_2}{2}\,. \qquad (7.58)$$

Assuming that $k_1 \geqslant k_2$ we have

$$\tau_0 = \sqrt{\mathrm{E}}\,, \qquad (7.59)$$

where $\mathrm{E} = \mathrm{H}^2 - \mathrm{K}$ is the Euler difference.

Thus $\sqrt{\mathrm{E}}$ *is the maximum geodesic torsion at the point under consideration. The minimum geodesic torsion is* $-\sqrt{\mathrm{E}}$.

It follows from the relation (7.57) that

$$\delta\tau_s = \frac{\delta k_2 - \delta k_1}{2}\sin 2\chi + (k_2 - k_1)\cos 2\chi\,\delta\chi\,, \qquad (7.60)$$

where $\delta\chi$ is the variation of the angle between the principal directions. Putting $\chi = 0, \frac{\pi}{2}$ in this relation we obtain

$$q \equiv \delta\tau_2 = -\delta\tau_1 = (k_1 - k_2)\,\delta\chi\,, \qquad (7.61)$$

where $\delta\tau_1$ and $\delta\tau_2$ are the variations of the geodesic torsions of the surface along the principal directions $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ (it should be borne in mind that before the deformation of the surface $\tau_1 = \tau_2 = 0$).

The quantity q will hereafter be called *the geodesic torsion* or simply *the torsion of the surface* under an infinitesimal bending. Obviously, q is a scalar function of the point of the surface.

Since

$$\delta \mathrm{K} = \delta(k_1 k_2) = k_1 \delta k_2 + k_2 \delta k_1 = 0\ , \tag{7.62}$$

$$2\delta \mathrm{H} = \delta(k_1 + k_2) = \delta k_1 + \delta k_2\ , \tag{7.63}$$

we have

$$\delta k_1 = k_1 p\ , \qquad \delta k_2 = -k_2 p\ , \tag{7.64}$$

where

$$p = \frac{2\delta \mathrm{H}}{k_1 - k_2} \equiv \frac{\delta \mathrm{H}}{\sqrt{\mathrm{E}}} \quad (k_1 \geqslant k_2)\ . \tag{7.65}$$

Evidently, p is a scalar. In view of (7.59) and (7.65) we have

$$p = \frac{\delta \tau_0}{\mathrm{H}} \equiv \frac{\delta \sqrt{\mathrm{E}}}{\mathrm{H}}\ , \tag{7.66}$$

i.e. Hp *is equal to the variation of the maximum geodesic torsion at the considered point.*

We have also from (7.65) and (7.66)

$$\delta \mathrm{H} = \frac{\sqrt{\mathrm{E}}}{\mathrm{H}} \delta \tau_0 = \sqrt{\mathrm{E}}\, p\ . \tag{7.67}$$

It the surface is referred to the coordinate system in curvature lines

$$\mathrm{I} = \mathrm{A}^2 d\xi^2 + \mathrm{B}^2 d\eta^2\ , \qquad \mathrm{II} \equiv \mathrm{A}^2 k_1 d\xi^2 + \mathrm{B}^2 k_2 d\eta^2\ . \tag{7.68}$$

If for $\boldsymbol{s}$ we take the tangent to the curvature line $\eta = \mathrm{const}$ we have

$$s^1 = \frac{1}{\mathrm{A}}\ , \qquad s^2 = 0\ , \qquad l^1 = 0\ , \qquad l^2 = -\frac{1}{\mathrm{B}}\ .$$

Hence, in view of the formulae (7.42) and (7.47)

$$\begin{gathered} \delta k_1 = \frac{\delta b_{11}}{\mathrm{A}^2}\ , \qquad \delta k_2 = \frac{\delta b_{22}}{\mathrm{B}^2}\ , \\ \delta \tau_1 = -\delta \tau_2 = \frac{\delta b_{12}}{\mathrm{AB}}\ , \end{gathered} \tag{7.69}$$

or taking into account the relations (7.64) and (7.61)

$$\begin{aligned} \delta b_{11} &= b_{11} p = \mathrm{A}^2 k_1 p\ , \\ \delta b_{22} &= -b_{22} p = -\mathrm{B}^2 k_2 p\ , \\ \delta b_{12} &= -\mathrm{AB} q\ . \end{aligned} \tag{7.70}$$

With respect to an arbitrary coordinate system these formulae have the form [14k] *

$$\frac{k_1-k_2}{2}\,\delta b_{\alpha\beta}=(\mathrm{H}b_{\alpha\beta}-\mathrm{K}a_{\alpha\beta})p+\frac{1}{2}(c_{\alpha\lambda}b^{\lambda}_{\beta}+c_{\beta\lambda}b^{\lambda}_{\alpha})q\,. \tag{7.71}$$

In fact, for the coordinate system in curvature lines these relations are identical with the relations (7.70). Hence, they hold also in an arbitrary coordinate system on the surface, since both sides of these relations are components of a covariant tensor of rank two.

Thus, the pair of scalar functions p and q the geometric meaning of which is defined by means of the relations (7.65), (7.66) and (7.61) completely describe the bending field of the surface.

7.8. In the case of a surface of positive curvature it is convenient to deal with *the scalar complex bending function*

$$w'_*=\sqrt{\mathrm{K}}\,p+iq\,, \tag{7.72}$$

which in view of the formulae (7.71), (7.26) and (6.26) of Ch. II is related to the complex bending function in the following way:

$$w'_*=e^{i\psi}\hat{w}'\,. \tag{7.73}$$

This scalar function satisfies the following differential equation:

$$\partial_{\bar z}w'_*-A_*w'_*-\bar{B}_*\overline{w}'_*=0\,, \tag{7.74}$$

where

$$A_*=-\partial_{\bar z}\ln\left(\sqrt{a\sqrt{\mathrm{K}}}\,e^{i\psi}\right),\qquad B_*=Be^{-2i\psi}\,, \tag{7.75}$$

* The right-hand sides of the relations (7.71) constitute a symmetric tensor apolar with respect to the tensor $b_{\alpha\beta}$. Relations between apolar symmetric tensors were established by Yefimov ([33c]; see also [37], Ch. XIX, §86, art. 4). The relation $c^{\alpha\lambda}c^{\beta\gamma}b_{\alpha\beta}\delta b_{\lambda\gamma}=0$ expresses the property of apolarity of the tensors $b_{\alpha\beta}$ and $\delta b_{\alpha\beta}$.

or

$$\partial_{\bar z}\widetilde{w}'_* - \overline{B_*(z)}\,\overline{\widetilde{w}'_*} = 0\,, \tag{7.75a}$$

where

$$\widetilde{w}'_* = \sqrt{a\sqrt{\mathrm{K}}}\,e^{i\varphi}w'_*\,. \tag{7.75b}$$

Employing this equation we can now give one more (the fourth) proof of the rigidity of an ovaloid.

Let the ovaloid belong to the class $D_{3,p}$, $p > 2$. If the bending field belongs to the class $D_{1,p}$, $p > 2$ then, obviously, $\widetilde{w}'_* \in D_{1,p}(E)$, $p > 2$, and, consequently, $\widetilde{w}'_* \in C_\alpha(E)$, $\alpha = \dfrac{p-2}{p}$. Moreover, in view of (3.68) near the point $z = \infty$

$$\widetilde{w}'_* = 0(|z|^{-4})\,.$$

Hence, according to the generalized Liouville theorem (3.11) $\widetilde{w}'_* \equiv 0$ which, evidently, means the rigidity of the ovaloid.

Now, making use of the results of the preceding chapter we can obtain some conditions of rigidity of convex regular surfaces with edges.

THEOREM 5.11. *Let the surface S of positive curvature of the class $D_{k+3,p}$ $(p > 2, k \geqslant 0)$ be bounded by $m+1$ smooth curves $L_0, L_1, \ldots, L_m$ of the class C^1_μ, $0 < \mu \leqslant 1$. Assume that the following conditions are satisfied*: (1) *there is only a finite number of umbilical points of the surface on L*; (2) *the variation of the mean curvature is zero along L, i.e.*

$$\delta\mathrm{H} = 0 \quad (\text{on } L)\,, \tag{7.76}$$

and (3) *there is at least one point M_0 on L at which the principal directions are preserved under an infinitesimal bending, i.e.*

$$q(M_0) = 0\,, \qquad M_0 \in L\,. \tag{7.77}$$

Under these conditions the surface is rigid.

PROOF. It is seen from the formulae (7.64) and (7.61) that the functions p and q are bounded everywhere on

the surface. Therefore it follows from the formula (7.65) that $\delta H = 0$ at an umbilical point of the surface, i.e. an umbilical point remains umbilical under an infinitesimal bending of the surface. Moreover, we have form (7.65)

$$\delta H = 0(|k_1 - k_2|) \,. \tag{7.78}$$

in the vicinity of an umbilical point. Since p is continuous on S, (7.65), (7.76) and (7.77) imply the boundary condition

$$p = 0 \quad (\text{on } L)\,, \qquad q(M_0) = 0\,, \qquad M_0 \in L. \tag{7.79}$$

Hence, in view of Theorem 4.6 we have

$$p \equiv 0\,, \qquad q \equiv 0\,.$$

This completes the proof.

Let us observe that this theorem does not hold for a spherical surface. In this case $\delta H \equiv 0$ everywhere and the condition $p = 0$ on L does not follows from (7.76).

Let us also note that a simply-connected domain will not be rigid if the condition (7.77) is not satisfied, while in the case of a multiply-connected surface ($m \geqslant 1$) this condition may prove to be superfluous. As a rule a multiply-connected surface of positive curvature will be rigid only if the first two conditions of the theorem are present.

We also observe that the theorem of rigidity proved above will be valid if we replace p for q, and conversely. Namely, instead of the conditions (2) and (3) we require that the following condition be satisfied: (2) along the boundary of the surface the principal directions are preserved, i.e.

$$q = 0 \quad (\text{on } L)\,,$$

and (3) at a fixed point M_0 of the boundary, distinct from umbilical, $\delta H = 0$.

In other words we have

THEOREM 5.12. *If under a small deformation of the surface the principal directions are preserved along the boundary of the surface and, moreover, at a non-umbilical*

point of the boundary the variation of the mean curvature vanishes, then the surface is rigid.

7.9. Computing the normal curvature of the surface in the tangent direction $\boldsymbol{s}$ we have in view of (7.55)

$$k_s = -\boldsymbol{s}\frac{d\boldsymbol{n}}{ds} = -(\boldsymbol{s}_1\cos\chi + \boldsymbol{s}_2\sin\chi)\left(\frac{d\boldsymbol{n}}{ds_1}\cos\chi + \frac{d\boldsymbol{n}}{ds_2}\sin\chi\right)$$
$$= -\boldsymbol{s}_1\frac{d\boldsymbol{n}}{ds_1}\cos^2\chi - \boldsymbol{s}_2\frac{d\boldsymbol{n}}{ds_2}\sin^2\chi\,,$$

where χ is the angle between $\boldsymbol{s}$ and $\boldsymbol{s}_1$. Hence, in view of (7.56) we have

$$k_s = k_1\cos^2\chi + k_2\sin^2\chi\,. \tag{7.80}$$

This formula is called *the Euler formula.*

Making use of the formulae (7.61) and (7.64) we obtain from (7.80)

$$\delta k_s = \delta k_1\cos^2\chi + \delta k_2\sin^2\chi + 2(k_2 - k_1)\sin\chi\cos\chi\,\delta\chi$$
$$= (k_1\cos^2\chi - k_2\sin^2\chi)p - \sin 2\chi q\,,$$

or, assuming that $k_1 \geqslant k_2$,

$$\delta k_s = \left(\sqrt{\mathrm{E}} + \mathrm{H}\cos 2\chi\right)p - \sin 2\chi q\,. \tag{7.81}$$

Replacing in the above formula $\boldsymbol{s}$ by $\boldsymbol{l}$ we have

$$\delta k_l = \left(\sqrt{\mathrm{E}} - \mathrm{H}\cos 2\chi\right)p + \sin 2\chi q\,. \tag{7.82}$$

Adding the relations (7.81) and (7.82) we obtain

$$\tfrac{1}{2}(\delta k_s + \delta k_l) = \sqrt{\mathrm{E}}\,p \equiv \delta\mathrm{H}\,. \tag{7.83}$$

Thus *a half of the sum of the variations of the normal curvatures of the surface corresponding to two mutually perpendicular tangent directions is equal to the variation of the mean curvature and, concequently, it is a scalar function of the point.*

From the formula (7.60) in view of (7.61) and (7.64) we obtain

$$\delta\tau_s = -\mathrm{H}\sin 2\chi p - \cos 2\chi q\,. \tag{7.84}$$

Making use of the formula (7.49) we have according to (7.81) and (7.84)

$$\frac{\partial \boldsymbol{V}}{ds} = q\boldsymbol{s}^* + k^* p\boldsymbol{t}^* , \tag{7.85}$$

where (see also p. 616)

$$\boldsymbol{s}^* = -\sin 2\chi \boldsymbol{l} - \cos 2\chi \boldsymbol{s} = -\boldsymbol{s}_1 \cos\chi + \boldsymbol{s}_2 \sin\chi , \tag{7.86}$$

$$k^*\boldsymbol{t}^* = (\sqrt{\mathrm{K}} + \mathrm{H}\cos 2\chi)\boldsymbol{l} - \mathrm{H}\sin 2\chi \boldsymbol{s} \equiv -k_2 \sin\chi \boldsymbol{s}_1 - k_1 \cos\chi \boldsymbol{s}_2 , \tag{7.87}$$

and

$$k^* = \sqrt{k_1^2 \cos^2\chi + k_2^2 \sin^2\chi} . \tag{7.88}$$

The unit vectors $\boldsymbol{s}^*$ and $\boldsymbol{t}^*$ belong to two mutually conjugate directions, and $\boldsymbol{s}^*$ is the mirror image of the unit vector $\boldsymbol{s}$ with respect to the principal direction $\boldsymbol{s}_2$. Moreover, $\boldsymbol{s}^*$ and $\boldsymbol{t}^*$ have the same orientation as $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ (Fig. 2).

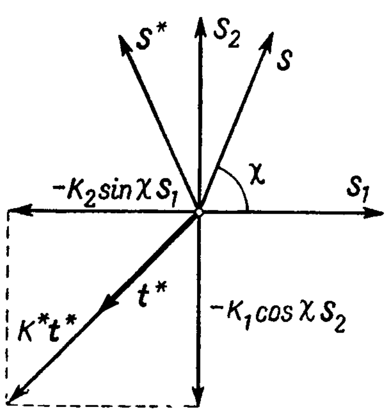

FIG. 2

We recall that two directions tangent to the surface—$\boldsymbol{s}$ and $\boldsymbol{t}$—which make with the principal direction $\boldsymbol{s}_1$ the angles θ and ϑ respectively, are said to be mutually conjugate if the relation $\boldsymbol{t}\boldsymbol{n}_s = 0$ is satisfied, i.e.

$$k_1 \cos\theta \cos\vartheta + k_2 \sin\theta \sin\vartheta = 0 .$$

§8. Conjunction conditions on the contact lines. Some criteria of rigidity of surfaces with edges. Bush constraints. Perfect clamping

8.1. In investigating infinitesimal bending of sectionally regular surfaces it should be taken into account that every curve of contact taking part simultaneously in the deformations of two adjacent regular sections of the surface, has a definite influence on the character of these deformations. The equations obtained in the preceding sections indicate that to a large degree, the nature of infinitesimal bending is determined by the nature of the surface. It is sufficient to indicate that surfaces of positive curvature lead to equations of elliptic type and surfaces of negative curvature to equations of hyperbolic type. Therefore infinitesimal bending of one surface can be incompatible with those of another surface. On a sectionally smooth surface the deformation of one regular section is taken up by the adjacent section, by means of the contact line which taking part in two distinct deformations deforms itself in a definite way. This fact leads to relations on the contact lines between the quantities describing infinitesimal bendings of adjacent regular sections of the surface. Therefore the adjacent regular surfaces in general cannot undergo arbitrary infinitesimal bendings. Their deformations should be adjusted in accordance with the condition that the deformation of the contact line under an infinitesimal bending of one surface should be identical with the deformation under an infinitesimal bending of the other adjacent surface.

Thus, the contact lines play the role of constraints restricting, in a definite way, the deformation of a sectionally regular surface. We shall see below that, with the help of contacting, certain constraints can be set up which make the surface rigid.

The relations which should be satisfied on the contact lines will be called *the conjunction conditions*. Below we

shall give a derivation of these conditions in both geometrical and kinematical forms.

8.2. Let L be the contact line of two regular surfaces S^+ and S^-. Along L we consider besides the natural triplet $\boldsymbol{s}, \boldsymbol{m}, \boldsymbol{b}$ two more triplets—$\boldsymbol{s}, \boldsymbol{n}^+, \boldsymbol{l}^+$ and $\boldsymbol{s}, \boldsymbol{n}^-, \boldsymbol{l}^-$ connected with the surfaces S^+ and S^-. We shall assume that these triplets have the same orientation as the natural triplet $\boldsymbol{s}, \boldsymbol{m}, \boldsymbol{b}$ (Fig. 3).

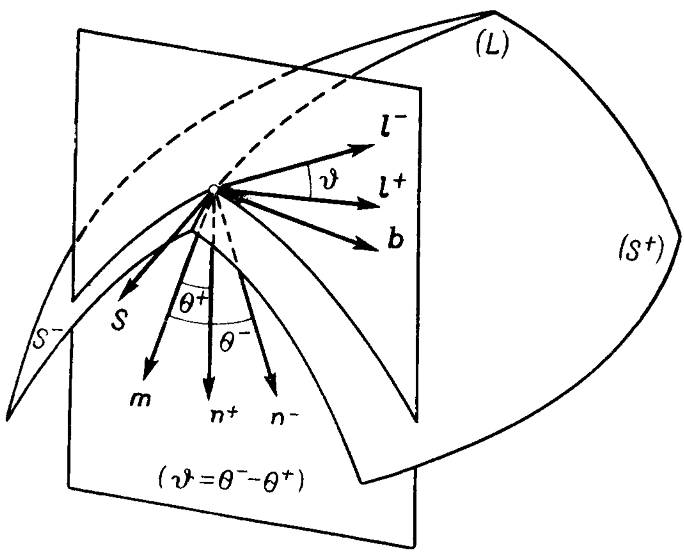

FIG. 3

Investigating infinitesimal bending of the sectionally regular surface $S = S^+ + S^-$ we have, in view of the continuity of deformation

$$\boldsymbol{U}^+ = \boldsymbol{U}^- \quad (\text{on } L)\,, \tag{8.1}$$

where $\boldsymbol{U}^+$ and $\boldsymbol{U}^-$ are the bending vectors of the surfaces S^+ and S^-, respectively. Differentiating the relation (8.1) with respect to the arc of the curve L and taking into account that $d\boldsymbol{U} = \boldsymbol{V} \times d\boldsymbol{r}$ we have

$$\boldsymbol{V}^+ \times \boldsymbol{s} = \boldsymbol{V}^- \times \boldsymbol{s}\,, \quad \text{i.e.} \quad \boldsymbol{V}^+ - \boldsymbol{V}^- = -\mu \boldsymbol{s}\,, \tag{8.2}$$

where μ is a function of the point of the curve L. Differentiating the relation (8.2) once more with respect to the arc s we obtain

$$\frac{d\boldsymbol{V}^+}{ds} - \frac{d\boldsymbol{V}^-}{ds} = -k\mu\boldsymbol{m} - \frac{d\mu}{ds}\boldsymbol{s},$$

or, in view of the formula (7.49),

$$\delta k_s^+ \boldsymbol{l}^+ + \delta\tau_s^+ \boldsymbol{s} - \delta k_s^- \boldsymbol{l}^- - \delta\tau_s^- \boldsymbol{s} = -k\mu\boldsymbol{m} - \frac{d\mu}{ds}\boldsymbol{s}, \quad (8.3)$$

where $k_s^+, k_s^-, \tau_s^+, \tau_s^-$ are the normal curvatures and geodesic torsions of the surfaces S^+ and S^- along the curve L, respectively. This vector relation is equivalent to the following three scalar relations:

$$\begin{aligned} \delta k_s^+ \cos\theta^+ - \delta k_s^- \cos\theta^- &= 0, \\ \delta k_s^+ \sin\theta^+ - \delta k_s^- \sin\theta^- &= k\mu, \\ \delta\tau_s^+ - \delta\tau_s^- &= -\frac{d\mu}{ds}. \end{aligned} \quad (8.4)$$

According to (7.43) the second of the last relations yields

$$\mu = \frac{\sin\theta^+}{k}\delta k_s^+ - \frac{\sin\theta^-}{k}\delta k_s^- \equiv \delta\theta^- - \delta\theta^+, \quad (8.5)$$

i.e.

$$\mu = \delta\vartheta, \quad \text{where} \quad \vartheta = \theta^- - \theta^+. \quad (8.6)$$

It is readily seen that ϑ is equal to the angle supplementing to π, the angle between the surfaces S^+ and S^-, i.e. it is equal to the angle formed by their normals $\boldsymbol{n}^+$ and $\boldsymbol{n}^-$: $\cos\vartheta = \boldsymbol{n}^+\boldsymbol{n}^-$ (Fig. 3).

Thus, on the contact line the rotation vector has a discontinuity

$$\boldsymbol{V}^+ - \boldsymbol{V}^- = -\delta\vartheta\boldsymbol{s}, \quad (8.7)$$

where $\delta\vartheta$ is *the variation of the contact angle.*

Making use of the relations (8.4) we have

$$\mu = \delta\vartheta = -\frac{\sin\vartheta}{k_s^+}\delta k_s^- \equiv -\frac{\sin\vartheta}{k_s^-}\delta k_s^+ \quad (8.8)$$

or, according to (7.43)

$$\delta\vartheta = -\frac{k\sin\vartheta}{k_s^+ k_s^-}\,\delta k\,. \tag{8.9}$$

The relation (8.6) can also be derived as follows: By virtue of (7.33)

$$\delta\boldsymbol{m} = \boldsymbol{V}^+ \times \boldsymbol{m} - \delta\theta^+\boldsymbol{b}\,, \qquad \delta\boldsymbol{m} = \boldsymbol{V}^- \times \boldsymbol{m} - \delta\theta^-\boldsymbol{b}\,.$$

whence

$$(\boldsymbol{V}^+ - \boldsymbol{V}^-) \times \boldsymbol{m} - \delta(\theta^+ - \theta^-)\boldsymbol{b} = 0\,,$$

or

$$\delta(\theta^- - \theta^+) \equiv \delta\vartheta = \boldsymbol{bm}(\boldsymbol{V}^+ - \boldsymbol{V}^-) \equiv -\mu\boldsymbol{bms} = \mu\,.$$

This was to be proved.

From (8.4) in view of (8.6) we have

$$\begin{aligned} \delta k_s^+ \cos\theta^+ - \delta k_s^- \cos\theta^- &= 0\,,\\ \delta\tau_s^+ - \delta\tau_s^- + \frac{d\,\delta\vartheta}{ds} &= 0\,. \end{aligned} \tag{8.10}$$

These relations will be called *the conjunction conditions along the contact line*. They were derived in the author's paper [14k].

The formulae (8.10) can also be derived in the following way. Since the curve L belongs to both surfaces S^+ and S^- and the deformation is continuous we can write, in view of (7.43) and (7.48),

$$\begin{aligned} \delta k &= \delta k_s^+ \cos\theta^+ = \delta k_s^- \cos\theta^-\,,\\ \delta\tau &= \delta\tau_s^+ - \frac{d\,\delta\theta^+}{ds} = \delta\tau_s^- - \frac{d\,\delta\theta^-}{ds}\,. \end{aligned} \tag{8.10a}$$

These relations immediately imply (8.10) if we take into account that $\delta\theta = \delta\theta^- - \delta\theta^+$.

8.3. In investigations of bending fields of sectionally regular surfaces it is therefore necessary to take into account not only the boundary conditions on the boundary of the surface but also the conjunction conditions on the

contact lines. This circumstance significantly complicates the problem. In this subsection we shall prove that even in the simplest cases we are faced with mathematical problems as yet very little investigated.

Let S^+ and S^- be sections of ovaloids which are contacted and together constitute a closed but in general not convex surface. Let us now choose on S^+ and S^- the isometric–conjugate nets of lines which map homeomorphically these surfaces onto the domains G^+ and G^- of the plane z, possessing a common boundary Γ which is a simple closed smooth curve; G^- is an infinite domain. It is obvious that such nets always exist. In this case the contact contour L is twice mapped homeomorphically onto the curve Γ, and to every point M of the contour L there correspond on Γ in general two distinct points t and $\nu(t)$ where $\nu(t)$ is the function establishing the homeomorphic mapping of the curve Γ onto itself.

If the domains G^+ and G^- are not fixed beforehand we can introduce on the whole closed sectionally smooth surface a common isometric–conjugate coordinate system by means of which the surface $S^+ + S^-$ is homeomorphically mapped onto the entire plane z.

It should be borne in mind that on both surfaces the isometric-conjugate coordinates are introduced in such a way that the equations for the complex functions of displacement and bending have the form

$$\partial_{\bar{z}} w + Aw + B\overline{w} = 0 \qquad (\text{in } G^+ + G^-)\,, \tag{8.11a}$$

$$\partial_{\bar{z}} \hat{w}' - A\hat{w}' - \overline{B}\overline{\hat{w}}' = 0 \qquad (\text{in } G^+ + G^-) \tag{8.11b}$$

respectively.

In the case under consideration the conjunction condition (8.10) or the equivalent condition (8.7) can also be written in the complex form, making use of the complex bending function $\tilde{w}'$ which satisfies the equation (8.11b). Introducing the function

$$U'(z) = \sqrt{a\sqrt{\mathrm{K}}}\,\hat{w}'(z)\,, \tag{8.11c}$$

we transform the equation (8.11b) to the form

$$\partial_{\bar z} U' - \bar B \bar U' = 0 \,. \tag{8.12}$$

In view of (8.11c) the formula (6.44) can be written in the form

$$\frac{dV}{ds} = \frac{2}{\sqrt{a\sqrt{\mathrm{K}}}} \operatorname{Im}\left\{U'(z)\frac{dz}{ds}\boldsymbol{r}_{\bar z}\right\}. \tag{8.12a}$$

Differentiating throughout the relation (8.7) with respect to the arc of the curve L we obtain

$$\frac{dV^+}{ds} - \frac{dV^-}{ds} = -\boldsymbol{k}m\delta\vartheta - \boldsymbol{s}\frac{d\delta\vartheta}{ds}. \tag{8.12b}$$

With the help of the formula (8.12a) this condition can also be written as follows:

$$2\operatorname{Im}\left\{\frac{U'^+(t)}{\sqrt{a^+\sqrt{\mathrm{K}^+}}}\left(\frac{dt}{ds}\right)^+(\partial_{\bar t}\boldsymbol{r})^+ - \frac{U'^-(z)}{\sqrt{a^-\sqrt{\mathrm{K}^-}}}\left(\frac{dz}{ds}\right)^-(\partial_{\bar z}\boldsymbol{r})^-\right\}$$
$$= -\boldsymbol{k}m\delta\vartheta - \frac{d\delta\vartheta}{ds}\boldsymbol{s}\,, \qquad z \in \Gamma\,, \tag{8.13}$$

where t and z are points of the curve Γ connected by the relation $t = \nu(z)$. In general, in what follows the quantities supplied by the signs "$+$" and "$-$" refer to the points $t = \nu(z)$ and z on Γ, respectively. Besides, it should be born in mind that the right-hand side of the relation (8.13) is taken at the point z. Furthermore, U'^+ and U'^- are solutions of the equation (8.12) in G^+ and G^-, and they are continuous in $G^+ + \Gamma$ and $G^- + \Gamma$, respectively. Besides, at infinity the following condition should be satisfied (see §7.8)

$$U'^-(z) = 0(|z|^{-4})\,. \tag{8.14}$$

Multiplying throughout the relations (8.13) scalarly by the complex vector $(\partial_t \boldsymbol{n})^+$ and taking into account that

$$(1)\ \partial_t\boldsymbol{n}\partial_{\bar t}\boldsymbol{r} = -\tfrac{1}{2}\sqrt{a\mathrm{K}}\,, \qquad (2)\ \partial_t\boldsymbol{n}\partial_t\boldsymbol{r} = 0\,, \tag{8.15}$$

we obtain on Γ the relation

$$U'^{+}[\nu(z)] = \beta_1 U'^{-}(z) + \beta_2 \overline{U'^{-}(z)} + \beta_3 \delta\vartheta + \beta_4 \frac{d\,\delta\vartheta}{ds}, \quad (8.16)$$

where β_j are given functions of the point of the curve Γ represented by the following formulae:

$$\begin{aligned}
\beta_1 &= -\frac{2(\partial_t \boldsymbol{n})^{+}(\partial_{\bar{z}}\boldsymbol{r})^{-}}{\sqrt[4]{\mathrm{K}^{+}}\sqrt{a^{-}\sqrt{\mathrm{K}^{-}}}}\frac{dz}{dt}, \qquad t = \nu(z)\,,\\
\beta_2 &= \frac{2(\partial_t \boldsymbol{n})^{+}(\partial_{z}\boldsymbol{r})^{-}}{\sqrt[4]{\mathrm{K}^{+}}\sqrt{a^{-}\sqrt{\mathrm{K}^{-}}}}\frac{d\bar{z}}{d\bar{t}}, \qquad t = \nu(z)\,,\\
\beta_3 &= \frac{2ik\,ds}{\sqrt[4]{\mathrm{K}^{+}}dt}\,\boldsymbol{m}(\partial_t \boldsymbol{n})^{+}\,,\\
\beta_4 &= \frac{2i\,ds}{\sqrt[4]{\mathrm{K}^{+}}dt}\,s(\partial_t \boldsymbol{n})^{+}\,, \qquad t = \nu(z)\,.
\end{aligned} \quad (8.16a)$$

If we now consider the formulae (8.8) and (7.50) we obtain for $\delta\vartheta$ the following expression:

$$\delta\vartheta = \frac{\sin\vartheta}{k_s^{+}\sqrt[4]{\mathrm{K}^{-}}}\,\mathrm{Re}\left\{\left(\frac{dz}{ds}\right)^2 U'^{-}(z)\right\} \quad (\text{on } \Gamma)\,. \quad (8.16b)$$

Introducing it into right-hand side of the relation (8.16) we arrive at the relation

$$\begin{aligned}
U'^{+}[\nu(z)] = \beta_1^{*} U'^{-}(z) + \beta_2^{*} \overline{U'^{-}(z)} &+ \\
&+ \beta_3^{*}\,\mathrm{Re}\left\{\left(\frac{dz}{ds}\right)\frac{dU'^{-}(z)}{ds}\right\}, \quad (8.17)
\end{aligned}$$

on Γ. Here

$$\begin{aligned}
\beta_1^{*} &= \beta_1 + \beta_3 \frac{\sin\vartheta}{2k_s^{+}\sqrt[4]{\mathrm{K}^{-}}}\left(\frac{dz}{ds}\right)^2 + \frac{1}{2}\beta_4\frac{d}{ds}\left[\frac{\sin\vartheta}{k_s^{+}\sqrt[4]{\mathrm{K}^{-}}}\left(\frac{dz}{ds}\right)^2\right],\\
\beta_2^{*} &= \beta_2 + \beta_3 \frac{\sin\vartheta}{2k_s^{+}\sqrt[4]{\mathrm{K}^{-}}}\left(\frac{d\bar{z}}{ds}\right)^2 + \frac{1}{2}\beta_4\frac{d}{ds}\left[\frac{\sin\vartheta}{k_s^{+}\sqrt[4]{\mathrm{K}^{-}}}\left(\frac{d\bar{z}}{ds}\right)^2\right], \quad (8.18)\\
\beta_3^{*} &= \beta_4 \frac{\sin\vartheta}{k_s^{+}\sqrt[4]{\mathrm{K}^{-}}}\,.
\end{aligned}$$

Thus, the problem of the rigidity of a closed surface made up of two sections of ovaloids leads to the problem of the determination of a function $U'(z)$ which in G^+ and G^- satisfies the equation (8.12), is continuous in $G^+ + \Gamma$ and $G^- + \Gamma$, while on the curve Γ the relation (8.17) and at infinity the condition (8.14) are satisfied. This problem will hereafter be called Problem **H**. It should be observed that this problem is reduced to an analogous problem for analytic functions if the surfaces S^+ and S^- are sections of convex algebraic surfaces of the second order, since then $B \equiv 0$ (§4). Hence, it is of interest to investigate Problem **H** at first in the class of analytic functions. The investigation of this case, apart from the obvious geometrical interest, is worth attention also from the point of view of the general problem. If we make use of the formula (4.3) of Ch. III, §4, the general case can be reduced to the above particular form. This fact is in a full agreement with the geometrical nature of the problem.

It was already indicated that, in general, we may confine ourselves to the case in which $\nu(t) = t$. It occurs if on the surface $S^+ + S^-$ (in general not convex) a common isometric—conjugate net of lines is introduced, such that these lines cross the contact line continuously but their tangents and the coordinate angle suffer discontinuities.

It is readily observed that Problem **H** is a particular case of the problem stated in §10 of Ch. IV (problems (10.7)–(10.8)). If Problem **H** has only the trivial solution $U' \equiv 0$, this means that the closed surface $S^+ + S^-$ is rigid. If now a non-trivial solution of Problem **H** exists the surface is non-rigid. Applying methods of singular integral equations it can be proved that *Problem* **H** *can have only a finite number of linearly independent solutions. Hence, a closed surface composed of two sections of ovaloids admits only a finite set of infinitesimal bendings.* This result implies that every surface of the above indicated type can be converted into a rigid surface by

subjecting it to some (finite) number of additional point constraints (Ch. IV, §6). It is however possible to indicate some classes of surfaces which are rigid without additional conditions. This property is possessed, for instance, by all convex sectionally continuous surfaces and by a number of classes of non-convex surfaces (see §9 and §11). Apparently, in general, in the above type of surfaces, cases of non-rigidity are encountered much seldomer than cases of rigidity. In other words, as a rule such surfaces are rigid. This assertion however is of too general a nature and its mathematical content is not sufficiently definite. It is of interest therefore to carry out further researches on more definite geometrical and analytic criteria of rigidity of sectionally regular surfaces of positive curvature. Below, in §§9 and 11, some criteria of this kind will be given.

We now examine the case in which the contact line coincides with the tangency line of the surfaces S^+ and S^-. In this case $\sin\vartheta \equiv 0$ (see Fig. 3) and the formulae (8.18) show that the condition (8.17) takes the form

$$U'^+[\nu(z)] = \beta_1 U'^-(z) + \beta_2 \overline{U'^-(z)}\,, \qquad z \in \Gamma\,. \tag{8.19}$$

An investigation of the solubility of this problem can be reduced to the analogous problem in the class of analytic functions. In fact, the required solution can be represented in the form (see §4 of Ch. III)

$$\left.\begin{aligned} U'^-(z) &= \Phi^+(z)\, e^{\omega^+(z)}\,, \quad \text{if} \quad z \in G^+\,, \\ U'^-(z) &= \Phi^-(z)\, e^{\omega^-(z)}\,, \quad \text{if} \quad z \in G^-\,, \end{aligned}\right\} \tag{8.20}$$

where Φ^+ and Φ^- are functions holomorphic in G^+ and G^-, respectively, and ω^+ and ω^- are functions of class $C_\alpha(E)$, $\alpha = \dfrac{p-2}{p}$, $p > 2$. Besides Φ^+ is continuous in $G^+ + \Gamma$ and Φ^- is continuous in $G^- + \Gamma$ and satisfies at infinity the condition

$$\Phi^-(z) = 0(|z|^{-4})\,. \tag{8.21}$$

Introducing the expressions (8.20) into (8.19) we obtain

$$\Phi^+[\nu(z)] = a_1(z)\Phi^-(z) + a_2(z)\overline{\Phi^-(z)}\,, \qquad z \in \Gamma\,, \quad (8.22)$$

where

$$a_1 = \beta_1 e^{\omega^-(z)-\omega^+(\nu(z))}\,, \qquad a_2 = \beta_2 e^{\overline{\omega^-(z)}-\overline{\omega^+(\nu(z))}}\,. \quad (8.23)$$

The problem (8.22) when $\nu(z) = z$ was investigated by N. P. Vekua, [14*b], for a system of analytic functions. His results imply that the problem can have only a finite number of linearly independent solutions of finite order at infinity. (In our case this order is $k = -4$). The problem (8.22) for $a_2 \equiv 0$ but $\nu(z) \neq z$ was investigated by Kveselava, [39b]. It is seen from (8.23) and (8.16a) that this case occurs when

$$\left(\frac{\partial n}{\partial t}\right)^+ \left(\frac{\partial r}{\partial z}\right)^- = 0 \qquad \text{for} \qquad t = \nu(z)\,. \quad (8.23a)$$

In view of (8.15) this relation holds for tangency of the second order. It is readily seen that the condition (8.23a) is invariant with respect to a change of the coordinate system on the surface. In this case the investigation of the problem can be completed.

In fact, the relation (8.22) takes now the form

$$\Phi^+[\nu(z)] = a_1(z)\Phi^-(z)\,. \quad (8.24)$$

Moreover, with the help of the formulae (8.23) and (8.16a) it can be found that $a_1(z)$ does not vanish anywhere on Γ and the index $\varkappa = 0$. These conditions occur in two cases, namely: (1) if $S^+ + S^-$ is an ovaloid and (2) if $S^+ + S^-$ are concave in the same direction.

According to a theorem of Kveselava all solutions of the problem (8.24) having at infinity poles of a finite order are given by the formulae

$$\begin{aligned} \Phi^+(z) &= \chi(z)\Phi_1(z)\,, & z \in G^+\,, \\ \Phi^-(z) &= \chi(z)\Phi_2(z) + \chi(z)P(z)\,, & z \in G^-\,, \end{aligned} \quad (8.25)$$

where $P(z)$ is a polynomial in z, Φ_1 and Φ_2 are functions holomorphic in G^+ and G^-, respectively, and are uniquely determined by the given polynomial $P(z)$; also $\Phi_2(\infty) = 0$ and $\Phi_1 \equiv \Phi_2 \equiv 0$ when $P(z) \equiv 0$. Finally $\chi(z)$ is the so-called canonical solution of the problem, which does not vanish anywhere on the plane except for the point at infinity where the following expansion holds:

$$\chi(z) = z^{-\varkappa}\left(a_0 + \frac{a_1}{z} + \dots\right), \qquad a_0 \neq 0 .$$

Since in our case $\varkappa = 0$, in view of (8.25) for the solution of the problem satisfying at infinity the condition (8.21) we obviously have $P(z) \equiv 0$. Consequently, $\Phi^+ \equiv \Phi^- \equiv 0$. This completes the proof.

In the case of simple tangency the condition (8.23a) is not satisfied. Then however a_1 and a_2 fulfil the inequality

$$|a_1| > |a_2| \qquad (\text{on } \Gamma)$$

Nevertheless it follows from the recent results of Bojarski that also in this case the problem (8.19) has no solution.

Thus, it has been established that *a closed surface composed of two sections of distinct ovaloids is rigid if the contact line is the tangency line.*

In conclusion we make the following two remarks:

Remark 1. The problem investigated above can also be reduced to a problem of the form (10.5)–(10.6) stated in §10 of Ch. IV. To this end it is sufficient to map the surfaces S^+ and S^- by means of appropriately chosen isometric-conjugate coordinate systems, onto one (the same for both S^+ and S^-) domain G bounded by a simple closed curve Γ.

In §11.11 the problem of the rigidity of a non-convex surface composed of two spherical sections will be investigated in this way.

Remark 2. The method given in this subsection can also be applied to the problem of the rigidity of closed surfaces

composed of a finite number of sections of ovaloids $S_0, S_1, \dots, S_m$. For definitness let us assume that S_0 is an m-connected surface, i.e. S_0 is an ovaloid with the openings $L_1, \dots, L_m$ contacted with simply-connected convex surfaces $S_1, \dots, S_m$, respectively. By an appropriate choice of isometric–conjugate coordinates we can map homeomorphically the surface S_0 onto a domain G_0 bounded by simple closed curves $\Gamma_1, \Gamma_2, \dots, \Gamma_m$, the curves $\Gamma_2, \dots, \Gamma_m$ being situated inside Γ_1. Subsequently by an appropriate choice of isometric—conjugate nets of lines on $S_1, \dots, S_m$ these surfaces can be mapped homeomorphically onto an infinite domain G_1 bounded by the curve Γ_1, and the domains $G_2, \dots, G_m$ bounded by $\Gamma_2, \dots, \Gamma_m$, respectively. In this case every contact line L_j which we assume to be sufficiently smooth, is mapped homeomorphically in two distinct ways onto the curve Γ_j, and to every point M of the curve L_j there correspond on Γ_j two points—z and $t = \nu_j(z)$ where $\nu_j(z)$ is the function establishing the homeomorphic mapping of Γ onto itself. Writing the conjunction conditions (8.7) in the complex form we obtain the boundary conditions of the form

$$U_j'[\nu_j(z)] = \beta_{1j}^* U_0'(z) + \beta_{2j}^* \overline{U_0'(z)} + \\ + \beta_{3,j}^* \operatorname{Re}\left[\left(\frac{dz}{ds}\right)^2 \frac{dU_0'(z)}{ds}\right], \quad (8.26)$$

$$(z \in \Gamma_j;\ j = 1, 2, \dots, m),$$

where β_{ij}^* are fully definite functions of the point of the contour $\Gamma = \Gamma_1 + \dots + \Gamma_m$, which are given by formulae of the form (8.18), U_j' satisfies the equation (8.12) in the domain G_j and is continuous in $G_j + \Gamma_j$ ($j = 0, 1, \dots, m$); at infinity U_1' is subject to the condition (8.14). The problem (8.26) is readily reducible to an equivalent singular integral equation of the ordinary type. Hence, it can be proved that the problem can have only a finite number of linearly independent solutions.

Thus, sectionally regular surfaces (non-convex) composed in the way indicated above, can have only a finite

set of infinitesimal bending (see p. 500). Therefore, such surfaces can always be converted into rigid surfaces by subjecting them to a finite number of point constraints, for instance by a rigid clamping of a finite number of points of the surface.

If all contact lines are tangency lines the boundary condition (8.26) is simplified and takes the form

$$U_j'[\nu_j(z)] = \beta_{1j} U_0'(z) + \beta_{2j} \overline{U_0'(z)}\,, \qquad z \in \Gamma_j \qquad (8.27)$$
$$(j = 1, \ldots, m)\,.$$

An investigation of the solubility of this problem can also be reduced to the case of analytic functions. If it is possible to discover that the problem (8.27) has no non-trivial solutions satisfying the condition (8.14), this result is equivalent to the rigidity of a closed surface composed in the above indicated way along the tangency lines, of a finite number of convex surfaces.

The problem of the rigidity of closed surfaces composed, in an arbitrary way, of a finite number of sections of convex regular surfaces leads to even more complicated boundary value problems for generalized analytic functions. Further researches on this topic are therefore of great importance.

8.4. In the following subsections of §8 we shall consider cases in which in consequence of contact of surfaces there arise constraints ensuring the rigidity of at least one of the contacting surfaces.

We shall say that *the surfaces S^+ and S^- are rigidly contacted along L if under infinitesimal bending of the surface the variation of the contact angle vanishes, i.e.*

$$\delta\vartheta = \delta\theta^- - \delta\theta^+ = 0 \qquad (\text{along } L)\,. \qquad (8.28)$$

Let us assume that at every point of the curve L we have either $k_s^+ \neq 0$ or $k_s^- \neq 0$. For instance, if one of the surfaces S^+ or S^- is of positive principal curvature the above condition is always satisfied. If this condition is satisfied the curve L is called *the normal contact curve.*

If at a point of the curve the condition $k_s^+ = k_s^- = 0$ occurs this point is said to be *a singular point of the curve L.* If the curve L has only singular points it is called *the singular contact line.* At every singular point the line is tangent to the asymptotic directions of the surfaces S^+ and S^-.

In what follows until otherwise stated, we assume that the curve L is a normal contact line.

Since according to (8.8)

$$\mu = \delta\theta = -\frac{\sin\theta}{k_s^+}\,\delta k_s^- \equiv -\frac{\sin\theta}{k_s^-}\,\delta k_s^+$$

the relation (8.28) holds if one of the following conditions is satisfied:

(1) $\sin\vartheta = 0$ (along L),

(2) $k_s^+ \neq 0\,,\quad \delta k_s^- = 0$ or $k_s^- \neq 0\,,\quad \delta k_s^+ = 0\,.$

Conversely, if $\sin\vartheta \neq 0$ the condition (8.28), in view of (8.29) and (8.10), always implies the conditions

$$\delta k_s^+ = 0\,,\quad \delta k_s^- = 0\,,\quad \delta\tau_s^+ = \delta\tau_s^- \quad \text{(along } L)\,, \tag{8.29a}$$

i.e.

$$dV^+ = dV^- \equiv \delta\tau_s s \quad \text{(along } L)\,. \tag{8.29b}$$

If the surfaces S^+ and S^- are tangent along a curve L then $\boldsymbol{n}^+ = \pm\boldsymbol{n}^-$ (on L) and consequently $\sin\vartheta = 0$, i.e. $\delta\vartheta = 0$ along L.

Thus we have

THEOREM 5.13. *The tangency line of surfaces is the line of rigid contact, i.e. under arbitrary infinitesimal bendings of these surfaces the tangency lines are preserved.*

In other words the tangency (along a curve) of surfaces is preserved under infinitesimal bendings. This is due to the fact that S^+ and S^- have a common strip with the basis L and consequently under infinitesimal bendings a rotation of an elementary area of the strip can be regarded as commom for S^+ and S^-.

Since in the case under consideration (the surfaces are tangent) $\cos\theta^+ = \pm\cos\theta^-$, on the tangency lines of the two

surfaces which are not asymptotic lines or their envelopes, the following conditions are satisfied:

$$\delta k_s^+ = \pm \delta k_s^- , \qquad \delta \tau_s^+ = \delta \tau_s^- \qquad (\text{on } L) . \tag{8.30}$$

8.5. Let us assume that the curve L for the surface S^- is an asymptotic line or an envelope of such lines, and on the surface S^+ it is nowhere tangent to the asymptotic directions. Then

$$\cos \theta^- = 0 , \qquad \cos \theta^+ \neq 0 \qquad (\text{on } L) . \tag{8.31}$$

The second condition is always satisfied if S^+ is a surface of positive curvature. Under these conditions the first relation (8.10a) yields

$$\delta k \equiv \delta k_s^+ \cos \theta^+ = 0 , \qquad \text{or} \qquad \delta k_s^+ = 0 \qquad (\text{on } L) . \tag{8.32}$$

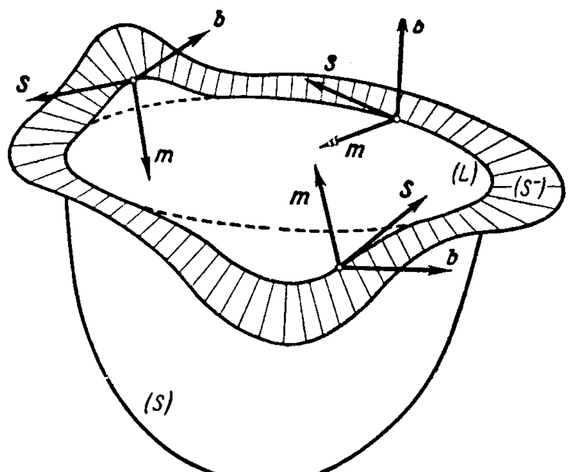

FIG. 4

In order to ensure the fulfilment of this condition it is sufficient to take for S^- an arbitrary asymptotic strip with the basis L, i.e. a strip containing L the normal of which coincides with the binormal of L at every point of this curve (Fig. 4).

We shall hereafter say that *the surface is bounded by asymptotic strips if every boundary curve of the surface is in contact with the corresponding asymptotic strip having as the basis the curve under consideration.*

Thus, *under infinitesimal bending of a surface bounded by asymptotic strips, the variation of the curvature of the boundary contour vanishes* ($\delta k = 0$). *If, moreover, the boundary contour is not tangent anywhere to the asymptotic directions of the surface, then along this contour the variation of the normal curvature vanishes* ($\delta k_s = 0$).

We observe that the conditions of the last part of this assertion are always satisfied for ovaloids, since on these surfaces there are no (real) asymptotic directions.

Let S be an ovaloid on which there are a finite number of openings and slits the contours of which we denote by $L_0, L_1, \ldots, L_m$. Let these contours be contacted with the surfaces $S_0, S_1, \ldots, S_m$, respectively, the contact of S with each of the surfaces S_k being rigid ($\delta\vartheta = 0$) on these sections of L_k which are not tangent to the asymptotic directions of the surface S_k. For this fact to take place it is sufficient that the contact lines $L_0, L_1, \ldots, L_m$ are the tangency lines (Theorem 5.13). In particular the surfaces S_k can be very narrow strips. If $\sin\vartheta \neq 0$, in a rigid contact the following boundary condition is satisfied:

$$\delta k = 0 \quad \text{or} \quad \delta k_s = 0\,.$$

In this case the ovaloid with openings will be called *the ovaloid with trimmed edges*. In particular if the openings of the ovaloid are plane curves and they are contacted with asymptotic (plane) strips we have the so-called truncated ovaloid. It is known, [33a], that *a truncated ovaloid is rigid.* We also have the following theorem:

THEOREM 5.14. *Let a regular ovaloid S with openings $L_0, L_1, \ldots, L_m$ be rigidly contacted with sectionally regular surfaces $S_0, S_1, \ldots, S_m$ along the contours of openings $L_0, L_1, \ldots, L_m$, respectively; these surfaces are not tangent anywhere to S along $L_0, L_1, \ldots, L_m$, i.e.* $\sin\vartheta \neq 0$ *everywhere*

on L. In this case under an infinitesimal bending of the surface $S^* = S+S_0+\dots+S_m$ the ovaloid S is rigid.

PROOF. Since we assume that along the contact lines there is no tangency, the boundary condition (8.32) is satisfied; in view of the formula (7.50) it can be written in the form

$$\delta k_s \equiv -\sqrt{a}\,\mathrm{Re}\left[\hat{w}'(z)\left(\frac{dz}{ds}\right)^2\right] = 0 \quad (\text{on } L)\,, \quad (8.33)$$

where $\hat{w}'$ is the complex bending function satisfying the equation

$$\mathfrak{C}'(w) \equiv \partial_{\bar{z}}w' - Aw' - \overline{B}\overline{w}' = 0 \quad (\text{in } G)\,. \quad (8.33\text{a})$$

Moreover, in view of its geometric meaning this function is evidently continuous in $G+\Gamma$ and satisfies also the conditions (6.42) ensuring the single-valuedness of the corresponding fields of displacement and rotation.

The index of the problem (8.33) can easily be computed (§3 of Ch. IV). We have

$$n = 2(m-1)\,, \quad (8.34)$$

where $m+1$ is the number equal to the connectedness of the surface. For definitness we assume that the contours of the openings of the ovaloid are smooth simple closed Jordan lines belonging to the class C^1_μ, $0<\mu\leqslant 1$. Moreover, we assume that the ovaloid belongs to the class $D_{k+3,p}$, $p>2$, $k\geqslant 0$. Then the coefficients of the equation (8.33a) belong to the class $D_{k,p}$, $p>2$ and the boundary Γ of G belongs to the class C^1_μ (see Ch. II, §6.9). Hence, we can apply to the problem (8.33) all the results of the preceding chapter.

Let us now consider the case of an ovaloid with one opening. Then $m=0$ and according to (8.34) $n=-2$. Hence, in view of Theorem 4.5 it follows that $\hat{w}'\equiv 0$. This completes the proof of our theorem for $m=0$. The case of a multiply-connected domain $(m\geqslant 1)$ will be examined below (§10.9).

8.6. Assume now that two convex simply-connected surfaces S' and S'' with edges L' and L'' are contacted

with the third surface S_0 the boundary strips of which S_0' and S_0'' are asymptotic strips with bases L' and L'', respectively. In this case, according to Theorem 5.14 the convex surfaces S' and S'' are rigid and S_0 in general is not rigid.

Thus, the compound surface $S_* = S' + S'' + S_0$ can undergo infinitesimal bendings only on account of S_0. In other words the presence of S_0 in S_* decreases rigidity of S_* and enables S_* to admit some bending deformations. Shells of such a structure can be of practical importance. The convex surfaces S' and S'' can carry the membrane loadings, the moments being taken up as bending deformations of *the weakening strip* S_0.

8.7. The asymptotic strips with the basis L lie on ruled surfaces of the form

$$\boldsymbol{r} = \boldsymbol{\varrho}(s) + [\alpha(s)\boldsymbol{s} + \beta(s)\boldsymbol{m}]t\,, \qquad \alpha^2 + \beta^2 = 1\,, \tag{8.35}$$

where s is the length of arc of the curve L, $\boldsymbol{\rho} = \boldsymbol{\rho}(s)$ is the equation of the curve L, α and β are given continuously differentiable functions of the length of arc s and t is a parameter. The condition (8.32) can therefore be ensured by contacting the contours of the ovaloid with ruled surfaces of the form (8.35).

We observe that for $\beta = 0, \alpha = 1$ we have a ruled surface constituted by the tangents of the curve L, i.e. a developable surface with the edge of regression L. For $\alpha = 0, \beta = 1$ we have a ruled surface constituted by the principal normals of the curve L. According to (8.8) and (8.10) the following conditions are satisfied on L:

$$\delta k_s^+ = 0\,, \qquad \delta\tau_s^+ = \delta\tau_s^-\,. \tag{8.36}$$

If S^+ is an ovaloid with the opening L then as we have seen above, it is rigid since along the opening $\delta k_s = 0$. Hence $\delta\tau_s = 0$ as well. Consequently, in view of the second relation (8.36) we have

$$\delta\tau_s^- = 0\,. \tag{8.37}$$

Thus, *a contact of an ovaloid with a ruled surface of the form* (8.35) *along the contour of the opening, ensures not only the rigidity of the ovaloid but moreover it imposes certain constraints on the ruled surface, equivalent to the condition* (8.37).

8.8. We now consider ovaloids in contact with cylindrical surfaces (Fig. 5).

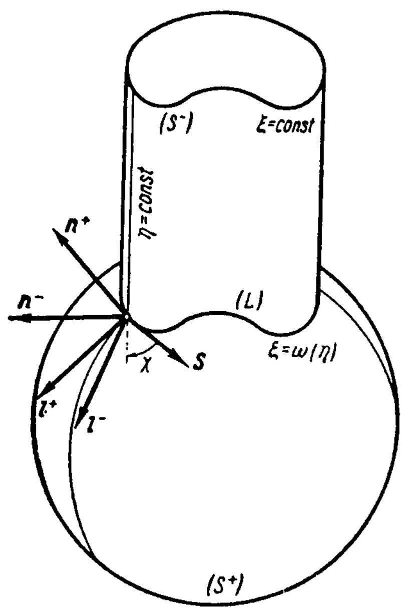

FIG. 5

We can choose on the cylinder a coordinate system such that the first fundamental quadratic form has the form $d\xi^2+d\eta^2 = ds^2$, and the lines $\eta = \text{const.}$ are the generators of the cylinder. Consequently

$$k_1 = 0\,, \quad k_2 = k_2(\eta) \neq 0\,. \tag{8.38}$$

Let us examine the case in which the contour L completely encloses the cylinder. Then the equation of the curve L on the cylinder has the form $\xi = \omega(\eta)$ where ω

is a single-valued continuously differentiable periodic function in η. Taking into account the formulae (7.81) and (7.84) we have

$$\delta k_s^- = -k_2 p^- \sin^2\chi^- - q^- \sin 2\chi^- ,$$

$$\delta \tau_s^- = -\frac{k_2}{2} p^- \sin 2\chi^- - q^- \cos 2\chi^- ,$$

where χ^- is the angle between the tangent of L and the generator of the cylinder, $\eta = \text{const.}$

Eliminating p^- from these equations we obtain (assuming $\sin\chi^- \neq 0$)

$$\cos\chi^- \, \delta k_s^- - \sin\chi^- \, \delta\tau_s^- = -q^- \sin\chi^- .$$

According to (8.10) this relation can be rewritten as follows:

$$\frac{\cos\chi^- \cdot \cos\theta^+}{\cos\theta^-} \delta k_s^+ - \sin\chi^- \cdot \delta\tau_s^+ - \sin\chi^- \frac{d\,\delta\vartheta}{ds} = -q^- \sin\chi^- . \tag{8.39}$$

Since $k_1 \equiv 0$ it follows from the second equation (6.33) that $q^- = \varphi(\eta)$, i.e. along every generator of the cylinder q^- has a constant value. Therefore if q^- is given along a curve L' situated on the cylinder S^- and enclosing it, the right-hand side of the relation (8.39) is also known on L. If we now express $\delta\vartheta$ in accordance with the formula (8.16b) and then δk_s^+ and $\delta\tau_s^+$ in accordance with the formulae (7.50), for the corresponding complex bending function w' which satisfies the equation (8.33a) we obtain the boundary condition

$$\mathrm{Re}\left[a\frac{dw'}{ds} + bw'\right] = f , \tag{8.40}$$

where a, b, f are known functions of the point of the contour L, the explicit expressions for which can evidently easily be written down. If L is a tangency line then $a \equiv 0$ and we have boundary conditions of the form $\mathrm{Re}[bw'] = f$.

8.9. The conjunction conditions on the contact lines can also be prescribed in a kinematic form, i.e. in terms of the components of the displacement vector and the rotation vector. If we decompose the displacement vector $\boldsymbol{U}$ on the unit vectors of the tangent, principal normal and binormal of the curve L, then it follows from the condition of continuity of the vector $\boldsymbol{U}$ that

$$u_s^+ = u_s^- , \quad u_m^+ = u_m^- , \quad u_b^+ = u_b^- . \qquad (8.41)$$

But in view of the formula (5.16)

$$\frac{du_s}{ds} = ku_m$$

and the second relation (8.41) is a consequence of the first one. Consequently, on the contact lines the following two conditions should be satisfied:

$$u_s^+ = u_s^- , \quad u_b^+ = u_b^- , \qquad (8.42)$$

which will be called *the kinematic conjunction conditions*.

Let L be the tangency line of the surfaces S^+ and S^-. If these surfaces constitute together a smooth surface then obviously the following conditions are satisfied:

$$u_s^+ = u_s^- , \quad u_l^+ = u_l^- \quad (\boldsymbol{l}^+ = \boldsymbol{l}^- = \boldsymbol{l}) . \qquad (8.43)$$

If now S^+ and S^- are tangent along L and do not together constitute a smooth surface, then we have

$$u_s^+ = u_s^- , \quad u_l^+ = \pm u_l^- \quad (\boldsymbol{l}^+ = \pm \boldsymbol{l}^- \equiv \boldsymbol{l}) . \quad (8.44)$$

If a closed surface is considered, composed of a finite number of convex regular surfaces, then the condition of continuity of the displacement field $\boldsymbol{U}^+ = \boldsymbol{U}^-$ on the contact lines leads to problems of the type considered above, in subsection 3. We shall not quote here the precise statement of these problems; they can easily be obtained by making use of the formulae (3.65) and (3.65a).

8.10. Let S^- be a ruled surface the equation of which has the form

$$\boldsymbol{r} = \boldsymbol{\varrho}(s) + \boldsymbol{\lambda}(s)t\,, \tag{8.45}$$

where $\boldsymbol{\rho} = \boldsymbol{\rho}(s)$ is the equation of the directrix L, $\boldsymbol{\lambda}(s)$ is a given unit vector of the generator of the surface which is a function of the length of arc s of the curve L, t is a parameter equal to the distance taken with the appropriate sign, from the point under consideration to the point of intersection of the corresponding generator with the directrix L. Let us represent the unit vector $\boldsymbol{\lambda}$ in the form

$$\boldsymbol{\lambda}(s) = \alpha(s)\boldsymbol{s} + \beta(s)\boldsymbol{l}^+ + \gamma(s)\boldsymbol{n}^+\,, \tag{8.46}$$

where $\alpha = \boldsymbol{\lambda s}$, $\beta = \boldsymbol{\lambda l}^+$, $\gamma = \boldsymbol{\lambda n}^+$. Then it follows from the condition of continuity of the displacement vector that

$$u_\lambda^+ \equiv \alpha u_s^+ + \beta u_{l^+} + \gamma u_0^+ = u_\lambda^-\,. \tag{8.47}$$

But we found above (Theorem 5.8) that u_λ^- is a function only of the length of arc s of the curve L, i.e. u_λ^- is constant along every generator of the ruled surface S^-. Let L_* be a new orthogonal directrix of the surface S^-. Let the surface S^- be in contact along the curve L_* with a rigid wall Σ orthogonal to S^-. It is then evident that $\boldsymbol{\lambda}$ is the normal of Σ and we have

$$u_\lambda^- = (\boldsymbol{\Omega}\times\boldsymbol{r})\boldsymbol{\lambda} + \boldsymbol{C}\boldsymbol{\lambda}\,, \tag{8.48}$$

where $\boldsymbol{\Omega}$ and $\boldsymbol{C}$ are constant vectors. In this case, in view of (8.47) and (8.48), we have the following condition on the contour L:

$$u_\lambda^+ \equiv \alpha u_s^+ + \beta u_{l^+} + \gamma u_0^+ = f(s) \qquad (f = \boldsymbol{\Omega r \lambda} + \boldsymbol{C\lambda})\,. \tag{8.49}$$

This is a boundary condition of the kinematic type.

If S^+ is a surface of positive curvature then introducing an isometric–conjugate net of lines, in view of (3.65) we can write the boundary condition (8.49) in the form

$$\operatorname{Re}\left[(\lambda^1 - i\lambda^2)w(z) + \frac{\lambda_0}{\mathrm{K}\sqrt{a}}\big(\sqrt{\mathrm{K}}w\big)_z\right] = f\,, \tag{8.50}$$

where λ^1, λ^2, $\lambda_0 = \gamma$ are the components of the unit vector λ, $\lambda = \lambda^\alpha \boldsymbol{r}_\alpha^+ + \lambda_0 \boldsymbol{n}^+$, and w is the complex displacement function satisfying the equation $\partial_{\bar{z}} w + Aw + B\bar{w} = 0$.

If $\gamma = 0$ we have

$$u_\lambda^+ \equiv \alpha u_s^+ + \beta u_l^+ = f\,. \tag{8.51}$$

In this case the ruled surface S^- is tangent along L to the surface S^+, and these surfaces constitute together a sectionally smooth surface.

For a surface of positive curvature the boundary condition (8.51) has the form

$$\mathrm{Re}[(\lambda^1 - i\lambda^2) w(z)] = f\,. \tag{8.52}$$

We have thus arrived at the boundary value problem investigated in the preceding chapter. Making use of the results obtained there we can establish a number of criteria for the rigidity of ovaloids with openings, in contact with ruled surfaces (see also §10 below).

8.11. Boundary conditions of the form (8.49) can also be obtained when the edge of a surface slides along another surface. This condition can for instance be set up by inserting into the openings *bushes*, the walls of which fit tightly the contours of the openings. In order to obtain these conditions in a natural way, on the basis of an examination of practicable mechanical and geometrical constraints, it is expedient to regard the surface as a thin elastic shell offering a considerable resistance to deformations of extension and shear and at the same time admitting infinitesimal bending deformations under comparatively small external forces. We agree to call such a surface *a flexible shell.*

Let us consider a flexible shell S bounded by a finite number of piecewise smooth simple Jordan curves the union of which will be denoted by L. We shall regard the surface S as double-sided and one side will be considered as positive, accordingly directing the normal $\boldsymbol{n}$

of the surface. As the positive direction on L we take that which leaves on the left the positive side of the surface. Among the boundary curves constituting L there may be both closed and open curves. The first will be called *openings* and the second *slits*. A slit has two edges—"the left edge" and "the right edge"—which will be denoted by the signs "+" and "−", respectively. Let $L_1, \dots, L_q$ be the slits and $L_{q+1}, \dots, L_m$ the openings. Then the boundary of S is

$$L = L_1^+ + L_1^- + \dots + L_q^+ + L_q^- + L_{q+1} + \dots + L_m .$$

Let there be inserted some elastic or rigid bodies into the openings and the slits; they will be called bushes. The union of the surfaces bounding the bushes will be denoted by Σ. Assume that the surfaces Σ are in a permanent contact with the surface S along the contour L, and in the course of deformation of S and Σ the contact is not violated. In other words we assume that the bushes fit the openings and slits so tightly that in the course of infinitesimal deformations there arise no clearences between the contour L and the surface Σ. This means that the displacements of the points of S and Σ in the direction of the normal to Σ are identical along the whole contact line, i.e.

$$\nu \boldsymbol{U} = \nu \widetilde{\boldsymbol{U}} , \qquad \text{or} \qquad u_\nu = \widetilde{u}_\nu \quad (\text{on } L) , \tag{8.53}$$

where $\boldsymbol{U}$ and $\widetilde{\boldsymbol{U}}$ are the displacement vectors of S and Σ, respectively, and ν is the unit vector of the normal to Σ (Fig. 6).

Let us observe that the relation (8.53) represents the condition that the contours of the openings and slits of the surface S are in a permanent contact with the surfaces Σ, and these contours can displace, sliding along the surfaces Σ. The surfaces Σ will therefore be termed ***the sliding surfaces*** and the constraints thus set up on the contours of the openings and slits of the surface S, ***the kinematic sliding constraints*** or ***the bush constraints***.

If the normal displacement u_ν of a point of the contact surface Σ is prescribed along the contours of the openings and slits, the condition (8.53) can be written in the form

$$u_\nu \equiv \nu_0 u_0 + \nu_l u_l = f \quad (\text{on } L)\,, \tag{8.54}$$

where $\nu_0 = \boldsymbol{n\nu}$, $\nu_l = \boldsymbol{\nu l}$ and $\boldsymbol{l}$ is the unit vector of the tangential normal of the contour L, $\boldsymbol{n}$ being the unit normal vector of the surface S.

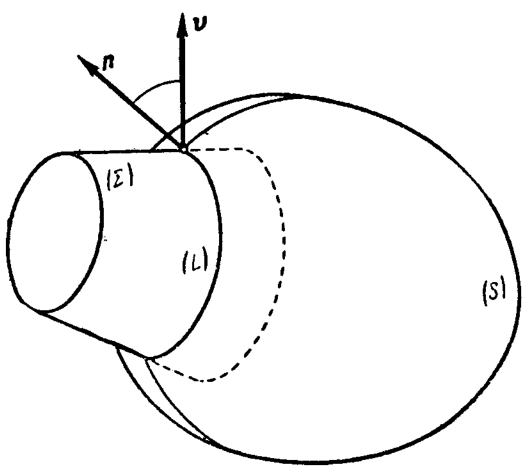

FIG. 6

For instance, a condition of the form (8.54) occurs when the bushes are perfectly rigid bodies. If, moreover, their surfaces are ideally smooth then (8.54) is the only condition which can be obtained for a bush constraint, since in this case arbitrary infinitesimal displacements tangent to Σ are possible, which do not violate the contact conditions (8.54).

If according to the formula (5.21) the normal displacement u_0 can be put in the form

$$u_0 = \frac{1}{k_s}\frac{du_s}{ds} + \operatorname{tg}\theta u_l\,, \tag{8.55}$$

the relation (8.54) takes the form

$$\nu_0 \frac{du_s}{ds} + a_0 u_l = k_s f \, , \tag{8.56}$$

where

$$a_0 = k_s(\nu_l + \nu_0 \operatorname{tg} \theta) = k \cos(\theta - \theta_0) \, , \qquad \cos\theta_0 = \boldsymbol{l}\nu \, . \tag{8.57}$$

Thus, we have obtained in a different way again the boundary condition of the form (8.49).

If the surfaces Σ are orthogonal to s, $\nu_0 = \nu \boldsymbol{n} = 0$, $\nu_l = \nu \boldsymbol{l} = 1$, and the condition (8.56) takes the form

$$u_l \equiv u_a l^a = f \, . \tag{8.58}$$

8.12. A boundary condition of the form (8.56) can also be obtained by clamping the edge of the surface.

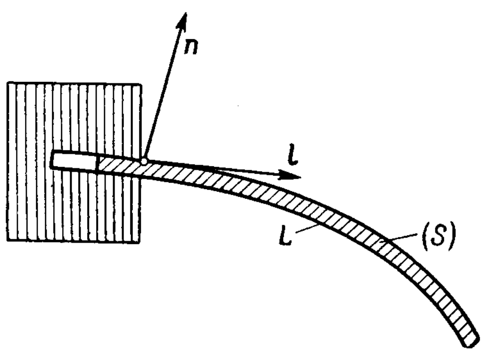

FIG. 7

Let a narrow boundary strip of the surface S be rigidly clamped between rigid walls of a sufficiently deep slit; we assume that the walls of the slit and the surfaces of the strip are ideally smooth (for instance there is a thin layer of a lubricant between them). Moreover, between the bottom of the slit and the contour L of the surface S there are fairly wide clearances (Fig. 7). Then the points of the contour L have only two degrees of freedom; in the course of deformation they can freely move along the

walls of the slit. Hence, along the contour L we have one condition, namely

$$\boldsymbol{n}\boldsymbol{U} = \boldsymbol{n}\tilde{\boldsymbol{U}}\,, \tag{8.59}$$

where $\boldsymbol{U}$ and $\tilde{\boldsymbol{U}}$ are the displacement vectors of the points of the surface S and the walls of the slit. Assuming that the normal displacement of the walls of the slit is a known function of the point of the contour L we can write the condition (8.59) in the form

$$\frac{du_s}{ds} + \sigma u_l = f \quad (\text{on } L)\,, \qquad \sigma = k \sin\theta\,, \tag{8.60}$$

where σ is the geodesic curvature of the contour L.

The above constraint to which the boundary condition (8.60) corresponds, will be called *a perfect clamping of the boundary strip of the surface (shell) S.*

§9. Some classes of rigid closed sectionally regular surfaces

In this paragraph we shall prove the rigidity of a fairly wide class of sectionally regular closed surfaces of non-negative curvature. This class contains convex sectionally regular closed surfaces, and also a family of non-convex closed surfaces. To this end we make use of the conjunction conditions on the contact lines derived in the preceding section, and an important integral identity of Blaschke [10]. First of all we shall give a derivation of this identity.

9.1. Let $\boldsymbol{V}$ and $\tilde{\boldsymbol{V}}$ be two rotation fields of a regular surface, bounded by a finite number of piecewise smooth curves $L_0, L_1, \ldots, L_m$, the union of which will be as before denoted by L. The side of S towards which the normal is directed will be taken as the positive side of the surface. The positive direction on L is this which leaves the positive side of the surface on the left.

Let us assume that the rotation fields are continuous in $S+L$ and belong to the class $D_{2,p}$, $p > 2$. Moreover, we assume that the surface S belongs to the class C^2.

Applying Green's identity we can write the formula

$$\int_L \boldsymbol{r}\boldsymbol{V}\,d\widetilde{\boldsymbol{V}} = \iint_S c^{\alpha\beta}(\boldsymbol{r}\boldsymbol{V}\widetilde{\boldsymbol{V}}_\beta)_\alpha dS\,, \qquad (\ )_\alpha \equiv \frac{\partial(\ )}{\partial x^\alpha}. \tag{9.1}$$

But

$$\begin{aligned} c^{\alpha\beta}(\boldsymbol{r}\boldsymbol{V}\widetilde{\boldsymbol{V}}_\beta)_\alpha &= c^{\alpha\beta}(\boldsymbol{r}\boldsymbol{V}\widetilde{\boldsymbol{V}}_{\alpha\beta}) + c^{\alpha\beta}(\boldsymbol{r}\boldsymbol{V}_\alpha\widetilde{\boldsymbol{V}}_\beta) + c^{\alpha\beta}(\boldsymbol{r}_\alpha\boldsymbol{V}\widetilde{\boldsymbol{V}}_\beta) \\ &= c^{\alpha\beta}(\boldsymbol{r}\boldsymbol{V}_\alpha\widetilde{\boldsymbol{V}}_\beta) = c^{\alpha\beta}\overset{*}{T}{}^\lambda_\alpha \overset{*}{\widetilde{T}}{}^\gamma_\beta \boldsymbol{r}\boldsymbol{r}_\lambda\boldsymbol{r}_\gamma = \hat{T}^{\alpha\lambda}\widetilde{T}^{\beta\gamma}c_{\alpha\beta}c_{\lambda\gamma}\boldsymbol{n}\boldsymbol{r}\,. \end{aligned}$$

In deriving this relation we have taken into account the formulae (6.1) and (5.4), and also the following relations:

$$c^{\alpha\beta}(\boldsymbol{r}\boldsymbol{V}\widetilde{\boldsymbol{V}}_{\alpha\beta}) = \boldsymbol{r}\boldsymbol{V}(\widetilde{\boldsymbol{V}}_{12} - \widetilde{\boldsymbol{V}}_{21})/\sqrt{a} = 0\,,$$

$$c^{\alpha\beta}\boldsymbol{r}_\beta\boldsymbol{V}\widetilde{\boldsymbol{V}}_\alpha = -c^{\alpha\beta}\overset{*}{\widetilde{T}}{}^\lambda_\alpha\boldsymbol{V}\boldsymbol{r}_\beta\boldsymbol{r}_\lambda = \widetilde{T}^{\alpha\beta}c_{\alpha\beta}\boldsymbol{n}\boldsymbol{V} = 0\,,$$

where $T^{\alpha\beta}$ and $\widetilde{T}^{\alpha\beta}$ are contravariant bending fields corresponding to the vectors $\boldsymbol{V}$ and $\widetilde{\boldsymbol{V}}$. Consequently, we have the identity

$$\int_L \boldsymbol{r}\boldsymbol{V}\,d\widetilde{\boldsymbol{V}} = \iint_S c_{\alpha\beta}c_{\lambda\gamma}T^{\alpha\lambda}\widetilde{T}^{\beta\gamma}\boldsymbol{n}\boldsymbol{r}\,dS\,, \tag{9.2}$$

valid for arbitrary two rotation fields $\boldsymbol{V}$ and $\widetilde{\boldsymbol{V}}$. If $\boldsymbol{V} = \widetilde{\boldsymbol{V}}$ we have the Blaschke formula

$$\int_L \boldsymbol{r}\boldsymbol{V}\,d\boldsymbol{V} = 2\iint_S \big(T^{11}T^{22} - (T^{12})^2\big)a\boldsymbol{r}\boldsymbol{n}\,dS\,, \tag{9.3}$$

or, according to the formulae (7.23)

$$\int_L \boldsymbol{r}\boldsymbol{V}\,d\boldsymbol{V} = 2\iint_S \frac{1}{a}(\delta b_{11}\delta b_{22} - \delta b_{12}\delta b_{12})\boldsymbol{r}\boldsymbol{n}\,dS\,. \tag{9.4}$$

Taking into account the relations (7.70) we have also

$$\int_L \boldsymbol{r}\boldsymbol{V}\,d\boldsymbol{V} = -2\iint_S (\mathrm{K}p^2 + q^2)\boldsymbol{r}\boldsymbol{n}\,dS\,. \tag{9.5}$$

9.2. We now apply the formula (9.5) to the proof of the rigidity of an ovaloid [10].

Assume that the ovaloid undergoes an infinitesimal bending the rotation vector $\boldsymbol{V}$ of which satisfies the conditions stated above. Let L be a simple closed piecewise smooth curve belonging to the ovaloid and dividing it into two sections S_1 and S_2. Applying to both these sections the formula (9.5) and adding term by term the resulting relations we obtain

$$\iint_S (\mathrm{K}p^2+q^2)\boldsymbol{rn}\,dS = 0\,. \tag{9.6}$$

We have here taken into account that the sum of curvilinear integrals along L vanishes in view of the continuity of the vector $\boldsymbol{V}$ and its derivatives $\boldsymbol{V}_\alpha$. Situating the origin of coordinates inside the ovaloids we have (the normal $\boldsymbol{n}$ being the outward normal)

$$\boldsymbol{rn} > 0 \qquad (\text{for } S)\,. \tag{9.7}$$

Besides, the principal curvature $\mathrm{K} = k_1k_2 \geqslant 0$ everywhere, and we assume that k_1 and k_2 can vanish simultaneously only on a set of measure zero. Under these conditions it follows from (9.6) that $\mathrm{K}p^2+q^2 \equiv 0$, i.e. $p \equiv q \equiv 0$. (The latter relations can be proved by applying the reasoning which will be given below, on p. 492). This completes the proof of the rigidity of regular ovaloids.

9.3. As was already indicated this method of proof can also be applied to a more general class of sectionally regular closed convex surfaces of non-negative curvature. Nevertheless, the previous reasoning cannot be carried over to a more general case without significant additions. Applying the formula (9.5) to regular sections of the surfaces and adding the relations thus obtained we arrive at a relation the right-hand side of which contains exactly the double integral (9.5) and the left-hand side is the sum of curvilinear integrals over the contact lines. As before, the rigidity of the surface follows immediately if it is proved that this sum vanishes. This is however not obvious. The difficulty is due to the fact that we cannot

a priori regard the rotation field as continuous on the whole surface. The formula (8.7) indicates that this field can have discontinuities of the first kind on the contact lines. Nevertheless, we shall find below that this difficulty can be overcome with the help of the formula (8.7). We shall thus prove below the rigidity of a very wide class of convex sectionally regular surfaces. The proof given in this subsection was given in the joint paper of Bojarski and the author [12].

Let S be a closed convex surface composed of a finite number of surfaces of non-negative curvature. Let us divide these surfaces into three groups. Group I contains those for which $k_1 = k_2 = 0$, i.e. plane figures. Group II contains the surfaces of zero curvature for which one of the principal curvatures is almost everywhere positive. Finally, group III contains the surfaces the Gaussian curvature of which satisfies the inequality $K \geqslant 0$ and the equality sign may hold on a (plane) set of measure zero.

We have seen above (§3.3) that infinitesimal bending of plane sections of the surface S is an indefinite problem. Nevertheless, we shall not exclude from our considerations the surfaces of group I, allowing them to have only trivial bendings

$$\boldsymbol{U} = \boldsymbol{\Omega} \times \boldsymbol{r} + \boldsymbol{C}, \quad \boldsymbol{V} = \boldsymbol{\Omega}, \tag{9.8}$$

where $\boldsymbol{\Omega}$ and $\boldsymbol{C}$ are constant vectors on each surface of group I. It is therefore evident that on each surface of group I we have

$$p \equiv q \equiv 0 \quad \text{(everywhere)}. \tag{9.9}$$

With respect to surfaces of group II we make the following additional assumption. Every asymptotic line of a surface of group II intersects at least at one point the boundary of a surface of group I or III. The contact lines of surfaces constituting S are assumed to be piecewise smooth. The union of these lines will be divided into two

groups, namely: (1) group α contains simple closed or open contact lines, (2) group β contains the contact lines on which there lies a finite number of points from which come out three or more contact lines. Let us agree to call these points vertices. Group β can consist of a number of connected sets $\beta_1, \ldots, \beta_n$, $n \geqslant 1$, and $\beta_i \beta_k = 0$ if $i \neq k$. The curves of every group β_j consist of a finite number of ribs, and each rib contains only two vertices—for the beginning and the end. We do not exclud the cases in which the rib is a closed curve, i.e. the beginning and the end coincide—the rib is a loop. On every rib and on every curve of group α, which we may call isolated ribs, we shall consider two edges—the left and the right, which will be denoted by the signs "+" and "−", respectively; the direction on L leaving the left edge on the left will be taken to be positive. Denoting by L the union of the contact lines we shall denote by L^+ and L^- the unions of the left and right edges.

Applying to each smooth section of the surface the Blaschke formula (9.5) and summing the relations over all sections we obtain

$$I = -2\iint_S (\mathrm{K}p^2 + q^2)\boldsymbol{rn}\,dS = \int_{L^+} \boldsymbol{r}\boldsymbol{V}^+ d\boldsymbol{V}^+ - \boldsymbol{r}\boldsymbol{V}^- d\boldsymbol{V}^- . \quad (9.10)$$

Here $\boldsymbol{V}^+$ and $\boldsymbol{V}^-$ are the limiting values of the vector $\boldsymbol{V}$ "from the left" and "from the right" of L. We shall as before assume that the origin of coordinates is situated inside S. Then $\boldsymbol{rn} > 0$ and, consequently,

$$I \leqslant 0 . \quad (9.11)$$

We have to prove that the inequality sign cannot occur, i.e. $I = 0$. Then it will follow that $\mathrm{K}p^2 + q^2 \equiv 0$, i.e. $\mathrm{K}p \equiv 0$, $q \equiv 0$. If $\mathrm{K} > 0$, $p \equiv q \equiv 0$. This means that the surfaces of group III are rigid. For the surfaces of group II, assuming for definitness that $k_1 \equiv 0$, $k_2 \neq 0$ we obtain from (6.33) that $k_2^2 B^2 p$ is independent of ξ, i.e.

$$k_2^2 B^2 p = \varphi(\eta) , \quad q \equiv 0 . \quad (9.12)$$

We now prove that $\varphi(\eta) \equiv 0$. The boundary L_* of the surface S_* under consideration (of group II) can evidently be represented in the form $L_* = L' + L''$ where L' is the union of the boundary arcs which are also boundary arcs of the adjacent surfaces belonging to the groups I and III, respectively; also none of the arcs belonging to L' contains an arc of the asymptotic line $\eta = \text{const.}$, nor they are tangent. L'' contains the boundary arcs of the surface S_* which are arcs of the line $\eta = \text{const}$. Let us denote by $S_{*\mathrm{I}}$ and $S_{*\mathrm{III}}$ the surfaces of group I and III, respectively, adjacent to S_*. Since $p \equiv q \equiv 0$ on $S_{*\mathrm{I}}$ and $S_{*\mathrm{III}}$, it follows from (8.10) that $\delta k_s^* = 0$ on L' (we assume that $\cos\theta_* \neq 0$ along L'). Since, according to (9.12) $q = 0$ on S_*, (7.81) implies that $p \equiv 0$ on L', since $\delta k_s^* \equiv$ $\equiv -k_2 \sin^2\chi \cdot p = 0$ on L', and $k_2 \sin^2\chi \neq 0$ on L'. But by hypothesis all lines $\eta = \text{const.}$ belonging to S_* intersect L'. Hence, in view of (9.12) $\varphi(\eta) \equiv 0$. It is therefore established that if $I = 0$, then

$$p \equiv q \equiv 0 \quad (\text{on } S) . \tag{9.13}$$

This proves the rigidity of every regular section of the surface S. It follows from (9.13) that on every regular section of S the vector $\boldsymbol{V}$ takes a constant value $\boldsymbol{\Omega}$. In order to prove the rigidity of the whole surface in the large we have still to establish that $\boldsymbol{\Omega}$ has the same constant value everywhere on S. But this fact follows from the relation (8.7), since in view of (9.13) and (8.8) $\delta\vartheta = 0$ on all contact lines.

Thus, it remains to prove that in (9.11) the inequality sign cannot occur. To this end it is sufficient to prove that

$$I \geqslant 0 . \tag{9.14}$$

The relation (9.10) can be written in the form

$$I = \int_{L^+} \boldsymbol{r}\boldsymbol{V}^+ d(\boldsymbol{V}^+ - \boldsymbol{V}^-) d\boldsymbol{V}^+ + \int_{L^+} \boldsymbol{r}\boldsymbol{V}^- d(\boldsymbol{V}^+ - \boldsymbol{V}^-) . \tag{9.15}$$

Integrating by parts we obtain

$$\int_{L^+} rV^- d(V^+ - V^-) = \int_{L^+} drV^-(V^+ - V^-) - \\ - \int_{L^+} V^-(V^+ - V^-)dr + \int_{L^+} r(V^+ - V^-)dV^- .$$

In view of the formula (8.7) the second integral in the right-hand side of the above relation vanishes and the first integral is equal to the increment of rV^-V^+ in moving once along all ribs in the positive direction

$$\delta = \int_{L^+} drV^-(V^+ - V^-) = \int_{L^+} drV^-V^+ = \{rV^-V^+\}_{L^+} . \quad (9.16)$$

Thus, the relation (9.15) has the form

$$I = \delta + \int_{L^+} r(V^+ - V^-)dV^+ + \int_{L^+} r(V^+ - V^-)dV^- ,$$

or, bearing in mind the formulae

$$V^+ - V^- = -\delta\vartheta s , \quad dV^\pm = \delta k_s^\pm l^\pm + \delta\tau_s^\pm s ,$$

$$\delta k_s^+ = -\frac{k_s^-}{\sin\vartheta}\delta\vartheta , \quad \delta k_s^- = -\frac{k_s^+}{\sin\vartheta}\delta\vartheta , \quad (9.17)$$

$$l^+ \times s = n^+ , \quad l^- \times s = n^- ,$$

we have

$$I = \delta + \int_L \delta\vartheta(\delta k_s^+ rn^+ + \delta k_s^- rn^-)ds$$

$$= \delta - \int_L \left(\frac{\delta k_s^-}{k_s^+}\right)^2 \sin\vartheta(k_s^- rn^+ + k_s^+ rn^-)ds . \quad (9.18)$$

Since $rn^+ > 0$, $rn^- > 0$, $\sin\vartheta \geqslant 0$, $k_s^\pm \leqslant 0$ (Fig. 8) we obtain

$$-\int_L \left(\frac{\delta k_s^-}{k_s^+}\right)^2 \sin\vartheta(k_s^- rn^+ + k_s^+ rn^-)ds \geqslant 0 . \quad (9.19)$$

It remains now to prove that

$$\delta \geqslant 0 . \quad (9.20)$$

For simplicity we shall confine ourselves to the case in which the surface S contains only ribs of group a. A proof of the inequality (9.20) for the general case is given in the paper [12] quoted above.

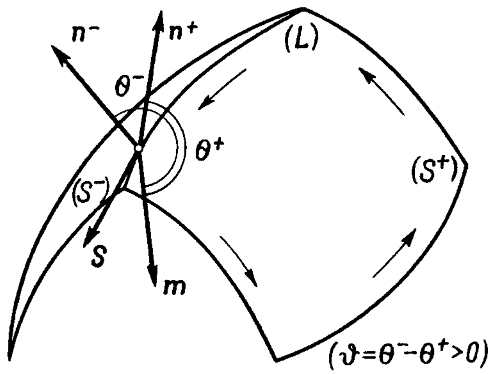

FIG. 8

Obviously $\delta = \delta_1 + \dots + \delta_n$ where δ_i is the increment of $\boldsymbol{r}\boldsymbol{V}^-\boldsymbol{V}^+$ on the rib L_i^+. If A_i and B_i are the beginning and the end of the rib L_i, then it is evident that

$$\delta_i = (r\boldsymbol{V}^-\boldsymbol{V}^+)_{B_i} - (\boldsymbol{V}^-\boldsymbol{V}^+)_{A_i}.$$

Since we assume that the vector $\boldsymbol{V}$ is continuous on every regular section of the surface S it is evident that $\delta_i = 0$ for every rib of group a. In fact, for a closed rib it is evident, since $A_i = B_i$. In the case of an open rib $\boldsymbol{V}_{A_i}^- = \boldsymbol{V}_{A_i}^+$, $\boldsymbol{V}_{B_i}^- = \boldsymbol{V}_{B_i}^+$ and, consequently, we again have $\delta_i = 0$. Hence, $\delta = 0$. In view of (9.19), therefore, we obtain making use of (9.18) that $I \geqslant 0$. The last result, by virtue of (9.11) yields the required relation $I = 0$.

We have thus proved

THEOREM 5.15. *A convex sectionally smooth surface of non-negative curvature is rigid if the contact lines are piecewise, smooth, simple, closed or open curves.*

We have indicated above that the theorem is preserved also in the presence of vertices and conical points [12].

9.4. It is readily seen that in the preceding section essentially a more general theorem is proved. In fact, the preceding reasoning remains valid if (1) the condition of convexity is replaced by the requirement that the surface under consideration is star-shaped with respect to an interior point O; this, obviously, ensures the fulfilment of the inequalities $\boldsymbol{rn}^+ \geqslant 0, \boldsymbol{rn}^- \geqslant 0$ along S, and (2) it is required that on the contact lines the following inequality holds:

$$k_s^+ rn^- + k_s^- rn^+ \leqslant 0 \quad (\text{on } L)\,. \tag{9.21}$$

Thus, we have the following ([113])

THEOREM 5.16. *Let us insert into the openings of a sectionally regular ovaloid S some ovaloids $S_0, S_1, \ldots, S_m$ in contact with S along the contours $L_0, L_1, \ldots, L_m$ of the openings. Assume that the surfaces $S_0, S_1, \ldots, S_m$ do not intersect. Further, assume that inside the surface $S_* = S + S_0 + \ldots + S_m$ there exists a point O (the origin of coordinates) from which the whole interior side of the surface S_* is visible. Finally, let us assume that the conditions* (9.21) *are satisfied on $L_0, \ldots, L_m$. Under these conditions the surface S_* is rigid.*

It is easy to give examples of non-convex closed surfaces satisfying the conditions of the last theorem. Let us consider the so-called truncated ovaloid which will result if we cut off from a regular ovaloid a finite number of caps with plane contours $L_0, L_1, \ldots, L_m$, and contact the latter with sections of a plane (covers). If we slightly press these covers inwards we can evidently obtain an infinite family of non-convex surfaces satisfying the conditions of the theorem.

Let us note that the conditions of the last theorem are sufficient to ensure rigidity of non-convex surfaces of non-negative curvature, but they are by no means necessary. As it was indicated in §8.3 a non-convex surface S composed of two sections of ovaloids S^+ and S^- tangent along the contact lines, is rigid. It is however readily

observed that such a surface is not starshaped with respect to an interior point and consequently does not satisfy the conditions of the theorem. It can be shown that these conditions are really not necessary, by examining the example of surfaces of revolution (§11). Apparently, a closed and non-convex surface S_* obtained as the result of contacting with an ovaloid S the ovaloids $S_0, \ldots, S_m$ imbedded into the ovaloid S and intersecting it only along one rib, but not intersecting each other, is always rigid.

9.5. Contrary to the above way of contacting, a contact of for instance two surfaces S and S_0 of positive curvature so that they constitute a closed surface S and S_0 does not enter the interior of S (or S does not enter the interior of S_0) can in general yield a non-rigid surface. Examples of such a contact of spherical segments and more general surfaces of revolution are given in §11 of this chapter.

In this connexion we give a theorem on infinitesimal deformations of compound surfaces having a plane of symmetry.*

Let the plane E cut off from an ovaloid S, in general sectionally regular, a simply-connected section S^+. By S^- we denote the mirror image of S^+ with respect to the plane E. Let us contact S^+ and S^- so that a closed surface S, in general non-convex, is obtained. The plane E is a plane of symmetry of S.

THEOREM 5.17. *The surface S admits such and only such non-trivial infinitesimal bendings which at the same time are the sliding bendings for S^+ (S^-) along the plane E.*

PROOF. Let us first consider the case in which S^+ is a regular surface. The plane $0\,x\,y$ of the Cartesian coordinate system is situated on the plane E. If $U^+ = (\xi^+, \eta^+, \zeta^+)$ is the displacement field of an infinitesimal sliding bending of S^+ along the plane E, then it is evident that in view of the system of equations (2.3), setting at the points

* This theorem was first proved by Bojarski.

of the surface S^- symmetric with respect to E, $\boldsymbol{U}^- = (\xi^-, \eta^-, \zeta^-) = (\xi^+, \eta^+, -\zeta^+)$ we obtain a new field $\boldsymbol{U} = \boldsymbol{U}^+$ on S^+ and $\boldsymbol{U} = \boldsymbol{U}^-$ on S^-, continuous on the whole surface S and trivial only when $\boldsymbol{U}^+$ is trivial on S^+. Let on the contrary $\boldsymbol{U} = \boldsymbol{U}^+$ on S^+ and $\boldsymbol{U} = \boldsymbol{U}^-$ on S^- be a continuous displacement field on S. Then the field $\boldsymbol{U}_1 = (\xi_1, \eta_1, \zeta_1) \equiv \{\xi^+(P) + \xi^-(P'), \eta^+(P) + \eta^-(P'), \zeta^+(P) - \zeta^-(P')\}$ where P' is the mirror image of P with respect to E, is evidently a field of an infinitesimal sliding bending on S^+. Thus, an arbitrary displacement field of the surface S generates a sliding field $\boldsymbol{U}_1$ on S^+. We shall now prove that under our assumptions the field $\boldsymbol{U}_1$ is trivial only if the field $\boldsymbol{U}$ is trivial on S. The field $\boldsymbol{U}_2 = \boldsymbol{U} - \frac{1}{2}\boldsymbol{U}_1$ has the components $(0, 0, \zeta)$ on L. According to a theorem on the truncated ovaloids * the field $\boldsymbol{U}_2$ is a field of an infinitesimal motion †. Hence $\boldsymbol{U} = \frac{1}{2}\boldsymbol{U}_1 + \boldsymbol{U}_2$ is trivial if and only if the field $\boldsymbol{U}_1$ is trivial. Moreover, we have proved that ζ is identical with the component of the trivial field on L, thus completing the proof.

If we make use of the generalization of the theorem on truncated ovaloids to the case of sectionally smooth ovaloids, the proof given above makes it possible to establish the validity of Theorem 5.17 for sectionally smooth S^+ and S^-.

The required generalization of the theorem on truncated ovaloids is immediately obtained if the method of proof of Yefimov ([33a]) be combined with the method of proof of Theorem 5.15.

We observe that the condition of convexity of S^+ is essential. We can compose of two spherical sections a surface of revolution S^+, even uniquely projectable on E,

* The remark that we can refer here to the theorem on truncated ovaloids was made by Yefimov. The original proof of Bojarski was in its later part in fact a proof (by a method distinct from Yefimov's) of the required here theorem on truncated ovaloids.

† This fact follows also from Theorem 5.14 or Theorem 5.19 (p. 515). It is sufficient to observe that $U_{2s} = 0$ on L.

which is not a part of a convex surfaces, such that the surface S composed of S^+ and its mirror image S^- with respect to the plane E, is non-rigid. The appropriate example can be computed by the method of §10.

The theorem is also not valid any more if S^+ has negative curvature on some of its sections. This means that the field U_1 can turn out to be trivial even if U is non-trivial.

§10. Some classes of rigid convex surfaces with edges

10.1. In the preceding sections we examined some classes of rigid closed surfaces and indicated many criteria of rigidity of surfaces with edges.

In this section new criteria of rigidity will be presented. We shall also consider some new forms of non-rigid constraints compatible with infinitesimal bendings and uniquely defining (non-trivial) displacement field.

Surfaces with edges (open surfaces) are always non-rigid, i.e. they admit non-trivial infinitesimal bendings if they are subject to no constraints. In fact, if we deal with a surface of positive curvature with edges, the rigidity problem is reduced to ascertaining whether there exists at least one complex function $\sqrt{K}p+iq$, continuous and bounded on S, which, as we have seen in §7.7, uniquely determines the bending field of the surface. But this function satisfies an equation of the form $\partial_{\bar{z}}w+Aw+B\overline{w}=0$ $(A, B \in L_p,$ $p>2)$ in the domain G onto which the surface is homeomorphically mapped by means of the isometric–conjugate coordinate system. In the case of an ovaloid with edges this domain does not cover the entire plane and its boundary contains continuation. The equa indicated above therefore always has continuous bounded non-trivial solutions. But to every such solution there corresponds a non-trivial bending field of the surface. Hence, surfaces of positive curvature with edges can be rigid only if some external constraints are present, which hereafter will be called *rigid constraints*. Obviously, not all constraints ensure rigidity of the surface. Every constraint evidently restricts

the possible forms of infinitesimal bendings but does not always exclude them completely. Of special interest are the nonrigid bonds which allow only a finite set of infinitesimal bendings, i.e. there exist a finite number of linearly independent displacement fields $\boldsymbol{U}^{(1)}, \ldots, \boldsymbol{U}^{(k)}$ which are compatible with the present constraints, and any displacement field compatible with these constraints is given in the form

$$\boldsymbol{U} = c_1 \boldsymbol{U}^{(1)} + \ldots + c_k \boldsymbol{U}^{(k)}, \tag{10.1}$$

where $c_1, \ldots, c_k$ are arbitrary real constants.

If at the same time all fields $\boldsymbol{U}^{(j)}$ are trivial the surface is *geometrically rigid*. In this case it is obvious that $k \leqslant 6$. If, now, among the fields $\boldsymbol{U}^{(j)}$ there are also non-trivial fields the surface will be non-rigid. It will always be so if $k > 6$. If the surface admits a finite set of non-trivial displacement fields we shall say that the constraints present are *almost-rigid*. The non-rigidity of this kind can also be described by the fact that the constraints admit a finite set of bending fields. In other words, there exist a finite number of linearly independent complex bending functions $\hat{w}'^{(1)}, \ldots, \hat{w}'^{(l)}$, and any other bending function compatible with the constraints present is a linear combination of the form

$$\hat{w}' = c_1 \hat{w}^{(1)} + \ldots + c_l \hat{w}'^{(l)} . \tag{10.2}$$

The number l can in this case be called *the degree of freedom of the almost-rigid constraints present.*

Below we shall consider many examples of constraints of this kind. By imposing on the surface a number of new point constraints (Ch. IV, §6) almost-rigid constraints can always be converted into rigid constraints.

When examining rigid constraints it should be taken into account that such constraints can possess various degrees of rigidity. If, rigid constraints being present, we impose new constraints on the surface, then evidently we obtain again rigid constraints. For instance, if a surface of positive curvature is in contact with an inextensible perfectly

flexible cord we shall have, as it was shown in §3.5, a rigid surface. If the same surfaces is in contact with one more inextensible cord and some other constraints are imposed, then it is evident that the rigidity of the surface will not be violated, and on the contrary, it will be increased. However it should be borne in mind that an increase of rigidity is not always expedient from the practical point of view. For instance, if the surface is a thin flexible shell and rigid constraints are applied, then every deformation of such a shell will result in appearance of extensions and shears, since infinitesimal bendings are excluded by the constraints present. This will create tension or compression and shear forces which under certain circumstances of concentration of stresses, can result in a considerable weakening and even destruction of constraints, or in creation of fissures and foldings. Thus, the presence of rigid constraints in certain cases can have a negative influence on the behaviour of the structure, especially if the constraints have small elasticity and flexibility or if the shell has a small resistance on the deformations due to compression or shear. Therefore, imposing on the shell (surface) some rigid constraints the above circumstances should be taken into account and constraints applied with circumspection.

In connexion with these considerations we introduce the concepts of *correct* or *optimum*, and *incorrect* or *superoptimum* rigidity. By incorrect rigid constraints we understand constraints which admit certain relaxation remaining, however, rigid. For instance such rigid constraints are set up if a surface of positive curvature is contacted with an inextensible cord. If we shorten the length of the cord we do not violate the rigidity of the surface, although the constraints are obviously weakened. The concept of correct or optimum rigidity will be introduced on the basis of an examination of linear constraints of type $R(U) = f$ where R is a homogeneous additive operator associating with every displacement field U a function given on a set of points of the surface and belonging to certain family

of functions. Of such form are many constraints set up by means of contacting or inserting bushes into openings (§8). Let us assume that the condition $R(U) = f$ enables us to associate uniquely with any given function f a definite bending field, the latter depending continuously on f, in a definite sense. Evidently, in this case for $f \equiv 0$ the bending field is reduced to the zero field. Under these conditions the constraints of the form $R(U) = f$ will be called *correct constraints*, and the corresponding rigid constraints $R(U) = 0$ will be called *correctly rigid*, or *optimally rigid constraints.*

We also observe that correct rigid constraints do not admit further weakenings, remaining rigid; but if they undergo weakenings entirely natural for the problem under consideration, new constraints result, which are always compatible with a displacement field of infinitesimal bendings of the surface.

In general there are difficulties in establishing optimum rigid constraints in practice. This is due to the fact that infinitesimal bendings of a surface are a very particular case of the general deformation of the surface and occur only if there are present geometrical and kinematical constraints of a very special type, the setting up and maintaining of these constraints requiring some special geometrical conditions and mechanical devices. In many cases, as we shall see below, such constraints can be set up by means of *contacting surfaces* and also by applying *bush constraints.* In the following subsections we shall investigate more thoroughly some classes of constraints which lead to the boundary value Problem A investigated in the preceding chapter.

10.2. Let there be given at every point of the boundary L of the surface S a direction $\boldsymbol{t}$ tangent to S, making the angle φ with the direction of the tangential normal $\boldsymbol{l}$, φ being a variable quantity, i.e. a function of point of the contour L. Assume that along L at every point the projections of the displacement vector $\boldsymbol{U}$ on the direction $\boldsymbol{t}$ are

given, i.e.

$$u_t \equiv u_l \cos\varphi + u_s \sin\varphi = f \quad (\text{on } L)\,, \tag{10.3}$$

where f is a known function of point of the contour L. Subsequently the functions prescribed on L will be regarded as continuous in the Hölder sense. Since the vector $\boldsymbol{U}$ can be represented by means of the complex displacement function w satisfying the equation (3.59), in view of the formula (3.65) we have

$$u_t \equiv \operatorname{Re}\left[w(z)\frac{d\bar{z}}{dt}\right], \quad \frac{dz}{dt} = t^1 + it^2\,, \quad t^\alpha = \frac{dx^\alpha}{dt}\,, \tag{10.4}$$

where t^α are contravariant components of the vector $\boldsymbol{t}$. The conditions of the form (10.3) can be obtained, as we have seen in §8, by contacting surfaces or by means of bush constraints. According to (10.4) the condition (10.3) can be written in the form

$$\operatorname{Re}\left[w(z)\frac{d\bar{z}}{dt}\right] = f \quad (\text{on } \Gamma)\,. \tag{10.5}$$

Thus we have arrived at a problem which hereafter will be termed *Problem* $\mathbf{A}_t$:

PROBLEM $\mathbf{A}_t$: *It is required to find a solution* $w(z)$ *of the equation*

$$\partial_{\bar{z}} w + A(z)w + B(z)\overline{w} = 0 \quad (\text{in } G)\,, \tag{10.6}$$

which is continuous in $G+\Gamma$ *and satisfies the boundary condition* (10.5).

Let us call this problem *the fundamental kinematic boundary value problem*. It belongs to the group of problems investigated in Ch. IV.

10.3. Before proceeding to an application of the results of Ch. IV to the above problem, let us examine some properties of the functions A and B, and the contour Γ in terms of the degree of smoothness of the surface S and its boundary L.

We assume that $S+L$ is a section of a surface S_0 of positive curvature belonging to the class $D_{k+3,p}$, $p > 2$, $k \geqslant 0$, the boundary curves $L_0, L_1, \ldots, L_m$ the union of

which constitutes the total boundary L of the surface, being of the class $C_{\mu}^{k'+1}$ $(0 \leqslant k' \leqslant k, 0 < \mu \leqslant 1)$. Under these assumptions, as we have seen in Chapter II (§6.8) a and $\psi \in D_{+1,p}(G_0)$ where G_0 is the domain on the plane $z = x+iy$ onto which the surface S_0 is homeomorphically mapped by means of a homeomorphism

$$x = x(\xi, \eta), \quad y = y(\xi, \eta) \tag{10.7}$$

of the second fundamental quadratic form

$$\Pi \equiv \mathrm{A}^2 k_1 d\xi^2 + \mathrm{B}^2 k_2 d\eta^2 . \tag{10.8}$$

Since by hypothesis $A^2, B^2, k_1, k_2 \in D_{k+1,p}$ the homeomorphism (10.7) belongs to the class $D_{k+2,p}$. According to the formulae (3.61) we readily observe that the coefficients of the equation (10.6)

$$A(z), \quad B(z) \in D_{k,p}(G_0) \quad (k \geqslant 0, \ p > 2) . \tag{10.9}$$

Hence, in view of Theorem 3.2 every continuous solution of the equation (10.6) belongs inside the domain G_0 to the class $D_{k+1,p}$ (G_0 is the homeomorphic image of the surface S_0). By virtue of the formula (3.65) we easily find that the corresponding displacement field $\boldsymbol{U}$ belongs to the class $D_{k,p}(G_0)$, and the tangent component of this field

$$U_s = -\frac{1}{\sqrt{aK}}(w\boldsymbol{n}_z + \overline{w}\boldsymbol{n}_{\bar z}) , \tag{10.10}$$

obviously belongs to the class $D_{k+1,p}(G_0)$.

The boundary Γ of the domain G which consists of the curves $\Gamma_0, \dots, \Gamma_m$ representing the homeomorphic images of $L_0, \dots, L_m$, belongs to the class $C_{\mu}^{k'+1}$. With respect to the angle φ defining the tangent direction $\boldsymbol{t}$ on the surface along L, we assume that it belongs to a class $C_{\nu}^{k''}(L)$ $(k'' \geqslant 0, 0 < \nu < 1)$. Then the function

$$\lambda_t(z) \equiv \frac{dz}{dt} , \tag{10.11}$$

is evidently continuous in the Hölder sense along Γ and vanishes nowhere on Γ. Hence, we can apply the results of the Chapter IV to Problem $\mathbf{A}_t$. Since we assume that the right-hand side f of the boundary condition (10.5) is Hölder continuous, the solution of Problem $\mathbf{A}_t$ belongs to the class $D_{k+1,p}(G)$, $p > 2$, inside G and it is continuous in the Hölder sense in $G+\Gamma$ (Theorems 3.2 and 4.1).

10.4. We now consider the homogeneous boundary value Problem $\mathring{\mathbf{A}}_t'$ adjoint to $\mathbf{A}_t$. It can be stated as follows:

PROBLEM $\mathring{\mathbf{A}}_t'$. *It is required to find the solution of the adjoint equation*

$$\partial_{\bar z} w' - A(z)w' - \overline{B(z)}\overline{w'} = 0 \qquad (\text{in } G)\,, \tag{10.12}$$

which is continuous in $G+\Gamma$ and satisfies the boundary condition

$$\mathrm{Re}\left[w'(z)\frac{dz}{ds}\frac{dz}{dt}\right] \equiv \mathrm{Re}[w'(z)\lambda_s(z)\lambda_t(z)] = 0\,. \tag{10.13}$$

The necessary and sufficient condition of solubility of Problem $\mathbf{A}_t$ consists in the fulfilment of the following relations (see Ch. IV, §2):

$$\int_L f w_j' \frac{dz}{\overline{\lambda_t(z)}} = 0 \qquad (j = 1, \ldots, l_t')\,, \tag{10.14}$$

where w_j' is the complete system of linearly independent solutions of the adjoint homogeneous Problem $\mathring{\mathbf{A}}_t'$ and l_t' is the number of solutions of this problem.

According to the formula (6.45) we have

$$\boldsymbol{t}\boldsymbol{T}_{(l)} \equiv T_{(u)} = -\sqrt{a}\,\mathrm{Re}\left[w'(z)\frac{dz}{ds}\frac{dz}{dt'}\right], \tag{10.15}$$

where $\boldsymbol{t}'$ is the direction tangent to S at the boundary point; it is perpendicular to $\boldsymbol{t}$ and $\boldsymbol{t}\times\boldsymbol{t}' = \boldsymbol{n}$. In deriving this formula we have made use of the relations

$$t_\alpha = c_{\alpha\beta}t'^{\beta} \qquad t^{\alpha} = c^{\alpha\beta}t_\beta'\,. \tag{10.16}$$

Taking into account the condition (10.13) we have

$$T_{(u)} = \frac{1}{i} w'(z) \frac{\dfrac{dz}{ds}}{\dfrac{d\bar{z}}{dt}} \equiv \frac{1}{i} w'(z) \frac{\lambda_s(z)}{\overline{\lambda_t(z)}}, \tag{10.17}$$

where w' is a solution of Problem $\mathring{A}_t'$.

In deriving the above relations we took into account the relationship

$$\frac{d\bar{z}}{dt}\frac{dz}{dt'} - \frac{dz}{dt}\frac{d\bar{z}}{dt'} = \frac{2i}{\sqrt{a}}, \tag{10.18}$$

proved in § 6 of Chapter II (formula (6.47)).

In view of (10.17) the relation (10.14) takes the form

$$\int_L f T_{(u)}^{(j)} dz = 0 \qquad (j = 1, \ldots, l_t'). \tag{10.19}$$

We shall see in the next chapter that Problem $\mathring{A}_t'$ has a statical interpretation, since w' is the complex stress function. But in the case of a simply-connected domain this function, as we have seen above in §6.6, can always be interpreted as the complex bending function connected with the rotation field in accordance with the formula (6.44). This formula implies that

$$\begin{aligned} t'\frac{dV}{ds} &\equiv -\delta k_s \sin\varphi + \delta\tau_s \cos\varphi \\ &\equiv \sqrt{a}\,\mathrm{Re}\left[w'(z)\frac{dz}{ds}\frac{dz}{dt}\right]. \end{aligned} \tag{10.20}$$

Thus, in the case of a simply-connected domain the boundary condition (10.13) can be written in the form

$$-\delta k_s \sin\varphi + \delta\tau_s \cos\varphi = 0. \tag{10.21}$$

In the case of a multiply-connected domain the last condition is equivalent to the condition (10.13) only when the function w' is subject to the additional conditions (6.42). Only then the function w' is the complex

bending function. The boundary condition $\delta k_s = 0$ (on L) which was examined above, in §8.5, is a particular case of the condition (10.21). In this case $\varphi = \frac{\pi}{2}$, i.e. $\boldsymbol{t}$ coincides with $\boldsymbol{s}$.

10.5. The index of the boundary value Problem $\mathbf{A}_t$ is equal to the increment of $\arg \frac{dz}{dt}$ when the point z moves once around the contour Γ of the domain G in the positive direction (§3 of Ch. IV). This number will be denoted by n_t.

Let us call two directions $\boldsymbol{t}$ *and* $\boldsymbol{t}^*$ *tangent to* S *directions of the same class* (*for instance directions of the class* $\boldsymbol{t}$), *if* $n_t = n_{t^*}$. Let φ^* be the angle between $\boldsymbol{t}$ and $\boldsymbol{t}^*$. If the absolute value of this angle at every point of L does not exceed π, i.e. $|\varphi^*| < \pi$ it is clear that $\boldsymbol{t}$ and $\boldsymbol{t}^*$ are of the same class. If $\boldsymbol{t}$ belongs to the class $\boldsymbol{l}$ (for instance if it coincides with $\boldsymbol{l}$ or $\boldsymbol{s}$) it is evident that

$$n_t = n_l = n_s = 1 - m \, . \tag{10.22}$$

If moreover the angle φ^* is Hölder continuous, i.e. $\varphi^* \in C_\nu(\Gamma)$ and its norm in the metric of $C_\nu(\Gamma)$ satisfies the condition $C_\nu(\varphi^*, \Gamma) < \varepsilon$ where ε is a sufficiently small positive number, we shall say that *the direction* $\boldsymbol{t}^*$ *is the normal perturbation of the direction* $\boldsymbol{t}$.

Let us now consider Problem $\mathbf{A}_{t^*}$ with the boundary condition

$$\operatorname{Re}\left[w^*(z)\frac{d\bar{z}}{dt^*}\right] = f^* \, , \tag{10.23}$$

where w^* is a solution of the equation

$$\partial_{\bar{z}} w^* + A^*(z) w^* + B^*(z) \overline{w}^* = 0 \, , \tag{10.24}$$

continuous in $G+\Gamma$, A^* and B^* being functions of the class $L_p(G+\Gamma)$, $p>2$, belonging as well as A and B to a compact in $L_p(G+\Gamma)$ family of functions, and satisfying the conditions

$$L_p(A-\overline{A^*}, G) < \varepsilon \, , \quad L_p(B-B^*, G) < \varepsilon \, , \quad p > 2 \, , \tag{10.25}$$

The right-hand side of the boundary condition (10.23) satisfies the condition

$$C_\nu(f-f^*, \Gamma) < \varepsilon\,, \qquad 0 < \nu \leqslant 1\,. \tag{10.26}$$

Thus, we pass from Problem $\mathbf{A}_t$ to another Problem $\mathbf{A}_{t^*}$ by varying the coefficients of the equation and the boundary condition, as well as the right-hand side of the boundary condition. Let us call *the boundary value Problem* $\mathbf{A}_{t^*}$ *the normal perturbation of Problem* $\mathbf{A}_t$.

We shall now prove that *if Problem* $\mathbf{A}_t$ *is quasi-correct, then every Problem* $\mathbf{A}_{t^*}$ *for sufficiently small values of* ε *is also quasi-correct.*

In fact, quasi-correctness of Problem $\mathbf{A}_t$ means that $l_t = 2n-m+1 \geqslant 0$ and $l_t' = 0$ (Ch. IV, §6.4). Then according to Theorem 6 of Bojarski (Appendix to Ch. IV) $l_{t^*} = 2n-m+1$ and consequently $l_{t^*}' = 0$. This was to be proved.

For $n > m-1$ Problem $\mathbf{A}_t$ is always quasi-correct and for $n \leqslant m-1$ it is in general incorrect. Therefore for $n > m-1$ Problem $\mathbf{A}_t$ is said to be *normally quasi-correct problem.*

It should be noted that the investigation of the problem of the normal correctness of Problem $\mathbf{A}_t$ is of considerable geometrical importance. An elucidation of this problem enables us for instance to examine the degree of stability of properties of the constraints (10.3). For example, if they are compatible with infinitesimal bendings of the surface it is interesting to know whether this property is preserved under a small perturbation of parameters determining the constraints. In practice constraints are always set up approximately; it is therefore important to know the extent of influence of errors which can occur in setting up the constraints, on the character of deformation of the surface (see also Ch. VI, §5.12).

10.6. We now procceed to the investigation of the problem of solubility of Problem $\mathbf{A}_t$. Let us denote by l_t and l_t' the numbers of solutions of the homogeneous

Problems $\mathring{\mathbf{A}}_t$ and $\mathring{\mathbf{A}}'_t$. In view of Theorem 4.8 these numbers are connected by the relation

$$l_t - l'_t = 2n+1-m\,, \quad \text{where} \quad n = n_t\,. \tag{10.27}$$

We first consider the case in which $n < 0$. Then according to Theorem 4.5 $l_t = 0$ and consequently

$$l'_t = -2n+m-1\,. \tag{10.28}$$

Thus, *in presence of bonds of the form* $u_t = 0$ *the surface is rigid if* $n_t < 0$. But rigid constraints of this type are not correct. Under normal perturbations they are carried into constraints of the form $u_{t*} = f^*$ which are not always compatible with infinitesimal bendings of the surface. The adjoint homogeneous Problem $\mathring{\mathbf{A}}'_{t*}$ has $l'_t = -2n+m-1$ solutions $w'^*_j(t)$ $(j = 1, \ldots, -2n+m-1)$ and consequently the following relations should be satisfied:

$$\int_L f^* w'^*_j \frac{dz}{\lambda_{t*}(z)} = 0 \qquad (j = 1, \ldots, -2n+m-1)\,, \tag{10.29}$$

These relations obviously do not hold for an arbitrary continuous function f^*. Hence, *for* $n_t < 0$ *the rigid* constraints $u_t = 0$ *are incorrect.* It should be observed however that *for normal perturbations of the form* $u_{t*} = 0$ *the property of rigidity of the constraints* $u_t = 0$ *is preserved.*

We note that the case examined above occurs for ovaloids with three or more openings $(m > 1)$ if the direction $\boldsymbol{t}$ belongs to the class $\boldsymbol{l}$ or $\boldsymbol{s}$. Then $n_t = 1-m < 0$ and the homogeneous of the form

$$u_t = 0 \qquad (\text{on } L) \tag{10.30}$$

are rigid.

Let $n > m-1$. Then according to Theorem 4.10 the homogeneous Problem $\mathring{\mathbf{A}}_t$ has exactly $l_t = 2n+1-m$ linearly independent solutions, and the non-homogeneous Problem $\mathbf{A}_t$ is always soluble.

Consequently the displacement field satisfying the conditions of Problem $\mathbf{A}_t$ has the form

$$\boldsymbol{U} = \boldsymbol{U}_0 + \sum_{k=1}^{2n+1-m} d_k \boldsymbol{U}_k \,, \tag{10.31}$$

where $\boldsymbol{U}_0$ is a displacement field compatible with the conditions of the non-homogeneous Problem $\mathbf{A}_t$, $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_{2n+1-m}$ are linearly independent displacement fields satisfying the homogeneous boundary condition $u_t = 0$ on L, and d_k are arbitrary real constants. Thus, Problem $\mathbf{A}_t$ in the case under consideration $(n > m-1)$ is quasi-correct. Making use of the results of §6.3 of Chapter 4 we can make the problem correct by adding new point conditions.

Let us arbitrarily fix k interior and k' boundary points of the surface, denoting them by $M_1, \ldots, M_k$ and $M'_1, \ldots, \ldots, M'_{k'}$, respectively. The numbers k and k' are subject to the following conditions:

$$(1) \quad 2k + k' = 2n+1-m \,, \qquad (2) \quad k' \geqslant m \,, \tag{10.32}$$

and on each of m arbitrarily chosen boundary curves (the number of them is $m+1$) an odd number of points are fixed. In §6 of Chapter 4 such a set of fixed points was called *the normally distributed* (k, k') *set*.

Let us now additionally prescribe at the points M_j and M'_j the tangential component of the displacement vector $\boldsymbol{U}$, i.e. let us additionally impose on the surface such constraints that they prescribe at these points the tangential components of the displacement vector. Taking into account that the components of the vector $\boldsymbol{U}$ in the direction $\boldsymbol{t}$ at the points M'_j should be given in accordance with the boundary condition of the problem, the above indicated additional conditions can be put in the form

$$\begin{gathered} u(M_j) = a_j \,, \qquad v(M_j) = b_j \qquad (j = 1, \ldots, k) \,, \\ u_{t'}(M'_j) = c_j \qquad (j = 1, \ldots, k') \,, \end{gathered} \tag{10.33}$$

where u and v are the tangent components of the displacement vector, and a_j, b_j and c_j are some given real constants.

The boundary value Problem $\mathbf{A}_t$ with the additional point conditions (10.33) taken into account, is a correct problem. This problem always has a solution and a unique one, the solution depending continuously on the data of the problem. These assertions readily follow from the results of §6 of Chapter IV. Thus, *for* $n > m-1$ *Problem* $\mathbf{A}_t$ *with the additional conditions* (10.33) *is normally correct.*

It remains to examine the cases of

$$0 \leqslant n \leqslant m-1, \quad m \geqslant 1. \tag{10.34}$$

It is observed that these cases can occur only for multiply-connected domain, i.e. for ovaloids with two or more openings.

For $n = 0$ two cases can take place, namely: (1) $l_t = 0$ or (2) $l_t = 1$. Accordingly, in view of (10.27) (1) $l'_t = m-1$ or (2) $l'_t = m$. Hence, for $m > 1$ Problem $\mathbf{A}_t$ is incorrect in both cases. In the first case ($l_t = 0$) the constraints of the form $u_t = 0$ are rigid but not correctly rigid. In fact, for solubility of the normally perturbed boundary value problem $u_{t*} = f^*$ conditions of the form (10.29) should be satisfied, the number of which is not smaller than $m-1$ ($m > 1$). In a similar way we find that in the second case ($l_t = 1$) Problem $\mathbf{A}_t$ is not quasi-correct. For $m = 1$ (ovaloid with two openings) in the first case ($l_t = 0$) Problem $\mathbf{A}_t$ always has a solution which is unique. In this case it is correct for normal perturbations of the bonds. This fact follows from Theorem 6 (Appendix to Chapter 4).

For $m = 1$ and $l_t = 1$ Problem $\mathbf{A}_t$ is evidently incorrect.

If for $n = 0$ Problem $\mathring{\mathbf{A}}_t$ has a non-trivial solution ($l_t = 1$) the corresponding displacement vector $\boldsymbol{U}$ has the property that its tangential displacement field does not vanish anywhere, either inside or on the boundary of the surface. This result follows immediately from the for-

mula (10.10) if it is taken into account that $w \neq 0$ in $G+\Gamma$, since it is a solution of the homogeneous Problem $\mathring{\mathbf{A}}_t$ with zero index (Theorem 4.6).

10.7. It should be observed that among non-trivial solutions of the homogeneous Problem $\mathring{\mathbf{A}}_t$ the number of which for $n > m-1$ is $2n+1-m$, there may be trivial displacement fields. Since the number of such fields does not exceed six we have

THEOREM 5.18. *If $2n-m > 5$ Problem $\mathring{A}_t$ has at least $2n-m-5$ non-trivial solutions to which there correspond non-trivial displacement fields. Consequently, in this case the surface is not (geometrically) rigid.*

Now, if $2n-m \leqslant 5$ it can turn out that all displacement fields, or a part of them, compatible with the boundary condition $u_t = 0$ are trivial. In the first case the surface is (geometrically) rigid.

Thus, *the inequality*

$$2n > m+5 \tag{10.35}$$

is a sufficient condition for the non-rigidity of a surface, and the inequality

$$2n \leqslant m+5 \tag{10.36}$$

is a necessary condition (but in general not sufficient) of geometrical rigidity.

In particular, for $m = 0$ *a (simply-connected) surface is certainly not rigid if $n > 2$. If however $0 \leqslant n \leqslant 2$, then in presence of only the constraints of the form $u_t = 0$ the surface is not kinematically rigid*, since Problem $\mathring{\mathbf{A}}_t$ has $2n+1$ solutions. Nevertheless, the possibility of geometric rigidity is not excluded, since in this case the necessary condition $2n+1 < 6$ is satisfied.

10.8. We now consider separately the case of Problem $\mathbf{A}_t$ in which the direction $\boldsymbol{t}$ belongs to the class of the tangent $\boldsymbol{s}$ or, which is equivalent, to the class of the tangential normal $\boldsymbol{l}$ of the contour L of the surface. In this case, according to (10.22) $n = n_t = 1-m$. For definiteness

let us consider the boundary condition $u_l = f$ which, as we have seen in §8.11, can be set up with the help of bush constraints. Inserting into the openings of the ovaloid rigid bushes the surfaces of which are orthogonal to the ovaloid and tightly fit the contours of the openings, we have along every boundary contour the following condition:

$$u_l = f_j \quad (\text{on } L_j) \quad (j = 0, 1, \dots, m), \tag{10.37}$$

where

$$f_j = \boldsymbol{l}\boldsymbol{\Omega}_j\boldsymbol{r} + \boldsymbol{l}\boldsymbol{C}_j \quad (j = 0, 1, \dots, m). \tag{10.38}$$

Here $\boldsymbol{\Omega}_j$ and $\boldsymbol{C}_j$ are constant vectors.

These conditions express firstly the presence of a continuous contact between the surfaces of the bushes and the contours of the openings, and, secondly, the fact that the surfaces of the bushes can displace only as rigid bodies. If the surfaces of the bushes are ideally smooth the contours of the openings of the ovaloid can freely move (slide) along these surfaces. Under these conditions the constraints of the form (10.37) will be called *orthogonal ideal bush constraints* or else *orthogonal sliding constraints.*

Since in what follows we shall consider only bush constraints of the above type, the words "orthogonal" and "ideal" will usually be omitted.

Besides constraints due to the presence of contact between the surfaces of the bushes and the contours of the openings of the ovaloid, which we shall call *internal constraints*, there can exist also *external constraints* between the bushes imposing certain restrictions on their configuration. These external constraints can be represented in the form of a number of equations or inequalities containing the components of the vectors $\boldsymbol{\Omega}_j$ and $\boldsymbol{C}_j$. Such relations will be termed *the conditions of external constraints.* It is natural to assume that these constraints do not oppose the motion of the whole system of the bushes and the ovaloid in the large. In other words the conditions of external constraints permit a change of the vectors $\boldsymbol{\Omega}_j$ and $\boldsymbol{C}_j$ on $\boldsymbol{\Omega}_j - \boldsymbol{\Omega}$ and $\boldsymbol{C}_j - \boldsymbol{C}$ where $\boldsymbol{\Omega}$ and $\boldsymbol{C}$ are arbitrary constant

vectors. This means that one of the bushes can be regarded as fixed. Therefore we shall hereafter assume that

$$\boldsymbol{\Omega}_0 = 0\,, \qquad \boldsymbol{C}_0 = 0\,. \tag{10.39}$$

Assume that the conditions of external constraints make it possible to express the remaining $6m$ constants—the components of the vectors $\boldsymbol{\Omega}_j$ and $\boldsymbol{C}_j$ ($j = 1, \ldots, m$)—by k independent parameters $\alpha_1, \ldots, \alpha_k$. Then we shall say that *the bush constraint has k degrees of freedom*. For $k = 0$ the constraint is said to be *the rigidly connected bush* constraint. In this case the union of the bushes constitutes one rigid system of bodies. For $k = 6m$ we have *an entirely free bush constraints*. In this case the external constraints between the bushes are absent and the right-hand side of the boundary condition of the problem has the form (10.38) where $\boldsymbol{\Omega}_j$ and $\boldsymbol{C}_j$ are entirely arbitrary constant vectors.

As the independent parameters $\alpha_1, \ldots, \alpha_k$ we can take k components of the vectors $\boldsymbol{\Omega}_j$ and $\boldsymbol{C}_j$. Then the remaining $6m-k$ components of these vectors are fully definite functions of the already fixed k components. In the case, in which $\boldsymbol{\Omega}_j$ and $\boldsymbol{C}_j$ *are linear homogeneous functions of the parameters $\alpha_1, \ldots, \alpha_k$ the bush constraints is said to be linear*.

In the case of a rigidly connected bush constraint, taking into account the assumption (10.39) we have the homogeneous boundary condition

$$u_l = 0 \qquad (\text{on } L)\,. \tag{10.40}$$

The index of this problem is $n = 1-m$. For a simply-connected surface $m = 0$ and the index $n = 1$. Acccording to Theorem 4.11 the problem has three linearly independent solutions. The adjoint homogeneous problem is that with the boundary condition (see (10.21); in our case $\varphi = 0$)

$$\delta\tau_s = 0 \qquad (\text{on } L)\,, \tag{10.41}$$

the index of which is $n = -2$. Consequently, the problem stated above has no solution. In general the boundary

value problem (10.21) has no continuous solution in the case of a simply-connected domain if the direction $\boldsymbol{t}$ belongs to the class $\boldsymbol{l}$.

Returning to the problem (10.40) we examine the case $m = 1$. Then the index $n = 0$ and two cases can occur, namely: (1) the problem has no solution and (2) it has one linearly independent solution. Finally, for $m > 1$ the index is negative and the problem has no non-trivial solutions. Thus, we have the following.

THEOREM 5.19. *In presence of a rigidly connected bush constraint the ovaloid with one opening admits three linearly independent displacement fields* $\boldsymbol{U}^{(1)}, \boldsymbol{U}^{(2)}, \boldsymbol{U}^{(3)}$, *an ovaloid with two openings is either rigid or admits one non-trivial displacement field. Finally, an ovaloid with three or more openings is always rigid.*

If it turns out that $\boldsymbol{U}^{(1)}, \boldsymbol{U}^{(2)}, \boldsymbol{U}^{(3)}$ are trivial displacement fields, i.e.

$$\begin{gathered} \boldsymbol{U}^{(j)} = \boldsymbol{\Omega}^{(j)} \times \boldsymbol{r} + \boldsymbol{C}^{(j)} \\ (j = 1, 2, 3), \end{gathered} \tag{10.42}$$

an ovaloid with one opening in presence of the constraint $u_l = 0$ is geometrically rigid. Hence, it is of interest to give examples when this takes place.

Let the contour of the opening be a plane curve and assume that the tangential normal of this curve is perpendicular (at all points) to the plane of the curve, similarly for instance to the hemisphere. Situating the coordinate plane oxy in the plane of the opening we easily find that the trivial displacement fields

$$\boldsymbol{U}^{(1)} = \boldsymbol{e}_1, \quad \boldsymbol{U}^{(2)} = \boldsymbol{e}_2, \quad \boldsymbol{U}^{(3)} = \boldsymbol{e}_3 \times \boldsymbol{r}, \tag{10.43}$$

satisfy the boundary condition $u_l = 0$.

It can also happen that only one or two of the three fields $\boldsymbol{U}^{(1)}, \boldsymbol{U}^{(2)}, \boldsymbol{U}^{(3)}$ are non-trivial. In the case of a doubly-connected domain ($m = 1$) it is possible to give an example in which the bush constraints are compatible with trivial

displacement fields. For instance, a sphere with two parallel plane (circular) openings admits a rotation about the axis in presence of bush constraints. As bushes, in this case cones with vertices at the centre of the sphere may be taken.

No matter whether the condition (10.42) is satisfied or not, we can ensure rigidity of an ovaloid with one opening in presence of the constraints $u_l = 0$ by adding point conditions of the form

$$u(M_1) = 0\,, \quad v(M_1) = 0\,, \quad u_s\,(M_1') = 0\,, \tag{10.44}$$

or

$$u_s(M_1') = 0\,, \quad u_s(M_2') = 0\,. \quad u_s(M_3') = 0\,, \tag{10.45}$$

where M_1 is an interior and M_1', M_2', M_3' are boundary points of the ovaloid, which can be fixed in an entirely arbitrary way. It is important to note that under these conditions an ovaloid with one opening is optimally rigid, and for $m > 1$ the rigidity is not correct.

In the case in which the bush constraints have k degrees of freedom, $0 < k \leqslant 6m$, it is necessary to investigate Problem $\mathbf{A}_l$ with the boundary condition of the form (10.37) where $\boldsymbol{\Omega}_j$ and $\boldsymbol{C}_j$ are given vector-functions of the parameters $a_1, \ldots, a_k$.

Let $m > 1$. Then the adjoint homogeneous Problem $\mathring{\mathbf{A}}_s'$ has in view of (10.28) $l_s' = 3m-3$ solutions. Hence, the necessary and sufficient condition of solubility of Problem $\mathbf{A}_l$ is the following:

$$\sum_{j=1}^{m} \int_{L_j} f_j T_{(u)}^{(p)} ds = 0 \tag{10.46}$$

$$p = 1, \ldots, 3m-3)\,,$$

where

$$T_{(u)}^{(p)} = \frac{1}{i}\, w_p'(z)\, \frac{\lambda_s(z)}{\overline{\lambda_t(z)}}\,, \qquad \lambda_t(z) = \frac{dz}{dt}\,. \tag{10.47}$$

Here w'_p are solutions of the homogeneous Problem $\mathrm{Å}'_l$, i.e. solutions of the equation (10.12) satisfying the boundary condition

$$\operatorname{Re}\left[w'(z)\frac{dz}{ds}\frac{dz}{dl}\right] = 0 \qquad (\text{on } \Gamma)\,. \tag{10.48}$$

According to the formulae (10.38) the relations (10.46) take the form

$$\sum_{j=1}^{m} \boldsymbol{\Omega}_j \boldsymbol{M}_{jp} + \boldsymbol{C}_j \boldsymbol{F}_{jp} = 0 \qquad (p = 1, \ldots, 3m-3)\,, \tag{10.49}$$

where

$$\left.\begin{aligned} \boldsymbol{F}_{jp} &= \frac{1}{i}\int\limits_{\Gamma_j} w'_p(z)\boldsymbol{r}_{\bar{z}}dz\,, \\ \boldsymbol{M}_{jp} &= \frac{1}{i}\int\limits_{\Gamma_j} w'_p(z)\boldsymbol{r}\times\boldsymbol{r}_{\bar{z}}dz\,. \end{aligned}\right\} \tag{10.50}$$

The mechanical meaning of these expressions will be elucidated in the next chapter.

Thus, *the fulfilment of the system of relations* (10.49) *is a necessary and sufficient condition for the existence of the solution of Problem* A_l *with the boundary condition* (10.37). In other words, an ovaloid with three or more openings admits an infinitesimal bending in presence of bush constraints with k degrees of freedom if and only if the parameters $a_1, \ldots, a_k$ determining the location of the bushes compatible with the external constraints, satisfy the system of equations (10.49).

This system of equations can be either linear or non-linear with respect to the parameters $a_1, \ldots, a_m$. Let us assume that we have linear bush constraints. Then the vectors $\boldsymbol{\Omega}_j$ and $\boldsymbol{C}_j$ are linear homogeneous functions of the parameters $a_1, \ldots, a_k$. Let ϱ be the rank of the matrix of the system (10.49).

If $\varrho = k$ it is evident that the system (10.49) has only the trivial solution $\boldsymbol{\Omega}_j = \boldsymbol{C}_j = 0$ $(j = 1, \ldots, m)$ and con-

sequently in this case the surface is rigid. If now $\varrho < k$ the system (10.49) has $k-\varrho$ solutions which can be represented as follows:

$$a_j = c_{j1}\beta_1 + \dots + c_{jk-\varrho}\beta_{k-\varrho} \qquad (j = 1, \dots, k)\,, \tag{10.51}$$

where c_{ji} are definite constants and $\beta_1, \dots, \beta_{k-\varrho}$ are independent parameters which we shall call *the allowable coordinates of the system of bushes.* To every system of values of these coordinates there corresponds a definite configuration of the bushes, and to every such configuration (and only to this configuration) there corresponds a definite (and unique) non-trivial displacement field compatible with the constraints present. Denoting by $\boldsymbol{U}^{(1)}, \dots, \boldsymbol{U}^{(k-\varrho)}$ the displacement fields corresponding to the allowable coordinates $(1, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 1)$, respectively, the general solution of Problem $\mathbf{A}_l$ in presence of external bush constraints has the form

$$\boldsymbol{U} = \beta_1\boldsymbol{U}^{(1)} + \dots + \beta_{k-\varrho}\boldsymbol{U}^{(k-\varrho)}\,. \tag{10.52}$$

It is readily seen that $\boldsymbol{U}^{(1)}, \dots, \boldsymbol{U}^{(k-\varrho)}$ are linearly independent vectors. If we now add to the prescribed conditions $k-\varrho$ more, suitably chosen additional conditions (for instance point conditions) we can obtain that Problem $\mathbf{A}_l$ has a unique solution. In particular, these (additional) conditions can so be chosen that the surface becomes correctly rigid.

We now emphasize some results which readily follow from the above considerations.

(1) *In presence of external bush constraints with k degrees of freedom an ovaloid with three or more openings is certainly not rigid if $k > 3m-3$. Then there exist $k-\varrho > > k-3m+3$ different configurations of the bushes to which there corresponds definite linearly independent displacement fields $\boldsymbol{U}^{(1)}, \dots, \boldsymbol{U}^{(k-\varrho)}$; the displacement fields corresponding to other allowable configurations of the system of bushes are given by the formula* (10.52) *where*

$\beta_1, \dots, \beta_{k-\varrho}$ are allowable coordinates of the system of the bushes.

(2) *In presence of external bush constraints with k degrees of freedom the surface can be rigid only if the necessary condition $k \leqslant 3m-3$ $(m > 1)$ is satisfied.*

This condition, however, can turn out to be insufficient. It would be of importance to give an appropriate example.

Finally, we examine the case of a doubly-connected domain ($m = 1$). Then $n = 0$ and the homogeneous problem either has no solution or has one solution. In the first case the surface admits always a non-trivial displacement field compatible with an arbitrary bush constraints $u_l = f$. In the second case such a displacement field exists only if the following (one) relation is satisfied:

$$\boldsymbol{\Omega}_1 \boldsymbol{M}_{11} + \boldsymbol{C}_1 \boldsymbol{F}_{11} = 0 . \tag{10.53}$$

Finally, let us observe that if the following relations were valid:

$$\boldsymbol{F}_{ji} = 0 , \quad \boldsymbol{M}_{ji} = 0 \tag{10.54}$$
$$(j = 1, \dots, m;\ i = 1, \dots, 3m-3) ,$$

the condition (10.49) would obviously be satisfied always. But the relations (10.54) imply that the statical field satisfying the boundary condition (10.48) is a bending field which satisfies the boundary condition (10.41). In the next subsection we shall prove that the geometrical problem with the boundary condition (10.41) has no non-trivial solution. In other words an ovaloid with edges, in presence of constraints of the form (10.41) is rigid. It follows that the fulfilment of all relations (10.54) simultaneously is impossible. Hence, the boundary condition (10.37) is incompatible with an infinitesimal bending if the values of the constant vectors $\boldsymbol{\Omega}_j$ and $\boldsymbol{C}_j$ are arbitrary.

10.9. We now investigate the geometrical boundary conditions of the form (10.21) which, as we have seen above, can be written in the form (10.13) where $w'(z)$ is a solution of the equation (10.12) and subject to the

additional conditions (6.42). The last conditions ensure existence of single-valued fields of displacement and rotation, which can be obtained by means of quadratures from the formula

$$\frac{dV}{ds} = 2\,\mathrm{Im}\left[w'(z)\frac{dz}{ds}r_{\bar{z}}\right]. \qquad (10.55)$$

In the case of a simply-connected surface the conditions (6.42) are always satisfied and therefore the problem is reduced to the determination of a solution of the equation (10.12) which is continuous in $G+\Gamma$ and satisfies the condition (10.13). If the direction $\boldsymbol{t}$ belongs to the class $\boldsymbol{l}$ the index $n = -2$ and, as it was already indicated, the problem (10.21) has no non-trivial solution.

Let us now investigate the case $m > 1$ assuming as before that $\boldsymbol{t}$ belongs to the class $\boldsymbol{l}$. If we do not take into account the condition (6.42) ensuring single-valuedness of the fields of displacement and rotation, we have the boundary value Problem $\mathring{\mathbf{A}}'_t$ adjoint to the kinematic Problem $\mathbf{A}_t$. This problem has a statical interpretation which will be elucidated in the next chapter. Since the index of Problem $\mathbf{A}_t$ is $n = 1-m$, in view of the formula (10.27) the number of linearly independent solutions of Problem $\mathring{\mathbf{A}}'_t$ is $l' = 3m-3$ $(m > 1)$. Denoting by $w'_1, \ldots, w'_{l'}$, the complete system of solutions of Problem $\mathring{\mathbf{A}}'_t$, the general solution of this problem has the form

$$w' = c_1 w_1 + \ldots + c_{l'} w'_{l'} \quad (l' = 3m-3). \qquad (10.56)$$

In order to obtain the solution of the homogeneous problem (10.21) it is necessary and sufficient that w' satisfies the relations (6.42) which can be written in the form

$$\sum_{j=1}^{3m-3} c_j \int_{L_k} w'_j \frac{\boldsymbol{t}}{\overline{\lambda_t(z)}}\, dz = 0, \quad \sum_{j=1}^{3m-3} c_j \int_{L_k} w'_j \frac{\boldsymbol{r}\times\boldsymbol{t}}{\overline{\lambda_t(z)}}\, dz = 0 \quad (10.57)$$
$$(k = 1, \ldots, m).$$

In deriving the above relations we have taken into account that for the solutions of Problem $\mathring{A}'_t$ the formula (6.45) takes the form

$$T_{(l)} = \frac{1}{i} w'(z) \frac{\lambda_s(z)}{\lambda_t(z)} t\,, \qquad \lambda_t(z) = \frac{dz}{dt} \qquad (\text{on } L)\,. \qquad (10.58)$$

Thus we have a homogeneous system of $6m$ equations for the determination of $3m-3$ real constants $c_1, \ldots, c_{3m-3}$. If this system of equations has no non-trivial solutions then (and only then) an ovaloid with $m+1$ openings subject to the constraints expressed by the boundary condition

$$-\delta k_s \sin\varphi + \delta\tau_s \cos\varphi = 0 \qquad (\text{on } L)\,, \qquad (10.21)$$

is rigid. It was already noted above that we assume that the angle φ is Hölder continuous and its absolute value does not exceed π, i.e. $0 \leqslant |\varphi| < \pi$. In other words the direction $\boldsymbol{t}$ belongs to the class $\boldsymbol{l}$.

For brevity this geometric problem will hereafter be denoted by $\boldsymbol{B}_{t'}$.

Taking into account the formula (10.58) the relations (10.57) can be written thus:

$$\sum_{j=1}^{3m-3} c_j \int_{L_k} T^{(j)}_{(lt)} \boldsymbol{t}\, ds = 0\,, \qquad \sum_{j=1}^{3m-3} c_j \int_{L_k} T^{(j)}_{(lt)} \boldsymbol{r} \times \boldsymbol{t}\, ds = 0 \qquad (10.59)$$

$$(k = 1, \ldots, m)\,,$$

where

$$T^{(k)}_{(lt)} = \frac{1}{i} w'_k(z) \frac{\lambda_s(z)}{\lambda_t(z)} \qquad (k = 1, \ldots, 3m-3)\,. \qquad (10.60)$$

We shall consider the case when the boundary condition (10.21) has the form

$$\delta k_s = 0 \qquad (\text{on } L)\,. \qquad (10.61)$$

As we found in §8.5 this boundary condition corresponds to an ovaloid with trimmed edges. In this case $\varphi = \dfrac{\pi}{2}$

and, consequently, $\boldsymbol{t} = \boldsymbol{s}, \boldsymbol{t}' = -\boldsymbol{l}$. Then the conditions (10.59) take the form

$$\sum_{j=1}^{m-3} c_j \int_{L_k} T^{(j)}_{(ls)} dr = 0\,, \quad \sum_{j=1}^{3m-3} c_j \int_{L_k} T^{(j)}_{(ls)} \boldsymbol{r} \times d\boldsymbol{r} = 0 \qquad (10.62)$$

$$k = 1, \dots, m)\,.$$

We now consider the case of an ovaloid with two openings. Then $m = 1$ and the index $n = 0$. According to Theorem 4.6 in this case the homogeneous Problem $\mathring{A}'_t$ either has no non-trivial solutions or has one linearly independent solution. In the first case the geometric problem with the boundary condition (10.61) evidently has no non-trivial solutions, i.e. if this condition is satisfied an ovaloid with two trimmed openings is rigid. Assume now that Problem $\mathring{A}_t$ has one non-trivial solution, i.e. there exists a complex stress function w' which satisfies the boundary condition

$$T_{(ll)} \equiv -\sqrt{a}\,\mathrm{Re}\left[w'(z)\left(\frac{dz}{ds}\right)^2\right] = 0 \qquad (\text{on } L)\,. \quad (10.63)$$

It was however indicated above that to the geometric condition (10.61) there correspond only these solutions of the problem (10.63) which additionally satisfy the conditions

$$(1) \quad \int_{L_1} T_{(ls)} d\boldsymbol{r} = 0\,, \qquad (2) \quad \int_{L_1} T_{(ls)} \boldsymbol{r} \times d\boldsymbol{r} = 0\,, \quad (10.64)$$

and in view of (10.60)

$$T_{(ls)} = \frac{1}{i} w'(z)\, e^{-2i\vartheta_s}\,, \qquad (10.65)$$

where ϑ_s is the angle of inclination of the tangent of Γ at the point z. But in view of Theorem 4.6 a non-trivial solution w' of Problem $\mathring{A}'_t$, if such exists, does not vanish anywhere in $G+\Gamma$. Consequently, $T_{(ls)} \neq 0$ everywhere on L (it is to be borne in mind that we consider for the

time being the case $m = 1$ and the index of the problem is zero).

Let us now take the origin of coordinates inside the ovaloid and let us consider the cone described by the position vector $\boldsymbol{r}(M)$ when the point M describes the contour L_1. Let $\boldsymbol{e}$ be the unit vector of an axis passing through the vertex of the cone (the origin of coordinates) and situated inside the opening L_1. Multiplying scalarly the second relation (10.64) by $\boldsymbol{e}$ we obtain

$$\int_{L_0} T_{(ls)}(\boldsymbol{r} \times \boldsymbol{s})\boldsymbol{e}\, ds = 0 \,. \tag{10.66}$$

However, this is impossible. In fact, the vector $\boldsymbol{r} \times \boldsymbol{s}$ is directed along the normal of the cone and consequently it makes with the unit vector $\boldsymbol{e}$ either always an acute angle or always an obtuse angle. Hence, the sign of the mixed product $(\boldsymbol{r} \times \boldsymbol{s})\boldsymbol{e}$ does not change along L_1. Since $T_{(ls)}$ does not vanish anywhere on L the sign of the integrand in the relation (10.66) does not change along L_1. Consequently, the relation (10.66) is impossible. This proves that the conditions (10.64) cannot be satisfied simultaneously if $T_{(ls)} \neq 0$ everywhere on L. In other words, the homogeneous geometrical problem (10.61) has no nontrivial solution in the case of a doubly-connected domain.

This result implies that *an ovaloid with two trimmed openings is rigid.*

We now examine the case of an ovaloid with three openings ($m = 2$). The index of the problem is $n = 2$ and the homogeneous Problem $\mathring{A}'_l$ has three linearly independent solutions. We shall now prove that in this case every solution of Problem $\mathring{A}'_l$ at least on one of the boundary contours is everywhere different from zero. According to the formula (4.17) of Chapter 4 the number of boundary zeros of the solution $w'(z)$ of Problem $\mathring{A}'_l$ does not exceed 4, since $n = 2$. According to Theorem 4.7 there is an even number of zeros of $w'(z)$ on every boundary contour. Since in the case under consideration there are

three boundary contours it follows that at least one contour is always free of zeros of $w'(z)$. Consequently, in view of (10.65) the corresponding tangent force $T_{(ls)} \neq 0$ everywhere on this contour. Now, as before, we prove that the second relation (10.64) cannot hold on the appropriate contour. This completes the proof of the assertion that the problem (10.61) in the case of a triply-connected surface has no solution.

Hence, *an ovaloid with three trimmed openings is rigid.*

The above reasoning cannot immediately be applied to the general case of $m > 2$. Nevertheless, also in this case it can be proved that an ovaloid with trimmed edges is rigid. In the case of a truncated ovaloid, i.e. when the openings are plane curves, the proof was given in a different way by Yefimov [33a].

§11. Infinitesimal bendings of surfaces of revolution

In this section we examine some problems of rigidity of surfaces of revolution. The comparative simplicity of the equations in this case makes it possible to investigate in more detail the problems of rigidity, not only for surfaces of revolution of positive curvature but also for a class of surfaces of mixed type. The global reduction to the canonical form of the equations of infinitesimal bending can readily be carried out in the case of surfaces of revolution. In consequence we obtain, in general, equations of mixed type in canonical form, and on the basis of geometrical considerations it is possible to establish in a natural way the character of singularities of the coefficients and the conjunction conditions on the curves of degeneracy of the type of equations.

11.1. We consider a surface of revolution having the equation

$$x = \varrho\cos\vartheta\,, \quad y = \varrho\sin\vartheta\,, \quad z = \varphi(\varrho)\,, \tag{11.1}$$

where $\varphi(\varrho)$ is a single-valued continuous function having sectionally continuous derivatives of the order $m \geqslant 3$ in the interval $0 \leqslant \varrho \leqslant a$, where a may be infinity.

Thus, we examine a surface generated by a revolution around the axis oz of a plane curve intersecting every straight line parallel to the axis oz at not more than one point. The first and the second fundamental quadratic forms are given by the formulae

$$ds^2 = (1+\varphi'^2)d\varrho^2 + \varrho^2 d\vartheta^2\,, \qquad \varphi' = \frac{d\varphi}{d\varrho}\,, \tag{11.2}$$

$$\mathrm{II} = \frac{\varphi''}{\sqrt{1+\varphi'^2}}\,d\varrho^2 + \frac{\varrho\varphi'}{\sqrt{1+\varphi'^2}}\,d\vartheta^2\,. \tag{11.3}$$

Hence

$$\mathrm{K} = \frac{\varphi'\varphi''}{\varrho(1+\varphi'^2)^2}\,, \qquad 2\mathrm{H} = \frac{\varrho\varphi'' + \varphi'(1+\varphi'^2)}{\varrho(1+\varphi'^2)^{3/2}}\,. \tag{11.4}$$

Assume that

$$\xi = \vartheta\,, \qquad \eta = \int\limits_{\varrho_0}^{\varrho} \sqrt{|\chi(\varrho)|}\,d\varrho\,, \qquad \chi = \frac{\varphi''(\varrho)}{\varrho\varphi'(\varrho)}\,, \tag{11.5}$$

and ϱ_0 is so chosen that the intergal is convergent. In general we assume that φ' and φ'' can vanish only at isolated points and that the integral is convergent for an arbitrary ϱ, $0 < \varrho < a$. Under these conditions it is evident that $\eta(\varrho)$ increases monotonically from $\eta_0 = \eta(0)$ to $\eta_a = \eta(a)$, and both η_0 and η_a can be equal to $-\infty$ and $+\infty$, respectively.

In the new variables ξ and η we have

$$\begin{aligned} \mathrm{I} &= ds^2 \equiv \mathrm{A}^2 d\xi^2 + \mathrm{B}^2 d\eta^2\,, \\ \mathrm{II} &= k_s ds^2 = \Lambda(d\xi^2 + \varepsilon(\eta)d\eta^2)\,, \end{aligned} \tag{11.6}$$

where

$$\varepsilon(\eta) = \operatorname{sign}\mathrm{K}(\eta) = \begin{cases} 1\,, & \text{if} \quad \mathrm{K}(\eta) > 0\,, \\ 0\,, & \text{if} \quad \mathrm{K}(\eta) = 0\,, \\ -1\,, & \text{if} \quad \mathrm{K}(\eta) < 0\,, \end{cases} \tag{11.7}$$

$$\mathrm{A} = \varrho\,, \qquad \mathrm{B} = \frac{\varrho}{l\sqrt{|\mathrm{K}|}}\,, \qquad \Lambda = \frac{\varrho^2}{l}\,, \qquad l = \frac{\varrho\sqrt{1+\varphi'^2}}{|\varphi'|} \tag{11.8}$$

and l is the length of section of the normal to the surface.

On the lines (parallels) where $K = 0$ the coefficient B^2 has a discontinuity.

On the basis of the formulae (3.34) we have

$$\begin{aligned} \Gamma^1_{11} &= \Gamma^1_{22} = \Gamma^2_{12} = 0\,, \qquad \Gamma^1_{12} = \frac{1}{A}\frac{dA}{d\eta} = \frac{1}{\varrho}\frac{d\varrho}{d\eta}\,, \\ \Gamma^2_{11} &= -\frac{A}{B^2}\frac{dA}{d\eta}\,, \qquad \Gamma^2_{22} = \frac{1}{B}\frac{dB}{d\eta}\,. \end{aligned} \tag{11.9}$$

Here ϱ is understood as a function in η represented by the relation (11.5). At the points where $K = 0$, Γ^1_{12} and Γ^2_{22} have discontinuities and Γ^2_{11} vanishes.

We can now write the basic equations of the infinitesimal bending of surfaces of revolution in the following form.

System of equations for the displacement field (3.39) and (3.40):

$$\frac{\partial\bar{u}}{\partial\xi} - \varepsilon(\eta)\frac{\partial\bar{v}}{\partial\eta} + b\bar{v} = 0\,, \qquad \frac{\partial\bar{u}}{\partial\eta} + \frac{\partial\bar{v}}{\partial\xi} + cu = 0\,, \tag{11.10}$$

$$u_0 = \frac{B}{A}\left(\frac{\partial\bar{u}}{\partial\xi} + b\bar{u}\right) \equiv \frac{\varepsilon(\eta)B}{A}\frac{\partial\bar{v}}{\partial\eta}\,, \tag{11.11}$$

where

$$b = \frac{A}{B^2}\frac{dA}{d\eta}\,, \qquad c = \frac{d}{d\eta}\ln\frac{B}{A^2}\,, \tag{11.12}$$

$$\bar{u} = \frac{u_1}{B} = \frac{A}{B}u_{(1)}\,, \qquad \bar{v} = \frac{u_2}{B} = u_2\,, \tag{11.13}$$

and $u_{(1)}$ and $u_{(2)}$ are the physical components of the displacement vector along the meridian and parallel.

System of equations for the bending field (6.33):

$$\frac{\partial\bar{u}'}{\partial\xi} - \frac{\partial\bar{v}'}{\partial\eta} + c\bar{v}' = 0\,, \quad \varepsilon(\eta)\frac{\partial\bar{u}'}{\partial\eta} + \frac{\partial\bar{v}'}{\partial\xi} + b\bar{u}' = 0\,, \tag{11.14}$$

where

$$\bar{u}' = -\varepsilon(\eta)Ap\,, \qquad \bar{v}' = -AB^2T^{12} = -Bq\,. \tag{11.15}$$

Characteristic equation (5.34): .

$$\frac{\partial^2\omega}{\partial\eta^2} + \varepsilon(\eta)\frac{\partial^2\omega}{\partial\xi^2} + M(\eta)\,\omega = 0\,, \tag{11.16}$$

where

$$\omega = |K|^{-\frac{1}{4}} v\,, \qquad v = \boldsymbol{Vn}\,, \tag{11.16a}$$

$$M = -\frac{1}{\lambda}\frac{d^2\lambda}{d\eta^2}\,, \quad \lambda = |K|^{-\frac{1}{4}}(1+\varphi'^2)^{-\frac{1}{2}} = |K|^{-\frac{1}{4}}\cos\theta\,, \tag{11.17}$$

and θ is the angle between the normal of the surface and the axis of revolution, $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$.

The equation (11.16) has three trivial solutions

$$\lambda\,, \ \mu = \lambda\varphi'\cos\vartheta\,, \qquad \nu = \lambda\varphi'\sin\vartheta\,, \tag{11.18}$$

which are proportional to the direction cosines of the normal to the surface.

11.2. In solving the system of equations (11.10) and (11.14) the conjunction conditions on the parallels where $K = 0$ should be taken into account. They follow from the continuity of the deformation (in the vicinity of such a curve of degeneracy, the surface is regarded as smooth).

Let $K = 0$ on a parallel $\eta = \eta'$. Bearing in mind the geometrical meaning of the quantities $u_{(1)}$ and $u_{(2)}$, as well as p and q (see §7.7) we have to regard them as continuous on the above line. Then we obtain from (11.13) and (11.15) the following conjunction conditions:

$$\bar{u}|_{\eta'-0} = \bar{u}|_{\eta'+0} = 0\,, \qquad \bar{v}|_{\eta'-0} = \bar{v}|_{\eta'+0} = 0\,, \tag{11.19}$$

$$\bar{u}'|_{\eta'-0} = -\bar{u}'|_{\eta'+0}\,, \qquad \left(\frac{\bar{v}'}{B}\right)_{\eta'-0} = \left(\frac{\bar{v}'}{B}\right)_{\eta'+0}. \tag{11.20}$$

If $K(\eta') = 0, K'(\eta') \neq 0$ it is easy to prove that near the point η' the function M has a singularity of the type

$$M(\eta) = \frac{M_*(\eta)}{(\eta-\eta')^2}\,, \tag{11.21}$$

where $M_*(\eta)$ is a continuous function, and

$$M_*(\eta') = -\frac{7}{36}. \tag{11.22}$$

Thus, the problem of the determination of the displacement field or the bending field of a surface of revolution, leads in an entirely natural way to the problem of the global investigation of properties of solutions of equations of mixed type, both for the system of partial differential equations of the first order (11.10) and (11.14) and for the equation of the second order (11.16). The type of these equations depends on the sign of $K(\eta)$. As $K(\eta) > 0$ the equations are of elliptic type, as $K(\eta) < 0$ and $K(\eta) = 0$ they are of hyperbolic and parabolic type, respectively. On the lines of degeneracy $(K(\eta) = 0)$ the conditions (11.19) and (11.20) should be accounted for, for the systems (11.10) and (11.14), respectively; for the characteristic equation of the second order (11.16), taking into account the continuity of the rotation field we have the following conditions:

$$v^+ = v^-, \qquad \left(\frac{dv}{d\varrho}\right)^+ = \left(\frac{dv}{d\varrho}\right)^-$$

or

$$\begin{gathered}(|K|^{\frac{1}{4}}\omega)^+ = (|K|^{\frac{1}{4}}\omega)^-\,,\\ \left[\sqrt{|K|}\,\frac{d}{d\eta}(|K|^{\frac{1}{4}}\omega)\right]^+ = \left[\sqrt{|K|}\,\frac{d}{d\eta}(|K|^{\frac{1}{4}}\omega)\right]^-.\end{gathered} \tag{11.23}$$

In addition, if the surface is sectionally smooth and open the conjunction conditions on the contact lines should also be taken into account, as well as the boundary conditions representing the constraints present.

In particular, if $M(\eta) = 0$ we obtain the simplest typical equation of mixed type of the second order

$$\frac{\partial^2\omega}{\partial\eta^2} + \varepsilon(\eta)\,\frac{\partial^2\omega}{\partial\xi^2} = 0\,. \tag{11.24}$$

The importance of the global investigation of the properties of this equation was first indicated by Lavrentyev [47a]. Fundamental properties of the solutions of (11.24) and a number of boundary value problems

for this equation were investigated in the papers of Bitsadse [9]. From his results it is possible to derive a number of conditions of rigidity for surfaces of revolution of the mixed type.

It is easy to verify that the relation $M(\eta) = 0$ is satisfied for instance by the following surfaces of revolution:

$$\begin{aligned} \varphi_1(\varrho) &= \gamma + \alpha\left[\varrho\sqrt{\varrho^2+\beta^2} + \beta^2\log\left(\varrho + \sqrt{\varrho^2+\beta^2}\right)\right], \\ \varphi_2(\varrho) &= \gamma + \alpha\left[\varrho\sqrt{\varrho^2-\beta^2} - \beta^2\log\left(\varrho + \sqrt{\varrho^2-\beta^2}\right)\right], \qquad (11.25) \\ \varphi_3(\varrho) &= \gamma + \alpha\left[\varrho\sqrt{\beta^2-\varrho^2} + \beta^2\arcsin\frac{\varrho}{\beta}\right], \end{aligned}$$

where α, β, γ are real constants.

It is also easy to indicate geometric problems which can be reduced to the Tricomi problem [85a, c].

Important criteria of solubility of boundary value problems for equations of the second order with a line of degeneration were first indicated by Keldysh [40b]. By applying his results a number of conditions can be derived which ensure rigidity of a surface of revolution bounded by parabolicity lines (lines on which $K = 0$). Further generalizations on one hand and more definite results for equations of more particular form on the other hand, of the above results were presented in the papers [16a], [43], [56b], [66b], [83].

It is of a considerable interest to carry out a further investigation of mixed problems and boundary value problems with a line of degeneracy for equations of the second order (11.16) or for the systems of equations (11.10) and (11.14), taking into account definite properties of the coefficients of these equations. In particular it would be interesting to investigate properties of solutions of the equation (11.16) when the coefficient $M(\eta)$ has the form (11.21). *

* It is well-known that for the mixed-type equations (or with lines of degeneracy), problems are stated mainly in connexion with problems of gas dynamics. It is of a great importance to connect more

11.3. In the case of a surface of revolution bounded by the parallels $\eta = \text{const}$ it is of interest to determine solutions of the equation (11.26) periodic with respect to ξ (the period being 2π). This problem leads to an investigation of ordinary differential equations of the form

$$\frac{d^2y}{d\eta^2}+[\varepsilon(\eta)n^2+\mathrm{M}(\eta)]y = 0 \tag{11.26}$$

$$(n = 0, 1, \ldots, \varepsilon(\eta) = \operatorname{sign}\mathrm{K}(\eta)) .$$

An absence of bounded solutions of these equations (except trivial) is a criterion of rigidity of the surface. In particular it is of interest to investigate the cases when the function $\mathrm{M}(\eta)$ has singularities of the form (11.21).

11.4. In the following subsections we shall derive some criteria of rigidity of various families of surfaces of revolution. The results stated below ware obtained by Metskhovrishvili [52], Bojarski [11h] and Sun Cheshen [81], many of these results being published here in complete form for the first time.

The function $\varphi(\varrho)$ has so far been regarded as single-valued. This is equivalent to the fact that the corresponding surface intersects every straight line parallel to the axis of revolution at not more than one point. It is desirable however to eliminate this restriction, since it is not satisfied for, say, closed surfaces of revolution.

closely this range of problems with problems of the infinitesimal bending of surfaces, and with the membrane theory of shells (see Chapter VI, §5). In many cases the comparative simplicity and clearness of these geometrical and mechanical problems can suggest a number of properties of the reguired solutions as well as some new methods of investigation. The examples usually quoted from gas dynamics have a great practical importance; they had a considerable influence on the progress of this important chapter of partial differential equations; they do not have however such a clear meaning. As we have seen above the statements of these problems follow from simple geometric consideration for the systems of first order equations (11.10), (11.14) and for the equation of the second order (11.16).

In such more general cases we can regard the surface as a union of open surfaces of revolution of the above considered type, which are in contact along the parallels bounding them. Writing the appropriate equations of infinitesimal bending for each of these surfaces we should complete them by the conjunction conditions representing the continuity of deformation along the contact lines. Thus, the problem is reduced to the solution of a set of systems of partial differential equations (11.10) and (11.14) or equations of the second order (11.16), taking into account the above indicated conjunction conditions on the contact lines and the boundary conditions representing the external constraints present.

However, for a number of surfaces of revolution, it is possible to introduce coordinate systems making it possible to write down the equations of infinitesimal bending for the whole surface, without splitting it up into sections, [41a, b]. This is the case for instance for ovaloids of revolution and the torus. We start with the case of a torus.

11.5. The equation of a torus can be written in the form

$$x = l(1+\varepsilon\cos\varphi)\cos\vartheta\,, \qquad y = l(1+\varepsilon\cos\varphi)\sin\vartheta\,,\ r = a\sin\varphi,$$

where a is the radius of the circle generating by rotation the torus, l the distance of the centre of rotation from the centre of the circle, $\varepsilon = \frac{a}{l}$ the eccentricity of the torus, φ internal latitude of the torus, ϑ the longitude of the meridian of the torus; for a full torus $0 \leqslant \varphi \leqslant 2\pi$, $0 \leqslant \vartheta \leqslant 2\pi$.

In this case the first and second fundamental quadratic forms of the surface of the torus have the form

$$\mathrm{I} \equiv ds^2 = a^2 d\varphi^2 + l^2(1+\varepsilon\cos\varphi)^2 d\vartheta^2\,,$$

$$\mathrm{II} = a\,d\varphi^2 + l\cos\varphi(1+\varepsilon\cos\varphi)\,d\vartheta^2\,.$$

Consequently. $A = a$, $B = l(1+\varepsilon\cos\varphi)$, $\varepsilon < 1$, and according to (3.34)

$$\Gamma^1_{11} = 0\ , \quad \Gamma^2_{11} = 0\ , \quad \Gamma^1_{12} = 0\ , \quad \Gamma^2_{22} = 0\ ,$$
$$\Gamma^2_{12} = -\frac{\varepsilon\sin\varphi}{1+\varepsilon\cos\varphi}\ , \quad \Gamma^1_{22} = \frac{\sin\varphi(1+\varepsilon\cos\varphi)}{\varepsilon}\ .$$

Since

$$k_1 = \frac{1}{R_1} = \frac{1}{a}\ , \quad k_2 = \frac{1}{R_2} = \frac{\cos\varphi}{l(1+\varepsilon\cos\varphi)}\ ,$$

the systems of equations (3.39) and (6.30) for a torus take the form

$$\begin{aligned} &\frac{\partial u}{\partial\varphi} - \frac{1}{\cos\varphi}\frac{\partial v}{\partial\vartheta} + \tan\varphi u = 0\ , \\ &\frac{\partial u}{\partial\vartheta} + \frac{1+\varepsilon\cos\varphi}{\varepsilon}\frac{\partial v}{\partial\varphi} + \sin\varphi v = 0\ , \end{aligned} \tag{11.27}$$

$$\begin{aligned} &\frac{\partial H_1}{\partial\vartheta} - \cos\varphi\frac{\partial N_2}{\partial\varphi} + 2\sin\varphi N_2 = 0\ , \\ &\frac{1+\varepsilon\cos\varphi}{\varepsilon}\frac{\partial H_1}{\partial\varphi} + \frac{\partial N_2}{\partial\vartheta} - 2\sin\varphi H_1 = 0\ , \end{aligned} \tag{11.28}$$

We also have

$$u_0 = \frac{\partial u}{\partial\varphi}\ , \quad N_1 = -\frac{\varepsilon\cos\varphi}{1+\varepsilon\cos\varphi}N_2\ .$$

The characteristic equation has the form

$$\frac{\partial}{\partial\varphi}\left[(1+\varepsilon\cos\varphi)\frac{\partial v}{\partial\varphi}\right] + \frac{\varepsilon}{\cos\varphi}\frac{\partial^2 v}{\partial\vartheta^2} + (1+2\varepsilon\cos\varphi)\vartheta = 0\ .$$

We now prove that *the surface of a (full) torus is rigid with respect to infinitesimal bendings*. The necessary and sufficient condition is that the system of equations (11.28) has no solution continuous on the whole torus, ensuring the single-valuedness of the displacement field.

Geometrical considerations imply that solutions of the system (11.28) continuous on the whole torus should be

periodic functions in φ and ϑ the periods being equal to 2π. It can be proved however that all such solutions of the system (11.28) have the form

$$N_1 = N_2 = 0 , \quad H_1 = c(1+\varepsilon\cos\varphi)^{-2} ,$$

where c is an arbitrary constant [52].

The condition (6.24) takes in our case the form

$$\int_L \boldsymbol{T}_{(l)} ds = \int_L \boldsymbol{H}_l d\boldsymbol{r} = 0 , \tag{11.29}$$

where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ are the unit vectors of Cartesian coordinate system ($\boldsymbol{e}_3$ is the unit vector of the axis of rotation). Taking for the integration line the circle $\vartheta = 0$ we can write the condition (11.29) in the form

$$\boldsymbol{e}_3 \int_0^{2\pi} \frac{\cos\varphi\, d\varphi}{(1+\varepsilon\cos\varphi)^2} - \boldsymbol{e}_1 \int_0^{2\pi} \frac{\sin\varphi\, d\varphi}{(1+\varepsilon\cos\varphi)^2} = 0 .$$

This condition however is not satisfied, since

$$\int_0^{2\pi} \cos\varphi(1+\varepsilon\cos\varphi)^{-2} d\varphi \neq 0 \quad \text{as} \quad \varepsilon \neq 0 .$$

The last result proves that the surface of a full torus does not admit infinitesimal bendings. The proof given above was obtained in the paper [52a]. In this paper conditions were also given for the rigidity for various sections of a torus, and problems of membrane equilibrium of a torus were examined [52a, b, c].

11.6. In this subsection we shall consider some cases of bush constraints on surfaces of revolution. We shall make use of a special coordinate system in which the axis of rotation of the surface coincides with the oz-axis. The points of the surface will be determined by the pair (z, ϑ) where ϑ is the angle of revolution. Then the position vector $\boldsymbol{r} = \boldsymbol{r}(z, \vartheta)$ of the surface S under consideration can be represented as follows:

$$\boldsymbol{r} = z\boldsymbol{k} + \varrho(z)\boldsymbol{e}(\vartheta) , \quad z_0 \leqslant z \leqslant z_0' , \quad 0 \leqslant \vartheta \leqslant 2\pi ,$$

where $\varrho(z)$ is the meridian of the surface, $\boldsymbol{k}$ the unit vector of the rotation axis, $\boldsymbol{e}(\vartheta)$ the unit vector perpendicular to $\boldsymbol{k}$, which as ϑ varies describes a circle with the axis $\boldsymbol{k}$ and the length of arc ϑ. Let $\boldsymbol{g}(\vartheta) = \boldsymbol{k} \times \boldsymbol{e}(\vartheta)$. The displacement vector $\boldsymbol{U}$ can be decomposed with respect to the triplet $\boldsymbol{e}, \boldsymbol{g}, \boldsymbol{k}$,

$$\boldsymbol{U} = \boldsymbol{U}(z, \vartheta) = u\boldsymbol{e} + v\boldsymbol{g} + w\boldsymbol{k} . \tag{11.30}$$

Then it is readily seen that the basic equation $d\boldsymbol{U}d\boldsymbol{r} = 0$ is equivalent to the following system of partial differential equations of the first order for the components u, v, w, [41a],

$$\begin{gathered} \varrho'(z)u_z + w_z = 0 , \\ v_\vartheta + u = 0 , \\ \varrho(z)v_z + \varrho'(z)(u_\vartheta - v) + w_\vartheta = 0 . \end{gathered} \tag{11.31}$$

Eliminating $u(z, \vartheta)$ and $w(z, \vartheta)$ from this system we obtain

$$-\varrho''(z)v_{\vartheta\vartheta} + \varrho(z)v_{zz} - \varrho''(z)v = 0 , \tag{11.31a}$$

The equations (11.31) and (11.31a) may have no singular lines when $z \to z_0$ or $z \to z_0'$.

Eliminating u from the system (11.30) and introducing new unknowns α and β in accordance with the formulae

$$\alpha = w - \varrho' v_\vartheta , \quad \beta = \frac{v}{\varrho} , \tag{11.32}$$

we obtain for α and β the system of equations

$$\alpha_z = p\beta_\vartheta , \quad \alpha_\vartheta = -q\beta_z \tag{11.33}$$

and

$$(p\beta_\vartheta)_\vartheta + (q\beta_z)_z = 0 , \tag{11.34}$$

where

$$p = -\varrho''\varrho , \quad q = \varrho^2 . \tag{11.35}$$

Let L_1 and L_1' be two parallels of the surface S, $z = z_1$ and $z = z_1'$, respectively, $z_0 < z_1 < z_1' < z_0'$, bounding the strip S_1 of the surface S. Let us assume that along L_1 and L_1' the surface S_1 is subject to bush constraints determined by a surface Σ (see §8.11). We shall not assume that the surface Σ is a surface of revolution but we do assume

that the normal ν of Σ along $L = L_1 + L_1'$ lies in the plane of the vectors $\boldsymbol{k}$, $\boldsymbol{e}$ i.e. $\nu = \nu_1\boldsymbol{k} + \nu_2\boldsymbol{e}$ where ν_1 and ν_2 are some functions of point of the curve L. According to §8.11 if Σ is fixed the conditions of bush constraints have the form $\boldsymbol{U}\nu = 0$ on L, or

$$w\nu_1 + u\nu_2 = 0\ . \tag{11.36}$$

In the variables a and β the last condition has the form

$$\tilde{\nu}_1 a + \tilde{\nu}_2 \beta_\vartheta = 0\ , \quad \text{where} \quad \tilde{\nu}_1 = \frac{\nu_1}{\varrho}\ , \quad \tilde{\nu}_2 = \varrho'\nu_1 - \nu_2\ . \tag{11.37}$$

Let us now assume that the surface S_1 under consideration is a sectionally regular convex surface. Let z_i, $i = 2, \ldots, n-1$, $z_1 < z_i < z_{i+1} < \ldots < z_n = z_1'$ be a sequence of the break points of the meridian $\varrho = \varrho(z)$. Let L_j be the parallel $z = z_j$ and I_j the interval (z_{i-1}, z_i). Then, inside $I_j, \varrho''(z) \leqslant 0$, i.e.

$$p \geqslant 0\ , \qquad q > 0 \tag{11.38}$$

and

$$\varrho'^{+}(z_i) \geqslant \varrho'^{-}(z_i)\ , \tag{11.39}$$

where the signs "+" and "−" denote the derivatives from the left and from the right, respectively.

Let $\boldsymbol{U}$ be a displacement field continuous on S_1 and belonging to the class C^2 in every strip between the parallels L_{i-1} and L_i, and moreover satisfying the bush constraints (11.37) on L.

Multiplying the equation (11.34) by β we obtain

$$p\beta_\vartheta^2 + q\beta_z^2 = (p\beta_\vartheta\beta)_\vartheta + (q\beta_z\beta)_z\ .$$

Integrating this relation over the rectangle $R_i: z_{i-1} < z < z_i$, $0 \leqslant \vartheta \leqslant 2\pi$, we have

$$J_i = \iint\limits_{R_i} (p\beta_\vartheta^2 + q\beta_z^2)\,dz\,d\vartheta = \iint\limits_{R_i} [(a_z\beta)_\vartheta - (a_\vartheta\beta)_z]\,d\vartheta\,dz$$

$$= \int\limits_{L_{i-1}} a\,d\beta + \int\limits_{L_i} a\,d\beta\ ,$$

the integration in both integrals being performed in the direction positive with respect to R_i. Summing these

relations with respect to i and denoting the limits of the quantities α and β on L_i from the right and from the left by the signs "$-$" and "$+$", respectively, we obtain

$$0 \leqslant J = \sum_{i=2}^{n} J_i = \int_{L_1} \alpha d\beta + \sum_{i=2}^{n-1} \int_{L_i} (\alpha^- d\beta^- - \alpha^+ d\beta^+) - \int_{L_1'} \alpha d\beta \,, \tag{11.40}$$

the integration in all integrals in (11.40) being performed in the same direction as on L_1, i.e. in the direction of decrease of the quantity ϑ.

In view of the continuity of deformation we have in view of (11.33)

$$\begin{aligned} \alpha^- d\beta^- - \alpha^+ d\beta^+ &= (\alpha^- - \alpha^+) d\beta^- \\ &= -[\varrho'^-(z_i) - \varrho'^+(z_i)]\beta_\vartheta^2 d\vartheta \text{ on } L_i \,, \end{aligned}$$

which yields

$$\int_{L_i} \alpha^- d\beta^- - \alpha^+ d\beta^+ \leqslant 0 \,, \qquad i = 2, \ldots, n-1 \,. \tag{11.41}$$

From the boundary condition (11.37) we obtain

$$\int_{L_1} \alpha d\beta = -\int_{L_1} \mu \alpha^2 d\vartheta \quad \text{and} \quad -\int_{L_1'} \alpha d\beta = \int_{L_1'} \mu \alpha^2 d\vartheta \,,$$

where

$$\mu = \frac{\tilde{\nu}_1}{\tilde{\nu}_2} = \frac{\nu_1}{\varrho(\varrho'\nu_1 - \nu_2)} \,.$$

We now assume that

$$\mu \leqslant 0 \quad \text{on} \quad L_1 \quad \text{and} \quad \mu \geqslant 0 \quad \text{on} \quad L_1' \,, \tag{11.42}$$

Then

$$\int_{L_1} \alpha d\beta \leqslant 0 \quad \text{and} \quad -\int_{L_1'} \alpha d\beta \leqslant 0 \,.$$

It follows from the last equation in view of (11.41) that $J \leqslant 0$. On the other hand, however, in view of (11.40) $J \geqslant 0$. Hence, $J = 0$, and $\beta_\vartheta = \beta_z \equiv 0$; it can easily be verified that the last result is possible only for trivial infinitesimal bendings.

If the contour L_1' is absent, i.e. when the section S_1 is simply-connected and a bush constraint is given only on L_1 and satisfies the conditions (11.36) and (11.42), then making use of the property of smoothness of the functions α and β in the vicinity of vertices of the surface, it is readily observed that also $\beta_\vartheta = \beta_z \equiv 0$.

Thus, we have proved

THEOREM 5.20. *A section S_1 of a convex sectionally regular surface of revolution bounded by one or two parallels, does not admit non-trivial infinitesimal bendings if it is subject to bush constraints generated by a surface Σ satisfying the conditions* (11.42).

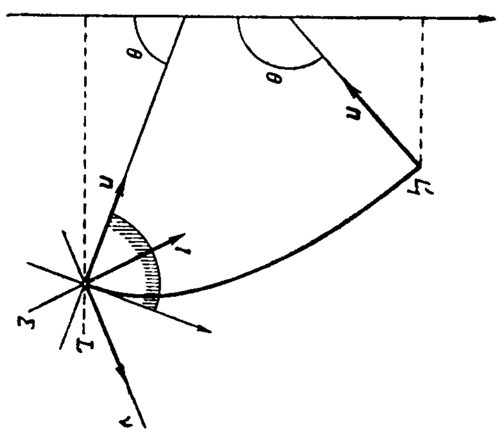

FIG. 9

If $\boldsymbol{n}$ is the inward normal of the surface S and φ is the angle between $\boldsymbol{\nu}$ and $\boldsymbol{n}$ it is readily derived by making use of the relation $\cot\theta = \varrho'(z)$ where θ is the angle between $\boldsymbol{n}$ and $\boldsymbol{k}$, that the condition (11.42) is satisfied if $\varphi > \frac{\pi}{2}$ as $\nu_1 > 0$, or $\varphi < \frac{\pi}{2}$ as $\nu_1 < 0$.

In Fig. 9 the appropriate region of change of the generators $\boldsymbol{l}$ of the surface Σ, orthogonal to the normal vector $\boldsymbol{\nu}$, is shaded. If all generators of the boundary strip of the surface Σ along the edge L are situated in the indicated

region, the bush constraint under consideration is certainly rigid. In the opposite case, as will be proved below, the bush constraint can turn out to be non-rigid. This fact however will occur in general only in exceptional cases.

11.7. If ν_1 and ν_2 are constant along L_1, i.e. when the boundary strip of the surface Σ passing through the contour L coincides with the strip of the cone of revolution the axis of rotation of which is the oz- axis, the assertion

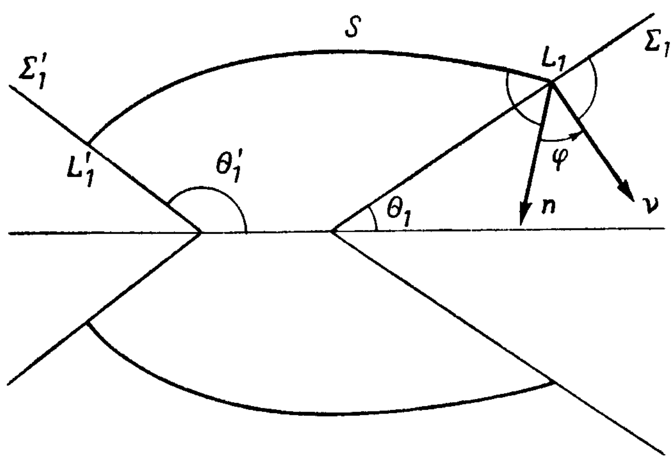

FIG. 10

proved above can be significantly strengthened (at the same time the proof is simplified). Namely, in this case the cones Σ and Σ_1' (obviously in the case under consideration all surfaces Σ can simply be regarded as cones of revolution) which lead to non-rigid constraints can be described more precisely. In this case both cones passing through L_1 and L_1' are completely determined by the angles θ_1 and θ_1', respectively, between the generators of the cones and the oz- axis (Fig. 10). It is obvious then that

$$\frac{\nu_1}{\nu_2} = \cot\theta_1 \quad \text{on} \quad L_1 \quad \text{and} \quad \frac{\nu_1}{\nu_2} = \cot\theta_1' \quad \text{on} \quad L_1'. \qquad (11.43)$$

THEOREM 5.21. *For an arbitrary* θ_1 *there exists a sequence* $\theta'_{1,k}$, *cot* $\theta'_{1,k} \to -\varrho'(z'_1)$ *when* $k \to \infty$, *such that the constraints determined by the cones* Σ_1 *and* $\Sigma'_{1,k}$ $(\theta'_1 = \theta'_{1,k})$, *respectively, are non-rigid. If for a fixed cone* Σ_1 *the cone* Σ'_1 *does not coincide with any of the cones given by the formula* (11.48) $(\theta'_1 = \theta'_{1,k})$ *the bush constraints determined by the pair of cones* (Σ_1, Σ'_1) *are rigid.*

PROOF. Let us expand the function $v(z, \vartheta)$ in the Fourier series with respect to ϑ, [41b]:

$$v(z, \vartheta) = \sum_{k=-\infty}^{\infty} \psi_k e^{ik\vartheta}.$$

The coefficients ψ_k of the above expansion, in view of (11.31) satisfy the equations

$$\varrho(z)\psi_k'' + (k^2-1)\psi_k \varrho'' = 0 \qquad (k = 0, \pm 1, \ldots). \tag{11.44}$$

The condition of bush constraints (11.36) yields the following boundary conditions for the functions $\psi_k(z)$, for every k:

$$\varrho\psi_k' + \varrho'(k^2-1)\psi_k + k^2\psi_k \cot\theta = 0 \qquad \text{for} \qquad z = z_1 \tag{11.45}$$

and

$$\varrho\psi_k' + \varrho'(k^2-1)\psi_k + k^2\psi_k \cot\theta = 0 \qquad \text{for} \qquad z = z'_1. \tag{11.46}$$

If the equation (11.44) for $k \geqslant 2$ has no continuous solutions in the interval (z_1, z'_1) which satisfy the boundary conditions (11.45) and (11.46), then it is evident that the bush constraints under consideration are rigid.

If the contour L_1 is absent, i.e. if the section of the surfaces under consideration contains a vertex, the boundary condition (11.45) on L_1 has to be replaced by the condition of boundedness of the solution of the equation (11.44) for $z \to z_0$. In view of the conditions $\varrho(z_0) = 0$, $\varrho'(z_0) = +\infty$ this leads to the relationships

$$\psi_k(z_0) = \psi_k'(z_0) = 0; \qquad \frac{\psi_k'(z)\,\varrho'(z)}{\varrho(z)} \to 0 \qquad \text{as} \qquad z \to z_0. \tag{11.47}$$

Integrating the equation (11.44) over the interval (z_1, z_1') we obtain

$$\psi_k' \varrho - \psi_k \varrho' \Big|_{z_1}^{z_1'} = -k^2 \int_{z'}^{z_1'} \psi_k \varrho'' dz ,$$

which in view of (11.45) leads to the condition

$$\psi_k(z_1')(\varrho'(z_1')+\cot\theta_{1,k}') - (\varrho'(z_1)+\cot\theta_1) = \int_{z_1}^{z_1'} \psi_k \varrho'' dz . \quad (11.48)$$

In view of the homogeneity of the equation (11.44) and the boundary conditions (11.45) we have taken above with no loss of generality $\psi_k(z_1) = 1$.

Thus, if the angle $\theta_{1,k}'$ satisfies the condition (11.48) the bush constraints under consideration are non-rigid; in fact, taking for $\psi_k(z)$ a solution of the equation (11.44) with the initial conditions

$$\psi_k(z_1) = 1 , \qquad \psi_k'(z_1) = \frac{\varrho'(z_1) - k^2[\varrho'(z_1) + \cot\theta_1]}{\varrho(z_1)} ,$$

we find that this solution according to (11.48) also satisfies the second boundary condition (11.46) and does not vanish identically. On the other hand, since the function $\cot\theta$ takes real values in the interval $(0, \pi)$ the relation (11.48) for any fixed θ_1 determines a sequence of angles $\theta_{1,k}'$, i.e. cones Σ_k', $k = 1, 2, \ldots$, which added to the cone Σ_1 set up non-rigid bush constraints.

If L_1' is absent [81], the relations (11.48) are regarded as conditions determining a sequence of angles $\theta_{1,k}$ for which the bush constraints set up by the cone Σ are non-rigid. In accordance with (11.48) these conditions take the form

$$\varrho'(z_1) + \cot\theta_{1,k} = -\int_{z_1}^{z_0'} \psi_k(z)\, \varrho''(z)\, dz . \qquad (11.49)$$

It still remains to prove the last assertion of the theorem. For simplicity we shall confine ourselves to the case in which L_1' is absent. Our condition $\psi_k(z_1) = 1$, the con-

ditions (11.47) and the equations (11.44) imply immediately that $\psi_k(z)$ is monotonous and $0 < \psi_k(z) < 1$ in the interval (z_1, z_0'). Further, taking into account (11.47) we obtain from (11.44) an integral equation for ψ_k

$$\psi_k(z) = k^2 \int_{z_0'}^{z} \int_{z_0'}^{u} \frac{\psi_k(s)\,\varrho''(s)}{\varrho(s)}\,ds\,du\,. \qquad (11.50)$$

It is however readily observed that in view of monotonic behaviour of $\psi_k(z)$ the assumption $\psi_{k_j}(z) > a > 0$ for a sequence $k_j \to \infty$ in an interval $z_1 < \bar{z} < z < \bar{\bar{z}} < z_0'$ at once leads to a contradiction with the equation (11.50). Hence, inside the inerval $(z_1, z_0')\,\psi_k(z) \to 0$ uniformly. But then, according to (11.49) $\varrho'(z_1) + \cot\theta_{1,k} \to 0$. This completes the proof.

In particular, we have also proved that $\varrho'(z_1) + \cot\theta_{1,k} > 0$. Now, from (11.49) it is possible to obtain for $\varrho'(z_1) + \cot\theta_{1,k}$ an upper estimate [81]. In fact, we infer immediately from the equation (11.44) that $\dfrac{\psi_k(z)}{z - z_1} \leqslant -\dfrac{1}{z_1 - z_0'}$ for all $z_1 < z < z_0'$. Substituting this estimate in (11.49) we have

$$0 \leqslant \varrho'(z_1) + \cot\theta_{1,k} \leqslant -\int_{z_1}^{z_0'} \frac{z - z_1}{z_1 - z_0'}\,\varrho''(z)\,dz$$
$$= -\varrho'(z_1) - \frac{\varrho(z_1)}{z_1 - z_0'}\,. \qquad (11.51)$$

It is readily observed that the last inequality has the following simple interpretation. Let us denote by Σ_0 and Σ_1 cones whose generators are perpendicular at the point A to the curve $\varrho = \varrho(z)$ and the straight line OA, respectively (Fig. 11). Then the generators of all cones Σ_k which set up non-rigid bush constraints are contained between the generators of the cones Σ_0 and Σ_1 (see Fig. 11).

11.8. In some cases—when the equations (11.44) can be explicitly solved—the formulae (11.48) make it possible to find exact values of the angles θ_k corresponding to non-rigid bush constraints.

Below we shall give examples of calculations of the angles θ_k. To begin with let us consider the case of infinitesimal bendings of a spherical section. We shall here apply a method distinct from that of the preceding section;

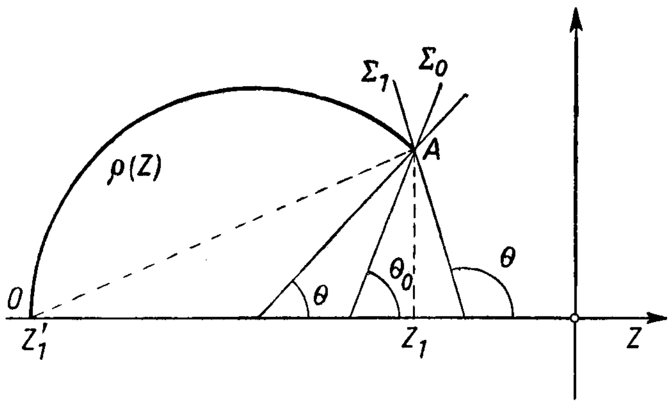

FIG. 11

the method of functions of a complex variable will be employed; in the general case it was given in the preceding sections.

Let (ξ, η) be an isometric–conjugate coordinate system on the sphere. It is known that such a system occurs if the sphere is projected from the south pole on the plane of the equator. Put $\zeta = \xi + i\eta$, ξ and η being connected with the geographic coordinates on the sphere by the formulae

$$\zeta = \tan\frac{\vartheta}{2}e^{i\varphi}, \tag{11.52}$$

where ϑ is the latitude and φ the longitude.

Let (x, y, z) be a system of Cartesian coordinates in space. The equation of the sphere S of radius r with centre at the point $(0, 0, q)$ referred to the isometric–conjugate coordinate system has the form

$$\boldsymbol{r} = \boldsymbol{r}(\zeta) = r\left(\frac{\zeta+\bar{\zeta}}{1+\zeta\bar{\zeta}}, \frac{i(\zeta-\zeta)}{1+\zeta\bar{\zeta}}, \frac{q}{r}+\frac{1-\zeta\bar{\zeta}}{1+\zeta\bar{\zeta}}\right). \tag{11.53}$$

Let us also write the expressions for the coefficients $a_{\alpha\beta}$ of the first fundamental quadratic form and the determinant a:

$$a_{11} = a_{22} = \frac{4r^2}{(1+\zeta\bar{\zeta})^2} \equiv \sqrt{a}\,, \qquad a_{12} = a_{21} = 0\,. \quad (11.54)$$

On the basis of the formulae (3.65a) we now obtain the following expression for the displacement vector:

$$U = \mathrm{Re}\left[-2\Phi(\zeta)\boldsymbol{n}_\zeta + \left(\Phi'(\zeta) - \frac{2\bar{\zeta}}{1+\zeta\bar{\zeta}}\Phi(\zeta)\right)\boldsymbol{n}\right], \quad (11.55)$$

where $\boldsymbol{n}$ is the unit vector of the inward normal, $\Phi(\zeta)$ a function holomorphic in ζ which according to (3.65b) and (11.54) is related to the complex displacement function by the formula

$$\Phi(\zeta) = \frac{w(\zeta)}{\sqrt{a}} = \frac{1}{4r^2} w(\zeta)(1+\zeta\bar{\zeta})^2\,. \quad (11.56)$$

The condition $\Phi \equiv 0$ is sufficient but in general not necessary in order that the infinitesimal bending corresponding to the function $\Phi(\zeta)$ be trivial, i.e. that the displacement field $\boldsymbol{U}$ be identical with the field of an infinitesimal motion.

Let us also note the following expressions for the vectors $\boldsymbol{r}_\zeta$ and $\boldsymbol{n}$:

$$\boldsymbol{r}_\zeta = r\left(\frac{1-\bar{\zeta}^2}{(1+\zeta\bar{\zeta}^2)}, \frac{-i(1+\bar{\zeta}^2)}{(1+\zeta\bar{\zeta})^2}, \frac{-2\bar{\zeta}}{(1+\zeta\bar{\zeta})^2}\right) = -r\boldsymbol{n}_\zeta\,, \quad (11.57)$$

$$-\boldsymbol{n} = \left(\frac{\zeta+\bar{\zeta}}{1+\zeta\bar{\zeta}}, \frac{i(\bar{\zeta}-\zeta)}{1+\zeta\bar{\zeta}}, \frac{1-\zeta\bar{\zeta}}{1+\zeta\bar{\zeta}}\right). \quad (11.58)$$

In a displacement of the centre of the sphere S along the z- axis the expressions for $\boldsymbol{n}_\zeta$ and $\boldsymbol{n}$ remain unaltered.

Employing the formulae derived above we examine the sliding of a spherical section along two co-axial circular cones (Fig. 12).

The parallel edges of the section S will be denoted by L_1 and L_2. Let θ_1 and θ_2 be the latitudes of L_1 and L_2 in geographic coordinates. Let us imagine that into the openings bounded by the curves L_1 and L_2 two circular cones Σ_1 and Σ_2 are inserted, their axes coinciding with

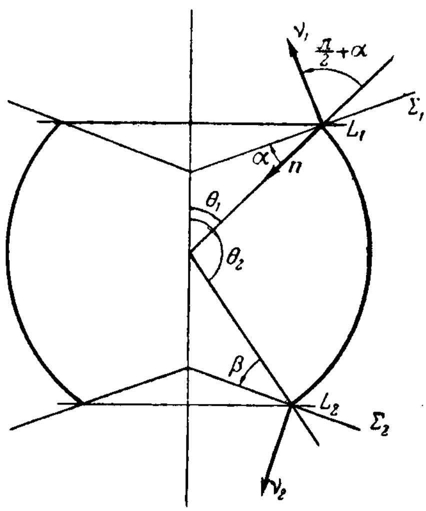

FIG. 12

the z- axis. The situation of Σ_1 and Σ_2 is completely determined by the angles α and β the orientation of which is taken to be counter-clockwise so that the angle α in Fig. 12 is regarded as negative. Evidently, we have $\theta_1-\pi \leqslant \alpha < \theta_1$ and $\theta_2-\pi \leqslant \beta < \theta_2$. The equality signs correspond to the case in which one or both cones Σ_1 and Σ_2 degenerate into cylinders with generators parallel to the z- axis. For $\alpha = \beta = 0$ we obtain orthogonal constraints to which Theorem 5.19 may be applied. If $\alpha = \theta_1-\frac{\pi}{2}$, $\beta = \theta_2-\frac{\pi}{2}$ the cones degenerate into planes and we have the case of slliding along parallel planes.

Let ν_1 and ν_2 be unit vectors of the normals of Σ_1 and Σ_2 at points of the contours L_1 and L_2, respectively. Then

the condition of bush constraints takes the form (we assume that the cones are rigid and fixed)

$$\begin{aligned} U\nu_1 &= 0 \quad \text{on} \quad L_1\,, \\ U\nu_2 &= 0 \quad \text{on} \quad L_2\,, \end{aligned} \tag{11.59}$$

where U is the displacement vector given by the formula (11.55). Let n_1 and n_2 be inward normals of S on the edges L_1 and L_2, respectively. It is clear (see Fig. 12) that

$$\begin{aligned} n_1\nu_1 &= \sin\alpha \quad \text{on} \quad L_1\,, \\ n_2\nu_2 &= \sin\beta \quad \text{or} \quad L_2\,. \end{aligned} \tag{11.60}$$

Similarly, it is readily observed that the following expresions for the components of the vectors ν_1 and ν_2 hold:

$$\begin{aligned} \nu_1 &= \left(\frac{\lambda_1(\zeta+\bar\zeta)}{2\varrho_1},\ \frac{\lambda_1 i(\bar\zeta-\zeta)}{2\varrho_1},\ \nu_1\right) \quad \text{on} \quad L_1\,, \\ \nu_2 &= \left(\frac{\lambda_2(\zeta+\bar\zeta)}{2\varrho_2},\ \frac{\lambda_2 i(\bar\zeta-\zeta)}{2\varrho_2},\ \nu_2\right) \quad \text{on} \quad L_2\,, \end{aligned} \tag{11.61}$$

where

$$\begin{aligned} \lambda_1 &= -\cos(\theta_1-\alpha)\,, \qquad \nu_1 = \sin(\theta_1-\alpha)\,, \\ \lambda_2 &= -\cos(\theta_2-\beta)\,, \qquad \nu_2 = \sin(\theta_2-\beta)\,. \end{aligned} \tag{11.62}$$

Hence, taking into account (11.57) we obtain for the scalar product $-n_\zeta\nu_i$, the following relation:

$$-n_\zeta\nu_i = \frac{\bar\zeta r}{\varrho_i(1+\varrho_i^2)^2}\left(\lambda_i(1-\varrho_i^2)-2\nu_i\varrho_i\right) \equiv \frac{\bar\zeta\mu_i}{(1+\varrho_i^2)^2} \tag{11.63}$$

$$(i = 1, 2)\,.$$

Taking into account the expression (11.55) for the components of the displacement vector, in view of (11.50), (11.63), the condition (11.59) takes the form

$$\operatorname{Re}(\Phi\bar\zeta\gamma_i+\Phi') = 0 \quad \text{on} \quad L_i \quad (\text{i.e. } |\zeta| = \varrho_i\,,\ i = 1, 2)\,, \tag{11.64}$$

where

$$\gamma_1 = \frac{2[\mu_1-\sin\alpha(1+\varrho_1^2)]}{(1+\varrho_1^2)^2\sin\alpha}\,,$$

$$\gamma_2 = \frac{2[\mu_2-\sin\beta(1+\varrho_2^2)]}{(1+\varrho_2^2)^2\sin\beta} \tag{11.65}$$

and $\Phi = \Phi(\zeta)$ is a function holomorphic in the ring $\varrho_1 < |\zeta| < \varrho_2$, which determines an infinitesimal bending of the section S. Thus, the determination of all infinitesimal bendings of the section S subject to the bush constraints described above, is reduced to the determination of all functions $\Phi = \Phi(\zeta)$ holomorphic in the ring $\varrho_1 < |\zeta| < \varrho_2$, in accordance with the condition (11.64).

By the substitution $\Phi = \zeta\Psi$ we can transform the condition (11.64) to the form

$$\operatorname{Re}(\Psi\tilde{\gamma}_i + \zeta\Psi') = 0 \quad \text{for} \quad |\zeta| = \varrho_i\,, \tag{11.66}$$

where $\tilde{\gamma}_i = 1 + \gamma_i\varrho_i^2$; taking into account the relations $\sin\theta_i = \dfrac{2\varrho_i}{1+\varrho_i^2}$, $\cos\theta_i = \dfrac{1-\varrho_i^2}{1+\varrho_i^2}$, we find that

$$\tilde{\gamma}_1 = \frac{\sin(\alpha-\theta_1)}{\sin\alpha}\,, \qquad \tilde{\gamma}_2 = \frac{\sin(\beta-\theta_2)}{\sin\beta}\,. \tag{11.67}$$

For the determination of $\Psi(\zeta)$ we have the boundary value problem of the form (11.66). Expanding $\Psi(\zeta)$ in the Laurent series in the ring $\varrho_1 < |\zeta| < \varrho_2$

$$\Psi(\zeta) = \sum_{k=-\infty}^{+\infty} a_k\zeta^k\,, \tag{11.68}$$

it is easy to show that this problem has a non-trivial solution if and only if

$$\frac{(\tilde{\gamma}_1+k)(\tilde{\gamma}_2-k)}{(\tilde{\gamma}_1-k)(\tilde{\gamma}_2+k)} = \left(\frac{\varrho_1}{\varrho_2}\right)^{2k} \tag{11.69}$$

for an integer k. The number $k = 0$ always satisfies the condition (11.69); this is a solution of the problem if at least one of the numbers $\tilde{\gamma}_1$ or $\tilde{\gamma}_2$ is different from zero. If both vanish we obtain two linearly independent solutions. Obviously, in both cases the solutions are eliminated by the number of degrees of freedom which the section S possesses under the above described constraints. The sec-

tion S has non-trivial infinitesimal bendings only if (11.69) is satisfied for some $k > 1$.

We examine in detail the case in which the cone Σ_1 is absent. Then we may take $\theta_1 = \varrho_1 = 0$ and our problem is reduced to the determination of all functions holomorphic in the circle $|\zeta| < \varrho = \varrho_2$ in accordance with the boundary condition

$$\operatorname{Re}[\Phi\bar{\zeta}\gamma + \Phi'] = 0\,, \qquad \gamma = \widetilde{\gamma}_2\,. \tag{11.70}$$

The function

$$\Phi_1(\zeta) = a_0 + a_1\zeta + a_2\zeta^2 \tag{11.71}$$

is a solution of (11.70) if the coefficients a_0, a_1, a_2 satisfy the system of equations

$$\begin{aligned} (\gamma\varrho^2 + 1)(a_1 + \bar{a}_1) &= 0\,, \\ \gamma a_0 + a_2(2 + \gamma\varrho^2) &= 0\,. \end{aligned} \tag{11.72}$$

Thus, for $\gamma\varrho^2 + 1 \neq 0$ we obtain three linearly independent solutions of the problem (11.70). The remaining solutions of this problem can be sought for in the form $\Phi = \zeta\Psi$, $\operatorname{Im}\Psi(0) = 0$. Then (11.70) takes the form

$$\operatorname{Re}(\Psi\gamma + \zeta\Psi') = 0 \quad \text{on} \quad |\zeta| = \varrho\,, \quad \widetilde{\gamma} = 1 + \gamma\varrho^2\,. \tag{11.73}$$

This condition is identical with (11.66). Hence, (11.73) has a non-trivial solution only if $\widetilde{\gamma} = -k$ for a positive integer k. All solutions of the problem (11.70) are given by the formula

$$\Phi(\zeta) = \Phi_1(\zeta) + a_k\zeta^{k+1}\,, \tag{11.74}$$

where a_k is an arbitrary complex number distinct from zero only when the condition $\widetilde{\gamma} = -k$ is satisfied, i.e.

$$\frac{\sin(\theta - \beta)}{\sin\beta} = k \tag{11.75}$$

for an integer k. Obviously, we should have $k > 1$. Three solutions of the form (11.71) yield three trivial displacement fields corresponding to the three degrees of freedom of the section as a rigid body under prescribed constraints. For

a fixed value of θ there exists a sequence of angles such that for $\beta = \beta_k$ the section S has two linearly independent non-trivial infinitesimal bendings. We have always $\beta_k > 0$ and $\beta_k \to 0$ as $k \to \infty$. For $\beta < 0$ the problem has the same number of solutions as under purely orthogonal constraints, when $\beta = 0$, i.e. three trivial bendings corresponding to the three analytic functions $\Phi = i\zeta, \zeta^2 - 1, i(\zeta^2 + 1)$.

The same facts are discovered in a detailed investigation of the condition (11.69). The whole reasoning concerning the condition (11.75) can here be repeated; in particular if $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ have the same sign, (11.69) cannot be satisfied for any $k \geqslant 1$.

11.9. Let us consider an ellipsoid of revolution S with the contour L, the meridian of which has the form $\varrho(z) = c\sqrt{R^2 - (z-R)^2}$, $0 \leqslant z \leqslant z_1$, where $c > 0$, $R > 0$ are arbitrary constants. The following theorem is valid:

THEOREM 5.22. *If the generators of the cone setting up a bush constraint on the opening $z = z_1$ of the surface S make the angle*

$$\theta_k = \cot^{-1}\left(-c\frac{R - k(z_1 - R)}{k\sqrt{R^2 - (z_1 - R)^2}}\right), \qquad k \geqslant 2\,, \tag{11.76}$$

with the z- axis, the surface S is not rigid. In all other cases the surface S is rigid.

PROOF. The equation (11.44) for our surface can be written in the form

$$\psi_k''(z) - (k^2 - 1)\frac{R^2}{[R^2 - (z - R^2)]^2}\psi_k(z) = 0\,. \tag{11.77}$$

The change of the variable

$$R - z = R\tanh t \tag{11.78}$$

leads to the following form of (11.77)

$$\psi_{kt}'' + 2\,\mathrm{th}\,\psi_k' + \psi_k - k^2\psi_k = 0\,. \tag{11.79}$$

Let $\varphi_k(t) = \coth\cdot\psi_k$; then (11.79) takes the form

$$\varphi_k'' - k^2\varphi_k = 0\,. \tag{11.80}$$

The linearly independent solutions of this equation have the form

$$\varphi_k^{(1)} = e^{kt}, \quad \varphi_k^{(2)} = e^{-kt}. \tag{11.81}$$

Hence, for two linearly independent solutions of the equation (11.79) we have

$$\psi_k^{(1)} = \frac{1}{\cos ht} e^{kt}, \quad \psi_k^{(2)} = \frac{1}{\cos ht} e^{-kt}. \tag{11.82}$$

Since $t \to +\infty$ as $z \to 0$ only the solution $\psi_k^{(2)}$ is the required one.

The solution of the equation (11.31) can be written as follows:

$$v(t, \vartheta) = \sum_{k=0}^{\infty} (a_k \cos k\vartheta + b_k \sin k\vartheta) \frac{1}{\cos ht} e^{-kt}, \tag{11.83}$$

$$t = th^{-1}\left(1 - \frac{z}{R}\right).$$

Since $\psi_k^{(2)}(z)$ should satisfy a boundary condition of the form (11.45) we have

$$\left\{c[k + \tan ht - (1-k^2)\tan ht] + \frac{k^2}{\cos ht} \cot\theta\right\} e^{-kt} = 0 \quad \text{on} \quad L,$$

i.e.

$$\cot\theta = -c\frac{R - k(z_1 - R)}{k\sqrt{R^2 - (z_1 - R)^2}} \quad (k \geqslant 2). \tag{11.84}$$

This completes the proof of the theorem.

Let us observe that it follows directly from (11.84) *that when* $z_1 = R + \frac{R}{k}$, $k = 2, 3$, *the surface* S *admits sliding along the plane* $z = z_1$. *In the case of the sphere* ($c = 1$) *this result was proved first by Rembs* [75].

In an analogous way we can prove the following theorem:

THEOREM 5.23. *Let us consider a surface of revolution* S *the meridian of which has the form* $\varrho(z) = Cz^a$, $0 \leqslant z \leqslant z_1$, *where* $0 < a < 1$; $C > 0$ *is an arbitrary constant.*

If the generators of the cone make the angle θ_k given by the formula

$$\theta_k = \cot^{-1}\left(-C\frac{1+2a(k^2-1)+\sqrt{4(k^2-1)\,a(a-1)+1}}{2k^2 z_1^{1-a}}\right), \quad k \geqslant 2, \qquad (11.85)$$

with the z- axis, the surface S is not rigid. In all other cases the surface S is rigid.

The results of this article were obtained by Sun' Che-shen [81a].

11.10. We found in §8.2 that in some cases the contacting of two surfaces of positive curvature into one closed surface by imbedding one into the other, leads to rigid non-convex surfaces. Below, making use of the equation (11.44) this problem will be investigated in more detail, for surfaces of revolution [11h].

THEOREM 5.24. *Let S_1 and S_2 be two co-axial convex sectionally smooth surfaces of revolution contacted along the*

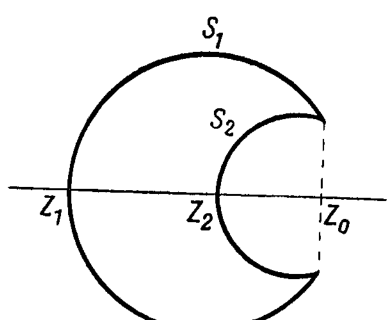

FIG. 13

common parallel L into one closed surface S for which L is a rib. Let S_2 be situated inside S_1 so that the whole surface S lies on one side of the plane of the parallel L. Then the surface S is rigid (Fig. 13).

PROOF. We shall confine ourselves to the case in which S_1 and S_2 are smooth surfaces (of the class C^2).

Let $\varrho_i = \varrho_i(z)$,

$$z_i < z < z_0 ,$$
$$\varrho_1(z_0) = \varrho_2(z_0) ,$$
$$\varrho_i'' \leqslant 0 , \quad i = 1, 2$$

be the parallels of the surfaces S_1 and S_2, respectively. According to our assumptions we have

$$z_1 \leqslant z_2 \quad \text{and} \quad \varrho_2(z) \leqslant \varrho_1(z)$$
$$\text{for} \quad z_2 \leqslant z < z_0 .$$

Let U_i be a displacement field on S_i satisfying the condition

$$U_1 = U_2 \quad \text{on} \quad L . \tag{11.86}$$

Decomposing U_i in accordance with the formula (11.30) and making use of the condition (11.86) we obtain form the system (11.31) for the components $\psi_{k,i}$ besides the equations

$$\psi_{k,i}''\varrho_i + (k^2-1)\psi_{k,i}\varrho_i'' = 0 , \quad i = 1, 2 ,$$
$$k = 1, 2, 3, \ldots, \tag{11.87}$$

also the following conjunction conditions for $z = z_0$:

$$\psi_{k,1}(z) = \psi_{k,2}(z_0) ,$$
$$\varrho_1'(z_0)\psi_{k,1}(z_0)\lambda + \varrho_1(z_0)\psi_{k,1}'(z_0)$$
$$= \varrho_2'(z_0)\psi_{k,2}(z_0)\lambda + \varrho_2(z_0)\psi_{k,2}'(z_0) , \quad \lambda = k^2-1 . \tag{11.88}$$

Our task is to prove that the systems (11.87) under the conjunction conditions (11.88) for $k > 1$ have no solutions distinct from zero.

In what follows we shall fix k and we shall set $\psi_{k,i} = \psi_i$, $i = 1, 2$.

Let $\psi_1(z_0) = \psi_2(z_0) \neq 0$. In view of the homogeneity of the equation (11.87) and the conditions (11.88) we may assume that $\psi_1(z_0) = \psi_2(z_0) = 1$. Boundedness of $\psi_i(z)$ at the point $z = z_i$ implies the relation

$$\psi_i(z_i) = 0 , \quad \frac{\psi_i\varrho_i'}{\varrho_i} \to 0 \quad \text{as} \quad z \to z_i \quad \text{and} \quad \psi'(z_1) = 0 . \tag{11.89}$$

Further, making use of the familiar fact that in view of the conditions $\varrho_i'' \leqslant 0$ the solutions of the equation (11.87) admit neither positive maximum nor negative minimum, it is easy to show that $\psi_1 > 0$ and $\psi_2 > 0$ as $z_2 < z < z_0$, both functions being convex downwards and strictly increasing, i.e. $\psi_i'(z) > 0$. It is also readily observed that $\frac{\psi_i}{\varrho_i}$ also increase, i.e.

$$\left(\frac{\psi_i}{\varrho_i}\right)' > 0 . \tag{11.90}$$

In fact, $(\psi_i'\varrho_i - \psi_i\varrho_i')' = -k^2\varrho_i''\psi_i > 0$, i.e. the function $\psi_i'\varrho_i - \psi_i\varrho_i' = \varrho_i^2\left(\frac{\psi_i}{\varrho_i}\right)'$ increases; but for $z = z_i$ it vanishes; hence, for $z > z_i$ it is everywhere positive.

Multiplying the first equation (11.87) by $\frac{\psi_2}{\varrho_1}$, the second by $\frac{\psi_1}{\varrho_2}$ and subtracting the results we obtain

$$(\psi_1'\psi_2 - \psi_1\psi_2')' = -\lambda\frac{\psi_1}{\varrho_1}\frac{\psi_2}{\varrho_2}(\varrho_1'\varrho_2 - \varrho_2'\varrho_1)' , \ \lambda = k^2 - 1 . \tag{11.91}$$

Integrating this relation over the interval $z_2 < z < z_0$, taking into account the conditions (11.89) on the end $z = z_2$ and integrating by parts we obtain

$$I = \lambda\int_{z_2}^{z_0}\left(\frac{\psi_1}{\varrho_1}\frac{\psi_2}{\varrho_2}\right)'(\varrho_1'\varrho_2 - \varrho_2'\varrho_1)\,dz \equiv \lambda\int_{z_2}^{z_0}\varphi\,dz , \tag{11.92}$$

where

$$I = \frac{1}{\varrho_1(z_0)}\Big[\big(\varrho_1'(z_0) - \varrho_2'(z_0)\big)\lambda - \varrho(z_0)\big(\psi_2'(z_0) - \psi_1'(z_0)\big)\Big] , \tag{11.93}$$

and φ denotes the integrand of the integral (11.92). We transform φ as follows:

$$\varphi = (\varrho_2^2 - \varrho_1^2)\left(\frac{\psi_1}{\varrho_1}\right)'\left(\frac{\psi_2}{\varrho_2}\right)' + \varrho_1^2\left(\frac{\psi_1}{\varrho_1}\right)'\left[\psi_2\left(\frac{1}{\varrho_2} - \frac{1}{\varrho_1}\right)\right]' +$$
$$+ \varrho_2^2\left(\frac{\psi_2}{\varrho_2}\right)'\left[\psi_1\left(\frac{1}{\varrho_2} - \frac{1}{\varrho_1}\right)\right]' . \tag{11.94}$$

The first term of the expression thus obtained is negative. The integrals of all other terms are also negative.

In fact,

$$\int_{z_2}^{z_J} \varrho_1^2 \left(\frac{\psi_1}{\varrho_1}\right)' \left[\psi_2\left(\frac{1}{\varrho_2} - \frac{1}{\varrho_1}\right)\right]' dz$$

$$= -\int_{z_2}^{z_J} \left[\varrho_1^2 \left(\frac{\psi_1}{\varrho_1}\right)'\right]' \psi_2 \left(\frac{1}{\varrho_2} - \frac{1}{\varrho_1}\right) dz \leqslant 0 ,$$

since $\dfrac{1}{\varrho_2} - \dfrac{1}{\varrho_1} > 0$, $\psi_2 > 0$ and the first bracket is also positive. Thus, we have established that $J < 0$ if only all terms in the expression for φ do not vanish identically. Now, in view of the boundary condition (11.88) $J = 0$. Hence, all three terms in the expression (11.94) vanish identically; it readily follows now that for $k > 1$, $\psi_i \equiv 0$, which contradicts the assumption.

Thus, the theorem is proved. The case of sectionally smooth surfaces is tackled in a similar way.

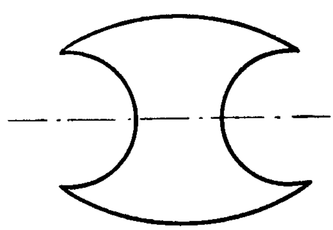

FIG. 14

We observe that it is possible to prove by a somewhat more complicated reasoning that if three co-axial surfaces of revolution are contacted as in Fig. 14, the closed surfaces S thus obtained is rigid.

If the surface S is composed of S_1 and S_2 in such a way that S_1 and S_2 are situated on distinct sides of the plane of the rib L, then the surface may be non-rigid. If S_2 is the mirror image of S_1 with respect to the plane E of the rib L, S will be non-rigid if and only if the surface S_1 admits sliding bendings along E (see Theorem 5.17).

We now give three examples of contacting of two surfaces into a closed non-convex surface. These examples make possible an exact characterization of the cases of rigid and non-rigid surfaces. The first and the third examples were given by Bojarski. The second was worked out by Sun' Che-shen.

11.11. Contacting of two spherical sections. This example will be computed by the method of §11.8.

Let r_1 and r_2 be radii of the sections S_1 and S_2, respectively. The equations of S_1 and S_2 will be taken in the form

$$\begin{aligned} \boldsymbol{r}_1(\zeta) &= r_1\left(\frac{\zeta+\bar{\zeta}}{1+\zeta\bar{\zeta}},\ \frac{i(\bar{\zeta}-\zeta)}{1+\zeta\bar{\zeta}},\ \frac{1-\zeta\bar{\zeta}}{1+\zeta\bar{\zeta}}\right), \\ \boldsymbol{r}_2(\zeta) &= r_2\left(\frac{\lambda(\zeta+\bar{\zeta})}{\lambda^2+\zeta\bar{\zeta}},\ \frac{i\lambda(\bar{\zeta}-\zeta)}{\lambda^2+\zeta\bar{\zeta}},\ \frac{q}{r_2}-\frac{\lambda^2-\zeta\bar{\zeta}}{\lambda^2+\zeta\bar{\zeta}}\right), \end{aligned} \tag{11.95'}$$

where $|\zeta| \leqslant \varrho$, $\lambda = \dfrac{\varrho_1}{\varrho_2}$, $\varrho_i = \tan\dfrac{\theta_i}{2}$ (Fig. 15). Taking the equation of S_2 in the form (11.95′) we obtained that in the parametric representation of S_1 and S_2 the parameter ζ varies in the circle $|\zeta| \leqslant \varrho$ and in both cases to a point of the boundary ζ, $|\zeta| = \varrho$, of this circle there corresponds the same point of the common edge of the sections S_1 and S_2.

From the formulae (11.55) for the displacement vectors $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ of the surfaces S_1 and S_2, making use of the conjunction condition $\boldsymbol{U}_1 = \boldsymbol{U}_2$ for $|\zeta| = \varrho$, after some transformations we obtain the relations

$$\left.\begin{aligned} \frac{2\sqrt{r_1}}{1+\varrho^2}\operatorname{Re}\left(w_1 - w_1'\frac{\zeta+\bar{\zeta}}{2}\right) &= \frac{2\sqrt{r_2}\lambda}{\lambda^2+\varrho_2}\operatorname{Re}\left(w_2 - w_2'\frac{\zeta+\bar{\zeta}}{2}\right), \\ \frac{2\sqrt{r_1}}{1+\varrho_1^2}\operatorname{Re}\left[i\left(w_1 + w_1'\frac{\bar{\zeta}-\zeta}{2}\right)\right] &= \frac{2\sqrt{r_2}\lambda}{\lambda^2+\varrho^2}\operatorname{Re}\left[i\left(w_2 + w_2'\frac{\bar{\zeta}-\zeta}{2}\right)\right], \\ -\frac{2\sqrt{r_1}}{1+\varrho^2}\operatorname{Re}\left[w_1\bar{\zeta} + w_1'\frac{1-\varrho^2}{2}\right] &= \frac{2\sqrt{r_2}\lambda}{\lambda^2+\varrho^2}\operatorname{Re}\left[w_2\bar{\zeta} + w_2'\frac{\lambda^2-\varrho^2}{2}\right], \end{aligned}\right\} \tag{11.95}$$

where w_1 and w_2 are the corresponding functions holomorphic in the circle $|\zeta| < \varrho$. Thus, the problem of the determination of all infinitesimal bendings of the surface $S = S_1 + S_2$ including also the trivial bendings, is reduced to the determination of the pairs (w_1, w_2) of functions holomorphic in the circle $|\zeta| < \varrho$, which satisfy the boundary conditons (11.95). Introducing new unknowns

$$\widetilde{w}_2 = \mu w_2\,, \qquad \mu = \frac{\sqrt{r_2\lambda}}{\sqrt{r_1}}\,\frac{1+\varrho^2}{\lambda^2+\varrho^2} = \frac{\sqrt{r_1}}{\sqrt{r_2}}\,, \qquad \Phi = w_1 - \widetilde{w}_2$$

we find that the two equations (11.95) are equivalent to the condition

$$\operatorname{Re}\{i\bar{\zeta}\Phi\} = 0 \qquad \text{on} \qquad |\zeta| = \varrho\,,$$

whence

$$\Phi = a_0 + C\zeta + \frac{\bar{a}_0\zeta^2}{\varrho^2}\,,$$

where a_0 is an arbitrary complex and C a real constant. Substituting $w_1 = \widetilde{w}_2 + \Phi$ into the third equation (11.95) we obtain for w_2 the non-homogeneous condition

$$\operatorname{Re}\left(\bar{\zeta}\widetilde{w}_2 + \frac{\lambda-\varrho^2}{2}\,\widetilde{w}_2'\right) = -\frac{\lambda}{1+\lambda}\operatorname{Re}\left(\Phi\bar{\zeta} + \Phi'\,\frac{1-\varrho^2}{2}\right). \qquad (11.96)$$

Since the function

$$v(\zeta) = v_0 + C_1\zeta + \frac{\bar{v}_0}{\lambda}\zeta^2\,,$$

where

$$C_1 = -\frac{(1+\varrho^2)\lambda C}{(1+\lambda)(\lambda+\varrho^2)} \qquad \text{and} \qquad v_0 = -\frac{a_0}{2}\,\frac{(1+\varrho^2)\lambda}{\varrho^2(1+\lambda)}\,,$$

is a solution of the non-homogeneous problem (11.96) and the function

$$\widetilde{v}(\zeta) = b_0 + iC_2\zeta - \bar{b}_0\zeta^2$$

is a solution of the homogeneous problem (11.96) for arbitrary constants b_0 and C_2 (C_2 is real), we obtain

from (11.96) for the function $\chi(\zeta)$ where $\widetilde{w}_2 = v+\widetilde{v}+ +\zeta\chi(\zeta)$, a boundary condition of the form

$$\mathrm{Re}[\chi(\zeta)+\gamma\xi\chi'(\zeta)] = 0\,, \qquad \mathrm{Im}\,\chi(0) = 0\,,$$
$$\gamma = \frac{\lambda-\varrho^2}{\lambda+\varrho^2}. \qquad (11.97)$$

We found in §11.8 that the problem (11.97) has the following solution distinct from zero: $\chi_k(\zeta) = a_k\zeta^k$; a_k is an arbitrary constant only if $\gamma = -\frac{1}{k}$ where k is an integer and $k \geqslant 0$. In this case the problem (11.97) has exactly two linearly independent solutions. All pairs (w_1, w_2) which are solutions of the problem (11.95) are given by the formulae

$$\begin{aligned} w_1 &= \Phi+[v+\widetilde{v}+\zeta\chi_k(\zeta)]\frac{1}{\mu}\,, \\ w_2 &= [w+\widetilde{v}+\zeta\chi_k(\zeta)]\frac{1}{\mu}\,, \end{aligned} \qquad (11.98)$$

in which the term $\chi_k(\zeta)$ is present only if

$$k = \frac{\varrho^2+\lambda}{\varrho^2-\lambda} \quad \text{where} \quad \varrho_1\varrho_2 = \frac{k+1}{k-1}\,, \qquad (11.99)$$

where k is a positive integer. If (11.99) does not take place the formulae (11.98) yield six linearly independent displacement fields which evidently should be identical with the six-parameter family of infinitesimal motions of the surface S.

For every fixed ϱ_2 S is non-rigid only for a countable sequence of values of ϱ_1, $\varrho_{1,k} = \frac{1}{\varrho_2}\frac{k+1}{k-1}$; in particular, always $\varrho_{1,k} > \frac{1}{\varrho_1}$; if $\varrho_1 \leqslant \frac{1}{\varrho_2}$ the surface S is convex. For $k\to\infty$ we have $\varrho_{1,k}\to\frac{1}{\varrho_2}$, i.e. non-rigid surfaces are encountered only in the non-convex case; for a fixed ϱ_2 there is only a countable number of such surfaces, and

they accumulate on approaching the sphere $\left(\varrho_1 = \frac{1}{\varrho_2}\right)$ $\left(\text{in Fig. 15 } \tan\frac{\theta_k}{2} = \varrho_{1,k}\right)$.

11.12. Let us consider a closed surface of revolution composed of two spherical sections S_1, S_2 and a cylinder T the meridians of which are given by the formulae (Fig. 16)

$$\varrho_1(z) = \sqrt{R_1^2-(z-z_1)^2}; \qquad \varrho_2(z) = \sqrt{R_2^2-(z+z_2)^2};$$
$$\varrho = \sqrt{R_1^2-h_1^2} = \sqrt{R_2^2-h_2^2}, \qquad h_1 = z_1-z_0, \qquad h_2 = z_2-z_0$$

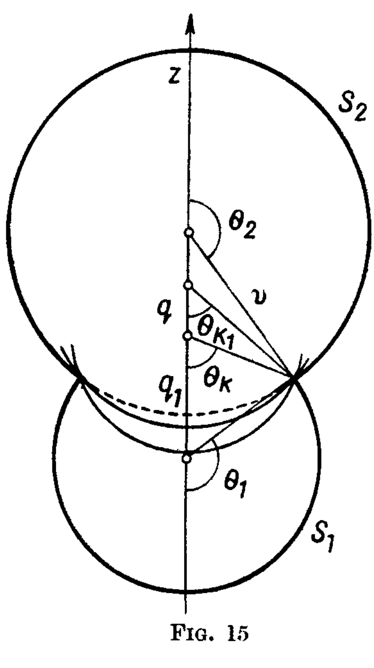

FIG. 15

respectively. S_1 and T are contacted along a circle L with center $z = z_0$ (on the z- axis) and radius $\varrho_1(z_0) = \varrho$. T and S_2 are contacted along a circle L_2 with center at $z = -z_0$ (on the z- axis) and radius $\varrho_2(-z_0) = \varrho$.

Let us denote by U_1, U_2, U_3 the displacement vectors of the surfaces S_1, S_2 and T, respectively.

In view of the continuity of the displacement vector $\boldsymbol{U}$ of the surface $S = S_1 + T + S_2$, on the contact lines L_1 and L_2 the following conditions are satisfied:

$$\boldsymbol{U}_1 = \boldsymbol{U}_3 \quad \text{on} \quad L_1\,,$$
$$\boldsymbol{U}_3 = \boldsymbol{U}_2 \quad \text{on} \quad L_2\,,$$

i.e.

$$\begin{gathered} u_1(z_0, \vartheta) = u_3(z_0, \vartheta)\,, \qquad v_1(z_0, \vartheta) = v_3(z_0, \vartheta)\,, \\ w_1(z_0, \vartheta) = w_3(z_0, \vartheta)\,, \\ u_3(-z_0, \vartheta) = u_2(-z_0, \vartheta)\,, \qquad v_3(-z_0, \vartheta) = v_2(-z_0, \vartheta)\,, \\ w_3(-z_0, \vartheta) = w_2(-z_0, \vartheta)\,. \end{gathered} \tag{11.100}$$

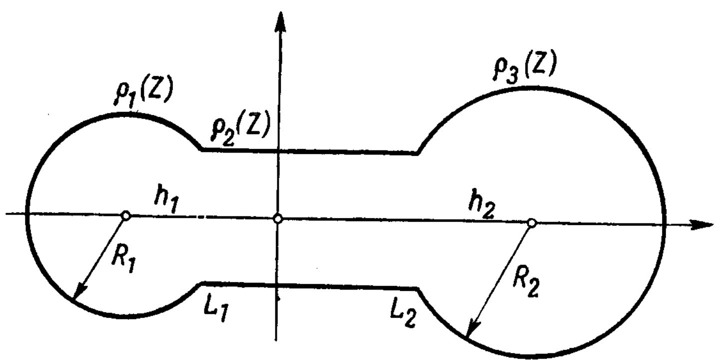

FIG. 16

Making use of the formula (11.83) the solutions of the equation (11.31) for the surfaces S_1, S_2 can be written in the form

$$\begin{aligned} &v_1(z, \vartheta) \\ &= \sum_0^\infty (a_k^{(1)} \cos k\vartheta + a_k^{(2)} \sin k\vartheta) \sqrt{1 - \left(\frac{z - z_1}{R_1}\right)^2}\, e^{-k \tan h^{-1}\left(\frac{z-z_1}{R_1}\right)}\,, \\ &v_2(z, \vartheta) \\ &= \sum_0^\infty (c_k^{(1)} \cos k\vartheta + c_k^{(2)} \sin k\vartheta) \sqrt{1 - \left(\frac{z + z_2}{R_2}\right)^2}\, e^{k \tan h^{-1}\left(\frac{z+z_2}{R_2}\right)}\,, \end{aligned} \tag{11.101}$$

and the solution (11.31) for the cylinder T:

$$v_3(z,\vartheta)$$
$$=\sum_0^\infty[(\alpha_k^{(1)}z+\beta_k^{(1)})\cos k\vartheta+(\alpha_k^{(2)}z+\beta_k^{(2)})\sin k\vartheta]\,. \qquad (11.102)$$

Taking into account (11.30) we obtain from (11.100), (11.101) and (11.102)

$$a_k^{(i)}\sqrt{1-\left(\frac{h_1}{R_1}\right)^2}\,e^{-k\tan h^{-1}\left(-\frac{h_1}{R_1}\right)}=\alpha_k^{(i)}z_0+\beta_k^{(i)} \qquad (11.103)$$
$$(i=1,2)\,,$$
$$a_k^{(i)}\left(-k+\frac{h_1}{R_1}k^2\right)e^{-k\tan h^{-1}\left(-\frac{h_1}{R_1}\right)}=\sqrt{R_1^2-h_1^2}\,\alpha_k^{(i)} \qquad (11.104)$$
$$(i=1,2)\,,$$
$$c_k^{(i)}\sqrt{1-\left(\frac{h_2}{R_2}\right)^2}\,e^{k\tan h^{-1}\left(\frac{h_2}{R_2}\right)}=-\alpha_k^{(i)}z_0+\beta_k^{(i)} \qquad (11.105)$$
$$(i=1,2)\,,$$
$$c_k^{(i)}\left(k-\frac{h_2}{R_2}k^2\right)e^{k\tan h^{-1}\left(\frac{h_2}{R_2}\right)}=\sqrt{R_2^2-h_2^2}\,\alpha_k^{(i)} \qquad (11.106)$$
$$(i=1,2)\,.$$

In view of the relations (11.103)–(11.106) we have the following

THEOREM 5.25. *For our surface*

$$S=S_1+T+S_2$$

non-trivial bendings exist only if one of the following conditions is satisfied:

(1) $h_1=\dfrac{R_1}{k}$ and $h_2=\dfrac{R_2}{k}$ $(k=2,3,\ldots)$;

(2) $z_0=\dfrac{1}{k}\,\dfrac{R_1^2-h_1^2}{2}\left(\dfrac{1}{kh_1-R_1}+\dfrac{1}{kh_2-R_2}\right)$

for

$$k\geqslant\max\left(\left[\frac{R_1}{h_1}\right],\ \left[\frac{R_2}{h_2}\right]\right)+1\equiv k_0\,,$$

or

$$k_1 \equiv \left[\frac{R_1+R_2}{h_1+h_2}\right] \geqslant k \geqslant \min\left(\left[\frac{R_1}{h_1}\right], \left[\frac{R_2}{h_2}\right]+1 \equiv k_2;\right.$$

$$(3) \quad z_0 \equiv 0\,, \qquad \frac{h_1+h_2}{R_1+R_2} = \frac{1}{k} \qquad (k = 2, 3, \ldots)\,.$$

PROOF. (1) It is almost obvious since from (11.104) and (11.106) we obtain $a_k^{(i)} = 0$, $i = 1, 2$, $k \geqslant 2$; substituting into (11.103) and (11.105) we cannot uniquely determine the other constants, i.e. $a_k^{(i)}, c_k^{(i)}, \beta_k^{(i)}$ do not vanish at all.

(2) It follows from (11.103)–(11.105) that

$$\frac{1}{R_1} a_k^{(i)} e^{k\tan h^{-1}\left(\frac{h_1}{R_1}\right)} - \frac{1}{R_2} c_k^{(i)} e^{k\tan h^{-1}\left(\frac{h_2}{R_2}\right)} = a_k^{(i)} z_0 \left(\frac{1}{\sqrt{R_1^2-h_1^2}} + \frac{1}{\sqrt{R_2^2-h_2^2}}\right), \qquad (11.107)$$

The relations (11.104)–(11.106) imply that

$$\frac{1}{R_1} a_k^{(i)} e^{k\tan h^{-1}\left(\frac{h_1}{R_1}\right)} - \frac{c_k^{(i)}}{R_2} e^{k\tan h^{-1}\left(\frac{h_2}{R_2}\right)} = \sqrt{R_1^2-h_1^2}\, \frac{a_k^{(i)}}{k} \left(\frac{1}{kh_1-R_1} + \frac{1}{kh_2-R_2}\right). \qquad (11.108)$$

Now, from (11.107)–(11.108)

$$a_k^{(i)} \left[z_0 - \frac{R_1^2-h_1^2}{2k}\left(\frac{1}{kh_1-R_1} + \frac{1}{kh_2-R_2}\right)\right] = 0\,.$$

Hence, if $z_0 = \dfrac{R_1^2-h_1^2}{2k}\left(\dfrac{1}{kh_1-R_1} + \dfrac{1}{kh_2-R_2}\right)$, $a_k^{(i)}$ remain arbitrary and consequently the remaining constants are also not determined.

Since $z_0 > 0$ we have

$$\frac{1}{kh_1-R_1} + \frac{1}{kh_2-R_2} > 0\,.$$

From the last relation restrictions on k follow readily.

(3) If $z_0 = 0$ we obtain from (11.103) and (11.105)

$$a_k^{(i)} \sqrt{1-\left(\frac{h_1}{R_1}\right)^2}\, e^{k \tan h^{-1}\left(\frac{h_1}{R_1}\right)} = c_k^{(i)} \sqrt{1-\left(\frac{h_2}{R_2}\right)^2}\, e^{k \tan h^{-1}\left(\frac{h_2}{R_2}\right)} . \qquad (11.109)$$

From (11.104) and (11.106) we have

$$a_k^{(i)} \left(-k+\frac{h_1}{R_1}k^2\right) e^{k \tan h^{-1}\left(\frac{h_1}{R_1}\right)} = c_k^{(i)} \left(k-\frac{h_2}{R_2}k^2\right) e^{k \tan h^{-1}\left(\frac{h_2}{R_2}\right)} . \qquad (11.110)$$

From (11.109) and (11.110)

$$\frac{R_2}{R_1} = \frac{\sqrt{1-\left(\frac{h_1}{R_1}\right)^2}}{\sqrt{1-\left(\frac{h_2}{R_2}\right)^2}} = -\frac{k\frac{h_1}{R_1}-1}{k\frac{h_2}{R_2}-1} ,$$

i.e.

$$\frac{h_1+h_2}{R_1+R_2} = \frac{1}{k} \qquad (k = 2, 3, \ldots) ,$$

In these cases all constants remain arbitrary.

In the particular case when $R_1 \equiv R_2 \equiv 1$, $z_2 = z_1$, non-trivial bendings exist only if one of the following conditions is satisfied:

(1′) $h = z_1 - z_0 = \dfrac{1}{k}$, $\quad k = 2, 3, \ldots;$

(2′) $z_0 = \dfrac{1-h^2}{k(kh-1)}$, $\quad k \geqslant \left[\dfrac{1}{h}\right] + 1 > \dfrac{1}{h}$.

We observe that the quantities z_0 ensuring existence of non-trivial bendings have an upper bound. For instance, for the case $R_1 = R_2 = 1$, $z_2 = z_1$, the upper bound is given by the formula

$$\bar{z} = \frac{1-h^2}{\left(\left[\frac{1}{h}\right]+1\right)\left(h\left[\frac{1}{h}\right]+h-1\right)} ,$$

i.e. for $h \neq \frac{1}{k}$, $z_0 > \bar{z}$ the surface S is always rigid.

11.13. Let $r = r(z)$, $z_1 < z < 0$, be the meridian of a convex surface S, $r(0) = r_0 > 0$.

Let us consider a family of convex surfaces of revolution S_λ with the meridian $\varrho_\lambda(z) = \varrho\left(\frac{z}{\lambda}\right)$, $\lambda > 0$, where $\varrho(z)$ is a function satisfying the conditions $\varrho(0) = r_0$, $\varrho(z) > 0$, $\varrho'' \leqslant 0$ in $0 < z < z_0$, $\varrho(z_0) = 0$.

Let $\chi_k(z)$ be a solution of the equation $\chi_k'' r + (k^2-1)r''\chi_k = 0$ in the interval $z_1 < z < 0$, and $\psi_k(z)$ a solution of the equation $\psi_k''\varrho + (k^2-1)\varrho''\psi_k = 0$ in the interval $0 < z < z_0$, regular for $z = z_1$ and $z = z_0$, respectively, and satisfying the condition $\psi_k(0) = \chi_k(0) = 1$. Then it can readily be verified that the surface ΣS_λ—the result of contacting S and S_λ—is non-rigid if and only if

$$\lambda = \lambda_k = \frac{(k^2-1)\varrho'(0) + r_0\psi_k'(0)}{(k^2-1)r'(0) + r_0\chi_k'(0)}.$$

For $k \to \infty$ we have $\lambda_k \to \frac{\varrho'(0)}{r'(0)}$. In particular if this ratio is negative there exist among the surfaces ΣS_λ at most a finite number of non-rigid surfaces. If $\frac{\varrho'(0)}{r'(0)} > 0$ there exist a countable number of non-rigid surfaces. Non-rigid surfaces accumulate as ΣS_λ tends to a smooth surface (i.e. $r'(0) = \varrho_\lambda'(0)$).

In all three examples, as a rule, in the case of contacting into a non-convex surface the compound surface is rigid. Non-rigidity is a comparatively rare case.

CHAPTER VI

PROBLEMS OF THE MEMBRANE THEORY OF SHELLS

IN THIS Chapter we shall consider problems of the membrane theory of shells; we shall also elucidate its connexion with the theory of generalized analytic functions and the problem of infinitesimal bendings of surfaces. As it was already indicated in the preceding chapter every infinitesimal bending can be interpreted as a membrane equilibrium state of stress of a shell. Hence, many formulae and relationships established in the preceding chapter admit a purely mechanical interpretation. Moreover, many results presented there acquire a natural interpretatation only after its mechanical nature is revealed. This is true for instance in respect of the integral relations (6.17) and (6.20) of Ch. V, and also some boundary value problems dealt with above. It is sufficient to note that the boundary value problems for a bending field have a definite statical interpretation (§5.11). Besides, it will be proved below (§3.2) that infinitesimal bendings of the middle surface of an elastic shell can be regarded as a deformed state corresponding to a purely bending state of equilibrium stress.

In the first two sections of this chapter the basic equations of the general theory of shells will be derived. The following sections are all devoted to problems of the membrane theory. Main results obtained on this topic are given in §5 where in particular we reproduce results contained in §9 of the author's paper [14a]. Princi-

pal attention is given to the formulation of conditions ensuring the existence of a membrane state of equilibrium of a shell; convex shells with an arbitrary number of openings are considered. We establish a number of peculiar properties of shells depending on the number of openings. We note here some results of §9 of [14a]. It was proved for the case of a convex shell with one opening, free of surface tractions, that the membrane state of equilibrium does not always occur if the distribution of forces along the openings is prescribed in accordance with an arbitrary law (obviously, tangential forces are not prescribed). In such a shell in this case a non-zero force field cannot exist if the normal forces vanish on the boundary. For a convex shell with two openings there exist cases when under an arbitrary distribution of normal forces on the contour the membrane state occurs in the shell. Similarly, it is possible to give examples in which the last fact does not take place. Finally, in the case of convex shells with more that two openings the membrane state of stress is possible under an arbitrary law of distribution of normal forces along the contours of the openings; in a shell with $m (m > 1)$ openings there can exist exactly $3m-3$ linearly independent (non-zero) membrane states of stress if the normal forces on the contour are absent. These results remain valid if in place of the normal forces on the contour, the force in an inclined direction is prescribed at every point; it is sufficient that the absolute value of the angle of inclination of the force to the principal normal does not exceed π (this angle may be a continuous function of point of the contour). In the same section (§5) we investigate problems of stability of the membrane state, the stability being understood in a sense used in engineering practice. It is known that if the problem is examined purely mathematically, the states of membrane equilibrium of a convex shell with edges are not stable. A violation of the mathematical conditions of existence of the membrane state, however, does not imply that the corresponding moment

state differs significantly from the membrane state. In many cases such states can practically be regarded as membrane states, since the moment components due to the appropriate stress fields are small and in practical problems may be neglected. Essentially it is on this fact that all practical applications of the membrane theory are based. It is evident however that the moment components may not be neglected for all violations of the mathematical conditions of the membrane state. Therefore, an attempt is made in §5 to investigate some general principles enabling us to estimate the bounds of the practical applicability of the membrane theory.*

* An extensive literature is devoted to the problems of the membrane theory of shells. First of all we mention the classical work of Love [50] where in particular problems of the membrane theory and infinitesimal bendings of surfaces are dealt with. General and particular problems of the membrane theory of shells and its connections with the infinitesimal bending of surfaces are investigated in the works of Vlasov [18], Goldenveyser [27], Novozhilov [65], Rabotnov [74], Sokolovski [80] (a fairly complete bibliography up to 1947 appears in [28]). In these works problems mainly of practical interest for shells of a special kind were solved. The choice of the shape of the shell and the boundary conditions were mostly determined by the possibilities of the method applied. A long time ago it was observed that in many cases the above problems are reduced to the Cauchy–Riemann system of equations, [80], [18a,b], but comparatively small benefit was derived from this fact, except for some problems for a closed spherical shell, [18a], [27a]. General methods of the theory of functions of complex variable developed mainly in the works of Muskhelishvili [60a, b] have only recently been introduced to the theory of shells. In papers of the author these methods were applied to the basic problems of the general (bending) theory of shallow shells, including spherical and cylindrical shells [14b] (see also [65*a, b, c], [78*a, b]). Goldenveyser investigated some problems of the membrane state of stress of spherical shells with edges [27a]. A general method of investigation of the basic problems of the membrane theory of convex shells was elaborated by the author. In the paper [14a] fundamental problems for convex shells of an arbitrary shape and with an arbitrary number of openings were solved in a general form; some results obtained there were already mentioned above, in the introduction (see also [4]).

In the treatment of the fundamentals of the general theory of shells we shall make use of tensor methods. In the investigation of problems of the membrane theory of shells of positive curvature we shall employ complex functions of stress and displacement, and also properties of generalized analytic functions.

§1. Forces and moments due to the stress field

1.1. By a shell we understand an elastic body bounded by two surfaces S^+ and S^- situated on distinct sides and at equal distances $h > 0$ from a surface S. We shall assume that S^+ is situated on the side of the positive normal $\boldsymbol{n}$ to S. The surfaces S^+ and S^- will be called the "upper" and the "lower" surfaces of the shell, and S is said to be the middle surface. The positive number $2h$ is called the thickness of the shell. The thickness is assumed to be small in comparison with other dimensions of the shell. If the middle surface S is not closed the shell is bounded also by lateral surfaces the union of which will be denoted by Σ. We shall assume that the surfaces S and Σ are orthogonal. Let L be the union of the lines of intersection of S and Σ; it is obvious that this line is the boundary of the middle surface S. It is also called the boundary or the edge of the shell.

If Q is a point of the shell we can take for its coordinates the three numbers * x^1, x^2, z where x^1, x^2 are the coordinates of the basis of the normal to S from the point Q, and z is the length of this normal with the appropriate sign; $z > 0$ if the point Q is situated in the direction of the positive normal to S, and $z < 0$ in the opposite case; obviously, $-h \leqslant z \leqslant h$ (Fig. 17).

Let us denote by $\boldsymbol{r} = \boldsymbol{r}(x^1, x^2)$ and $\boldsymbol{n} = \boldsymbol{n}(x^1, x^2)$, respectively, the position vector of a point (x^1, x^2) of the surface S and the normal unit vector to S at the point

* In this case it should in general be assumed that the shell is sufficiently thin.

(x^1, x^2). Then the position vector of the point $Q(x^1, x^2, z)$ is given by the relation (Fig. 17)

$$\boldsymbol{R} = \boldsymbol{r}(x^1, x^2) + z\boldsymbol{n}(x^1, x^2) . \tag{1.1}$$

Hence, we have immediately that *the base vectors of the special coordinate system* x^1, x^2, z are the following:

$$\begin{aligned} \boldsymbol{R}_\alpha &\equiv \frac{\partial \boldsymbol{R}}{\partial x^\alpha} = \boldsymbol{r}_\alpha - z b^\beta_\alpha \boldsymbol{r}_\beta \qquad (\alpha = 1, 2) , \\ \boldsymbol{R}_3 &\equiv \frac{\partial \boldsymbol{R}}{\partial x^3} = \boldsymbol{n} \qquad (x^3 \equiv z) . \end{aligned} \tag{1.2}$$

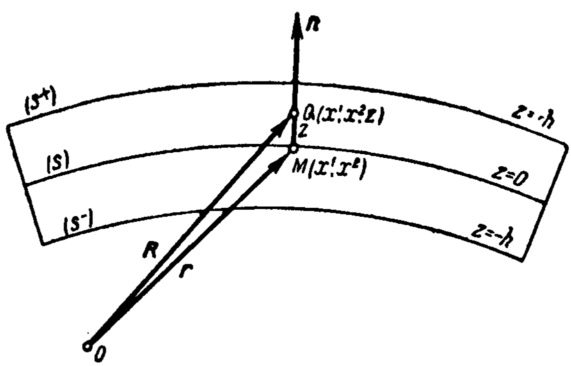

FIG. 17

We now compute the square of the distance between two neighbouring points $Q(x^1, x^2, z)$ and $Q'(x^1 + dx^1, x^2 + dx^2, z + dz)$. We have

$$ds^2 = d\boldsymbol{R}\, d\boldsymbol{R} = \boldsymbol{R}_i \boldsymbol{R}_k\, dx^i dx^k = g_{ik}\, dx^i dx^k , \tag{1.3}$$

where

$$\begin{aligned} g_{\alpha\beta} = g_{\beta\alpha} = \boldsymbol{R}_\alpha \boldsymbol{R}_\beta = a_{\alpha\beta} - 2z b_{\alpha\beta} + z^2 b^\lambda_\alpha b_{\lambda\beta} \quad (\alpha, \beta = 1, 2) , \\ g_{13} = g_{31} = g_{23} = g_{32} = 0 , \qquad g_{33} = 1 . \end{aligned} \tag{1.4}$$

Evidently, $g_{\alpha\beta}$ is a covariant tensor of rank two for an arbitrary fixed value of z. Since the mean curvature H and the principal curvature K of the surface are given by the formulae

$$2\mathrm{H} = b^\alpha_\alpha \equiv k_1 + k_2 , \qquad \mathrm{K} = \frac{b_{11} b_{22} - b_{12}^2}{a_{11} a_{22} - a_{12}^2} \equiv k_1 k_2 , \tag{1.5}$$

we easily find

$$\begin{aligned} \boldsymbol{R}_1\boldsymbol{R}_2\boldsymbol{R}_3 &= \sqrt{g} = \sqrt{g_{11}g_{22}-g_{12}^2} \\ &= \sqrt{a}(1-2\mathrm{H}z+\mathrm{K}z^2) = \sqrt{a}(1-k_1z)(1-k_2z)\,. \end{aligned} \tag{1.6}$$

The system (1.6) indicates that the ratio g/a is independent of the choice of the coordinate system x^1, x^2 on the surface S.

Let us consider the contravariant tensor of rank two

$$\begin{gathered} g^{11} = \frac{g_{22}}{g}\,,\quad g^{12} = g^{21} = -\frac{g_{12}}{g}\,,\quad g^{22} = \frac{g_{11}}{g}\,, \\ g^{13} = g^{31} = g^{23} = g^{32} = 0\,,\qquad g^{33} = 1\,, \end{gathered} \tag{1.7}$$

and also *the reciprocal base vectors of the system* x^1, x^2, z:

$$\boldsymbol{R}^i = g^{ik}\boldsymbol{R}_k \qquad (i = 1, 2, 3)\,. \tag{1.8}$$

It can easily be shown that

$$\boldsymbol{R}^i\boldsymbol{R}_k = \delta^i_k\,,\qquad \boldsymbol{R}^i\boldsymbol{R}^k = g^{ik}\,, \tag{1.9}$$

$$\begin{gathered} \boldsymbol{R}_1\times\boldsymbol{R}_2 = \sqrt{g}\boldsymbol{R}^3\,,\qquad \boldsymbol{R}_2\times\boldsymbol{R}_3 = \sqrt{g}R^1\,, \\ \boldsymbol{R}_3\times\boldsymbol{R}_1 = \sqrt{g}\boldsymbol{R}^2\,. \end{gathered} \tag{1.10}$$

1.2. Let us consider on S two neighbouring points $M(x^1, x^2)$ and $M'(x^1+dx^1, x^2+dx^2)$ connected by an infinitesimal arc $\overset{\frown}{MM'}$. Let $\boldsymbol{s}$ be the unit vector of the tangent to the arc $\overset{\frown}{MM'}$ at the point M, and $\boldsymbol{l}$ the unit vector of the tangential normal of the arc $\overset{\frown}{MM'}$; we have $\boldsymbol{l} = \boldsymbol{s}\times\boldsymbol{n}$. Denote by Σ^l the area of the normal cross-section of the shell, which is perpendicular to $\boldsymbol{l}$ and contained between the neighbouring normals to S at the points M and M'. Let $P_{(l)}\,(x^1, x^2, z)$ be the force resultant of the stresses on the area Σ^l; it is applied at the point with the coordinates x^1, x^2, z. Then to the element of the area Σ^l contained between the lines $z = \text{const}$, $z+dz = \text{const}$ and denoted hereafter by $d\Sigma^l_z$, the following force due to the stresses is applied (Fig. 18):

$$\boldsymbol{P}_{(l)}(x^1, x^2, z)\,d\Sigma^l_z\,.$$

Denoting by $\boldsymbol{T}_{(l)}\,ds$ the resultant vector of these forces we have obviously

$$\boldsymbol{T}_{(l)}ds = \int\limits_{-h}^{h} \boldsymbol{P}_{(l)}(x^1, x^2, z)\,d\Sigma_z^l\,, \qquad ds = |\widetilde{MM'}|\,. \quad (1.11)$$

The vector $\boldsymbol{T}_{(l)}$ is called *the force* applied to the lateral area Σ^l with the normal $\boldsymbol{l}$ (Fig. 19). Apparently, the force is taken per unit lenght.

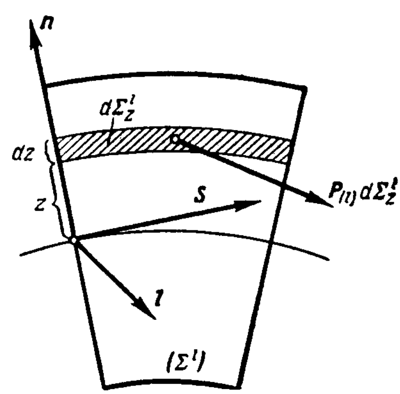

FIG. 18

Denoting now by $\boldsymbol{M}_{(l)}\,ds$ the resultant moment of the forces $\boldsymbol{P}_{(l)}\,d\Sigma_z^l$ with respect to the point $M(x^1, x^2)$ of the middle surface S, we evidently have (Fig. 19)

$$\boldsymbol{M}_{(l)}ds = \int\limits_{-h}^{h} \boldsymbol{n} \times \boldsymbol{P}_{(l)}(x^1, x^2, z)\,d\Sigma_z^l \equiv \boldsymbol{n} \times \int\limits_{-h}^{h} \boldsymbol{P}_{(l)}z\,d\Sigma_z^l\,. \quad (1.12)$$

The vector $\boldsymbol{M}_{(l)}$ is *the moment* of the stresses applied to the lateral area with the normal $\boldsymbol{l}$. The vector $\boldsymbol{M}_{(l)}$ is also taken per unit length.

The set-the force $\boldsymbol{T}_{(l)}$ and the moment $\boldsymbol{M}_{(l)}$—are statically equivalent to the system of stress forces applied to the lateral area Σ^l. It is usually assumed that in a sufficiently thin elastic shell the stress and deformation

states do not alter significantly if a continuously distributed system of stress forces on every lateral area Σ^l is replaced by a statically equivalent system of stress forces. Hence, in the investigation of the deformation of an elastic shell we usually confine ourselves to the determination of the force field $\boldsymbol{T}_{(l)}$ and the moment

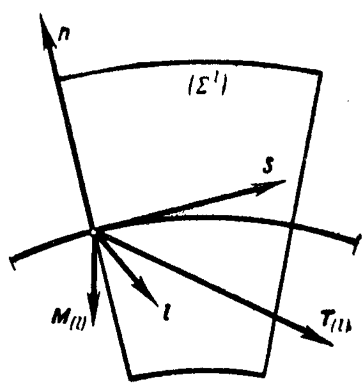

FIG. 19

field $\boldsymbol{M}_{(l)}$. In many cases in this way a practically admissible approximation is obtained. The field of stress forces can be computed in accordance with the formulae

$$\boldsymbol{P}_{(l)}(x^1, x^2, z) = \frac{1}{2h}\boldsymbol{T}_{(l)}(x^1, x^2) + \\ + \frac{3z}{2h^3}\boldsymbol{M}_{(l)}(x^1, x^2) \times \boldsymbol{n}(x^1, x^2)\,. \qquad (1.13)$$

The first term of the right-hand side of the above relation is called *the membrane component of the stress field of the shell*; the second term is called *the moment component of this field.*

1.3. In what follows, when the direction $\boldsymbol{l}$ coincides with the direction of the reciprocal base vector $\boldsymbol{r}^\alpha$, instead of $\boldsymbol{P}_{(l)}, \boldsymbol{T}_{(l)}$ and $\boldsymbol{M}_{(l)}$ we shall write $\boldsymbol{P}_{(\alpha)}, \boldsymbol{T}_{(\alpha)}, \boldsymbol{M}_{(\alpha)}$, respectively.

Let $d\Sigma_z^1$ and $d\Sigma_z^2$ be two rectangles constructed on the vectors $\boldsymbol{n}dz$, $\boldsymbol{R}_2 dx^2$ and $\boldsymbol{n}dz$, $\boldsymbol{R}_1 dx^1$, respectively. The corresponding areas are obviously given by the formulae

$$\begin{aligned} d\Sigma_z^1 &= |\boldsymbol{R}_2| dx^2 dz = \sqrt{g_{22}}\, dx^2 dz = \sqrt{\frac{g_{22}}{a_{22}}}\, ds_2 dz , \\ d\Sigma_z^2 &= |\boldsymbol{R}_1| dx^1 dz = \sqrt{g_{11}}\, dx^1 dz = \sqrt{\frac{g_{11}}{a_{11}}}\, ds_1 dz , \end{aligned} \tag{1.14}$$

where $ds_1 = \sqrt{a_{11}}\, dx^1$, $ds_2 = \sqrt{a_{22}}\, dx^2$ are elements of arcs of the coordinate lines $x^2 = \text{const.}$ and $x^1 = \text{const.}$, respectively. For simplicity it is assumed that $dx^1 > 0$, $dx^2 > 0$, $dz > 0$.

On the basis of the above formulae we obtain from (1.11)

$$\sqrt{a_{22}}\, \boldsymbol{T}_{(1)} = \int_{-h}^{h} \boldsymbol{P}_{(1)} \sqrt{g_{22}}\, dz , \quad \sqrt{a_{11}}\, \boldsymbol{T}_2 = \int_{-h}^{h} \boldsymbol{P}_{(2)} \sqrt{g_{11}}\, dz . \tag{1.15}$$

Since

$$a_{11} = aa^{22} , \quad a_{22} = aa^{11} , \quad g_{22} = gg^{11} , \quad g_{11} = gg^{22} , \tag{1.16}$$

introducing the notation

$$\boldsymbol{T}^1 = \sqrt{a^{11}}\, \boldsymbol{T}_{(1)} , \qquad \boldsymbol{T}^2 = \sqrt{a^{22}}\, \boldsymbol{T}_{(2)} , \tag{1.17}$$

$$\boldsymbol{P}^1 = \sqrt{g^{11}}\, \boldsymbol{P}_{(1)} , \qquad \boldsymbol{P}^2 = \sqrt{g^{22}}\, \boldsymbol{P}_{(2)} , \tag{1.18}$$

we have

$$\boldsymbol{T}^\alpha = \int_{-h}^{h} \boldsymbol{P}^\alpha \sqrt{\frac{g}{a}}\, dz \qquad (\alpha = 1, 2) . \tag{1.19}$$

In a similar way we obtain

$$\boldsymbol{M}^\alpha = \int_{-h}^{h} \boldsymbol{n} \times \boldsymbol{P}^\alpha z \sqrt{\frac{g}{a}}\, dz \equiv \boldsymbol{n} \times \int_{-h}^{h} \boldsymbol{P}^\alpha z \sqrt{\frac{g}{a}}\, dz , \tag{1.20}$$

where

$$\boldsymbol{M}^1 = \sqrt{a^{11}}\, \boldsymbol{M}_{(1)} , \qquad \boldsymbol{M}^2 = \sqrt{a^{22}}\, \boldsymbol{M}_{(2)} . \tag{1.21}$$

1.4. Since, according to the formula (1.6), g/a is independent of the choice of the coordinate system on the surface, we shall see later (p. 574) that in passing from one coordinate system x^1, x^2, z to another system $\bar{x}^1, \bar{x}^2, z$ the vectors $\boldsymbol{T}^\alpha$ and $\boldsymbol{M}^\alpha$ transform in accordance with the formulae

$$\overline{\boldsymbol{T}}^\alpha = \overline{\boldsymbol{T}}^\beta \frac{\partial \bar{x}^\alpha}{\partial x^\beta}, \qquad \overline{\boldsymbol{M}}^\alpha = \boldsymbol{M}^\beta \frac{\partial \bar{x}^\alpha}{\partial x^\beta}. \tag{1.22}$$

We shall now prove that the forces and moments applied to a lateral area Σ^l with the normal $\boldsymbol{l}$ are given by the formulae

$$\boldsymbol{T}_{(l)} = \boldsymbol{T}^\alpha l_\alpha, \quad \boldsymbol{M}_{(l)} = \boldsymbol{M}^\alpha l_\alpha \quad (l_\alpha = \boldsymbol{l}\boldsymbol{r}_\alpha). \tag{1.23}$$

We now choose a new coordinate system $\bar{x}^1, \bar{x}^2$, such that the lateral area Σ^l with the normal $\boldsymbol{l}$ lies on the coordinate surface $\bar{x}^1 = \text{const}$. Then, obviously,

$$\boldsymbol{l} = \frac{\bar{\boldsymbol{r}}^1}{|\bar{\boldsymbol{r}}^1|} = \frac{1}{\sqrt{\bar{a}^{11}}} \frac{\partial \bar{x}^1}{\partial x^\alpha} \boldsymbol{r}^\alpha \equiv l_\alpha \boldsymbol{r}^\alpha,$$

i.e.

$$l_\alpha = \frac{1}{\sqrt{\bar{a}^{11}}} \frac{\partial \bar{x}^1}{\partial x^\alpha} \quad (\alpha = 1, 2).$$

According to the formulae (1.17) and (1.22)

$$\boldsymbol{T}_{(l)} = \overline{\boldsymbol{T}}_{(l)} = \frac{\overline{\boldsymbol{T}}^{(1)}}{\sqrt{\bar{a}^{11}}} = \frac{1}{\sqrt{\bar{a}^{11}}} \frac{\partial \bar{x}^1}{\partial x^\alpha} \boldsymbol{T}^\alpha = l_\alpha \boldsymbol{T}^\alpha.$$

In a similar way we can obtain the second formula (1.23).

§2. Basic system of equilibrium equations of a shell

2.1. We now consider an infinitesimal prism $\mathfrak{B}$ constructed by means of the vectors $\boldsymbol{R}_1 dx^1$, $\boldsymbol{R}_2 dx^2$ and $\boldsymbol{R}_3 dz$ with common vertex at the point $Q(x^1, x^2, z)$; for definitness we assume that $dx^1 > 0$, $dx^2 > 0$, $dz > 0$ (Fig. 20).

Let $d\Sigma^\alpha$ be the side of the prism on which $x^\alpha = \text{const}$, $d\Sigma^l$ the side passing through the ends of the vectors

$\boldsymbol{R}_1 dx^1$ and $\boldsymbol{R}_2 dx^2$, $\boldsymbol{l}$ denoting the unit vector of the outward normal of this side. Since according to (1.10)

$$\begin{aligned} \boldsymbol{R}_3 dz \times \boldsymbol{R}_1 dx^1 &= \sqrt{g}\,\boldsymbol{R}^2 dx^1 dz \,, \\ \boldsymbol{R}_2 dx^2 \times \boldsymbol{R}_3 dz &= \sqrt{g}\,\boldsymbol{R}^1 dx^2 dz \,, \\ \boldsymbol{R}_3 dz \times (\boldsymbol{R}_1 dx^1 - \boldsymbol{R}_2 dx^2) &= \boldsymbol{l}\, d\Sigma^l \,, \end{aligned} \tag{2.1}$$

we have

$$\boldsymbol{l}\, d\Sigma^l = \sqrt{g}\,\boldsymbol{R}^1 dx^2 dz + \sqrt{g}\,\boldsymbol{R}^2 dx^1 dz \tag{2.2}$$

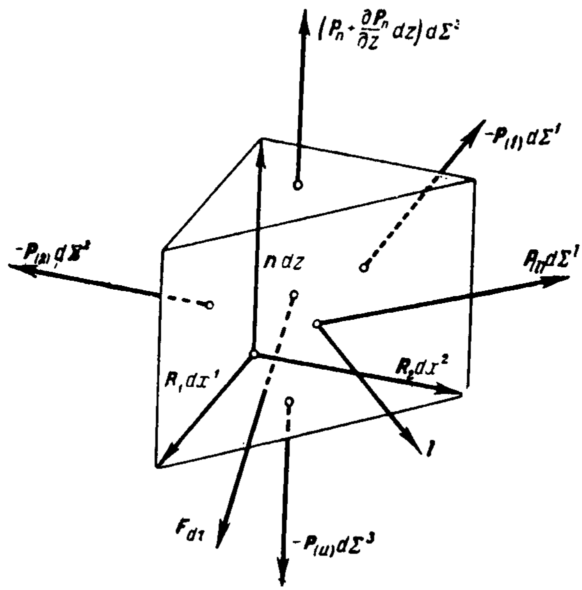

FIG. 20

or, bearing in mind the formulae (1.14)

$$\boldsymbol{l}\, d\Sigma^l = \frac{\boldsymbol{R}^1}{\sqrt{g^{11}}} d\Sigma^1 + \frac{\boldsymbol{R}^2}{\sqrt{g^{22}}} d\Sigma^2 \,. \tag{2.3}$$

Thus,

$$d\Sigma^1 = \sqrt{g^{11}}\, l_1' \, d\Sigma^l \,, \qquad d\Sigma^2 = \sqrt{g^{22}}\, l_2' \, d\Sigma^l \,, \tag{2.4}$$

where

$$l_\alpha' = \boldsymbol{l}\boldsymbol{R}_\alpha = l_\alpha - z b_\alpha^\lambda l_\lambda \,, \qquad l_\alpha = \boldsymbol{l}\boldsymbol{r}_\alpha \,. \tag{2.5}$$

On the prism $\mathfrak{B}$ the following surface forces are acting:
$\boldsymbol{P}_{(l)} d\Sigma^l$, $-\boldsymbol{P}_{(1)} d\Sigma^1$, $-\boldsymbol{P}_{(2)} d\Sigma^2$, $-\boldsymbol{P}_{(n)} d\Sigma^3$, $\left(\boldsymbol{P}_{(n)} + \frac{\partial \boldsymbol{P}_{(n)}}{\partial z} dz\right) d\Sigma^3$.

Besides we have the body force $\boldsymbol{F}d\tau$; $d\Sigma^3$ is the area of the basis of the prism and $d\tau$ its volume. Equating the sum of these forces to zero we obtain

$$\boldsymbol{P}_{(l)}d\Sigma^l = \boldsymbol{P}_{(1)}d\Sigma^1 + \boldsymbol{P}_{(2)}d\Sigma^2 - \frac{\partial \boldsymbol{P}_{(n)}}{\partial z}dz\,d\Sigma^3 + \boldsymbol{F}d\tau\,.$$

Dividing the above equation throughout by $d\Sigma^l$ and taking into account that

$$(1)\quad d\tau = dz\,d\Sigma^3 \quad\text{and}\quad (2)\quad \frac{d\tau}{d\Sigma^3}\to 0\,,$$

when the prism contracts to a point, we obtain in view of (2.4) and (1.18) the relation

$$\boldsymbol{P}_{(l)} = \sqrt{g^{11}}\boldsymbol{P}_{(1)}l_1' + \sqrt{g^{22}}\boldsymbol{P}_{(2)}l_2' \equiv \boldsymbol{P}^\alpha l_\alpha'\,. \tag{2.6}$$

Since $\boldsymbol{P}_{(l)}$ is independent of the choice of the coordinate system, for two arbitrary coordinate systems x^1, x^2, z and $\bar{x}^1, \bar{x}^2, z$ we have

$$\boldsymbol{P}_{(l)} = \boldsymbol{P}^\alpha l_\alpha' = \overline{\boldsymbol{P}}^\alpha \bar{l}_\alpha'\,. \tag{2.7}$$

Taking into account that $\bar{l}_\alpha' = l_\beta' \dfrac{\partial \bar{x}^\beta}{\partial x^\alpha}$ we obtain

$$\left(\boldsymbol{P}^\alpha \frac{\partial \bar{a}^\beta}{\partial x^\alpha} - \overline{\boldsymbol{P}}^\beta\right)\bar{l}_\beta' = 0\,.$$

Since $\boldsymbol{l}$ is an arbitrary direction tangential to S we have the relation

$$\overline{\boldsymbol{P}}^\beta = \boldsymbol{P}^\alpha \frac{\partial \bar{x}^\beta}{\partial x^\alpha} \qquad (\beta = 1, 2)\,. \tag{2.8}$$

Thus, the vectors $\boldsymbol{P}^\alpha$ are transformed in passing from a coordinate system x^1, x^2, z to an arbitrary coordinate system $\bar{x}^1, \bar{x}^2, z$ according to the law of transformation of the differentials dx^α, i.e. as contravariant vectors. Therefore, the vectors $\boldsymbol{P}^i$ will be called *the contravariant vectors of the stress forces*. If these vectors are determined, the formulae (2.7) yield at once the physical stress forces.

The formulae (1.19), (1.20) and (2.8) imply at once the relations (1.22) employed above in the derivation of the formulae (1.23).

2.2. Resolving the vectors $\boldsymbol{P}^i$ with respect to the base vectors $\boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3$ we obtain

$$\boldsymbol{P}^i = p^{ik}\boldsymbol{R}_k \quad (i = 1, 2, 3;\ \boldsymbol{P}^3 = \boldsymbol{P}_3,\ \boldsymbol{R}_3 = \boldsymbol{n}), \quad (2.9)$$

where $p^{ik} = \boldsymbol{P}^i\boldsymbol{R}^k$ is a contravariant tensor of rank two. This tensor is symmetric, i.e. $p^{ik} = p^{ki}$.

In fact, this property can easily be proved in a Cartesian coordinate system; but the property of symmetry is invariant with respect to the choice of the coordinate system.

In view, of (1.2) we obtain from (2.9)

$$\boldsymbol{P}^i = (p^{i\alpha} - zp^{i\beta}b^\alpha_\beta)\boldsymbol{r}_\alpha + p^{i3}n . \quad (2.10)$$

Substituting these expressions into (1.19) and (1.20) we have

$$\boldsymbol{T}^\alpha = \boldsymbol{T}^{\alpha\beta}\boldsymbol{r}_\beta + \boldsymbol{T}^\alpha\boldsymbol{n}, \quad \boldsymbol{M}^\alpha = M^{\alpha\beta}\boldsymbol{r}_\beta \quad (\alpha = 1, 2), \quad (2.11)$$

where

$$T^{\alpha\beta} = \int\limits_{-h}^{h} (p^{\alpha\beta} - zp^{\alpha\lambda}b^\beta_\lambda)\sqrt{\frac{g}{a}}\,dz, \quad T^\alpha = \int\limits_{-h}^{h} p^{\alpha3}\sqrt{\frac{g}{a}}\,dz, \quad (2.12)$$

$$M^{\alpha\beta} = c^{\cdot\beta}_{\lambda\cdot}S^{\alpha\lambda}, \quad S^{\alpha\lambda} = \int\limits_{-h}^{h} (p^{\alpha\lambda} - zp^{\alpha\beta}b^\lambda_\beta)z\sqrt{\frac{g}{a}}\,dz, \quad (2.13)$$

and $c^{\cdot\beta}_{\alpha} = c_{\alpha\lambda}a^{\lambda\beta}$.

The contravariant tensors of rank two $T^{\alpha\beta}$, $M^{\alpha\beta}$ and the contravariant vector T^α will be called *the contravariant stress tensor, contravariant moment tensor and the contravariant shear force vector*, respectively.

The force and the moment acting on an area with the normal $\boldsymbol{l}$, in view of (1.23) and (2.11) can also be represented in the form

$$\boldsymbol{T}_{(l)} = N_l\boldsymbol{l} + H_l\boldsymbol{s} + T_l\boldsymbol{n}, \quad (2.14)$$

$$\boldsymbol{M}_{(l)} = G_l\boldsymbol{l} + M_l\boldsymbol{s}, \quad (2.15)$$

N_l and H_l are called *the normal and tangential force*, respectively, T_l *the shear force*, G_l and M_l *the twisting and bending moments*, respectively.

It is readily observed that these quantities are given by the following formulae:

$$N_l = T^{\alpha\beta} l_\alpha l_\beta\,, \qquad H_l = T^{\alpha\beta} l_\alpha s_\beta\,, \qquad T_l = T^\alpha l_\alpha\,, \quad (2.16)$$

$$G_l = M^{\alpha\beta} l_\alpha l_\beta = -S^{\alpha\beta} l_\alpha s_\beta\,, \qquad M_l = M^{\alpha\beta} l_\alpha s_\beta = S^{\alpha\beta} l_\alpha l_\beta\,. \quad (2.17)$$

2.3. The basic system of equilibrium equations of a continuous medium can be written in an arbitrary coordinate system in the vector form

$$\frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}\, \boldsymbol{P}^k}{\partial x^k} + \boldsymbol{F} = 0\,, \qquad (2.18)$$

where $\boldsymbol{F}$ is the body force. The validity of this relation in a Cartesian coordinate system is obvious. But in view of the property (2.8) of the vectors $\boldsymbol{P}^k$ (it should be borne in mind that $\overline{\boldsymbol{P}}^3 = \boldsymbol{P}^3 = \boldsymbol{P}_3$) it can easily be established that the left-hand side of the relation (2.18) is entirely independent of the choice of the coordinate system. Consequently, the relation (2.18) is valid in an arbitrary curvilinear spatial coordinate system.

Multiplying both sides of the relation (2.18) by $\sqrt{g/a}\,dz$ and then integrating with respect to z we obtain

$$\frac{1}{\sqrt{a}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{a} \int_{-h}^{h} \boldsymbol{P}^\alpha \sqrt{\frac{g}{a}}\, dz \right) + \\ + \boldsymbol{P}^3 \sqrt{\frac{g}{a}} \bigg|_{z=-h}^{z=+h} + \int_{-h}^{h} \boldsymbol{F} \sqrt{\frac{g}{a}}\, dz = 0$$

or, in view of the formula (1.19)

$$\frac{1}{\sqrt{a}} \frac{\partial \sqrt{a}\, \boldsymbol{T}^\alpha}{\partial x^\alpha} + \boldsymbol{X}(x^1, x^2) = 0\,, \qquad (2.19)$$

where

$$\boldsymbol{X} = (1+2\mathrm{H}h+\mathrm{K}h^2)\boldsymbol{P}_3(x^1, x^2, h)-(1-2\mathrm{H}h+\mathrm{K}h^2)\times$$

$$\times\boldsymbol{P}_3(x^1, x^2, -h)+\int\limits_{-h}^{h}(1-2\mathrm{H}z+\mathrm{K}z^2)\boldsymbol{F}(x^1, x^2, z)\,dz\,. \quad (2.20)$$

It follows that for thin or shallow shells we may put with a sufficient degree of approximation

$$\boldsymbol{X} = \boldsymbol{P}_3(x^1, x^2, h)-\boldsymbol{P}_3(x^1, x^2, -h)+\int\limits_{-h}^{h}\boldsymbol{F}(x^1, x^2, z)\,dz\,. \quad (2.21)$$

Thus, $\boldsymbol{X}$ is the resultant vector of the surface and body forces measured per unit area.

The vector $\boldsymbol{X}$ can be regarded as a known function of the point (x^1, x^2) of the surface S, for it can be expressed by the surface forces $\boldsymbol{P}_3(x^1, x^2, h)$, $\boldsymbol{P}_3(x^1, x^2, -h)$ and the body force $\boldsymbol{F}$, which are regarded as known. In particular, $\boldsymbol{X} = 0$ if these forces are absent.

Taking the vector product of equation (2.18) with the vector $z\sqrt{g}\,\boldsymbol{n}$ we have

$$\boldsymbol{n}\times\frac{\partial}{\partial x^\alpha}(z\sqrt{g}\boldsymbol{P}^\alpha)+\boldsymbol{n}\times z\frac{\partial}{\partial z}(\sqrt{g}\boldsymbol{P}^3)+\boldsymbol{n}\times z\sqrt{g}\boldsymbol{F} = 0$$

or

$$\frac{\partial}{\partial x^\alpha}(\boldsymbol{n}\times\boldsymbol{P}^\alpha z\sqrt{g})-\boldsymbol{n}_\alpha\times\boldsymbol{P}^\alpha z\sqrt{g}+\frac{\partial}{\partial z}(\boldsymbol{n}\times\boldsymbol{P}^3 z\sqrt{g})-$$

$$-\boldsymbol{n}\times\boldsymbol{P}^3\sqrt{g}+\boldsymbol{n}\times\boldsymbol{F}z\sqrt{g} = 0\,. \quad (2.22)$$

Taking into account the relations

$$z\boldsymbol{n}_\alpha = -zb_\alpha^\lambda\boldsymbol{r}_\lambda \equiv \boldsymbol{R}_\alpha-\boldsymbol{r}_\alpha\,, \qquad \boldsymbol{n} = \boldsymbol{R}_3\,, \qquad \boldsymbol{R}_k\times\boldsymbol{P}^k = 0\,, \quad (2.23)$$

we have

$$-\boldsymbol{n}_\alpha\times\boldsymbol{P}^\alpha z\sqrt{g}-\boldsymbol{n}\times\boldsymbol{P}^3\sqrt{g} = \sqrt{g}(\boldsymbol{r}_\alpha\times\boldsymbol{P}^\alpha-\boldsymbol{R}_\alpha\times\boldsymbol{P}^\alpha-$$

$$-\boldsymbol{R}_3\times\boldsymbol{P}^3) = \sqrt{g}(\boldsymbol{r}_\alpha\times\boldsymbol{P}^\alpha-\boldsymbol{R}_k\times\boldsymbol{P}^k) = \sqrt{a}\sqrt{\frac{g}{a}}\boldsymbol{r}_\alpha\times\boldsymbol{P}^\alpha\,.$$

Hence, the relation (2.22) takes the form

$$\frac{1}{\sqrt{a}}\frac{\partial}{\partial x^\alpha}\left[\sqrt{a}\left(\boldsymbol{n}\times\boldsymbol{P}^\alpha z\sqrt{\frac{g}{a}}\right)\right]+\boldsymbol{r}_\alpha\times\boldsymbol{P}^\alpha\sqrt{\frac{g}{a}}+$$
$$+\frac{\partial}{\partial z}\left(\boldsymbol{n}\times\boldsymbol{P}^3 z\sqrt{\frac{g}{a}}\right)+\boldsymbol{n}\times\boldsymbol{F}z\sqrt{\frac{g}{a}}=0\,.$$

Integrating the above relation with respect to z and taking into account the formulae (1.19) and (1.20) we obtain

$$\frac{1}{\sqrt{a}}\frac{\partial\sqrt{a}\boldsymbol{M}^\alpha}{\partial x^\alpha}+\boldsymbol{r}_\alpha\times\boldsymbol{T}^\alpha+\boldsymbol{Y}=0\,, \tag{2.24}$$

where

$$\boldsymbol{Y}=h(1-2\mathrm{H}h+\mathrm{K}h^2)\boldsymbol{n}\times\boldsymbol{P}_3(x^1,x^2,h)+$$
$$+h(1+2\mathrm{H}h+\mathrm{K}h^2)\boldsymbol{n}\times\boldsymbol{P}_3(x^1,x^2,-h)+$$
$$+\int_{-h}^{h}\boldsymbol{n}\times\boldsymbol{F}(1-2\mathrm{H}z+\mathrm{K}z^2)z\,dz\,. \tag{2.25}$$

For thin or shallow shells we may assume with a sufficient degree of approximation that

$$\boldsymbol{Y}=h\boldsymbol{n}\times\boldsymbol{P}_3(x^1,x^2,h)+$$
$$+h\boldsymbol{n}\times\boldsymbol{P}_3(x^1,x^2,-h)+\int_{-h}^{h}\boldsymbol{n}\times\boldsymbol{F}z\,dz\,. \tag{2.26}$$

Thus, $\boldsymbol{Y}$ represents the moment resultant of the surface and body forces measured per unit area. Obviously, $\boldsymbol{Y}$ is a given vector function of x^1, x^2.

The equations (2.19) and (2.24) constitute *the basic system of equilibrium equations of a shell.* Making use of d'Alembert's principle we can also derive the system of equations of vibrations of a shell. We shall not however write it down here, since it will not be used in the subsequent considerations.

2.4. The vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ which appear in the equations (2.19) and (2.24), are functions of position on the middle surface. As was already indicated above the first vector is the resultant vector of the surface and body forces

and the second the resultant moment of these forces acting on the shell. Thus, in the theory of shells instead of a given distribution of the system of external body and surface forces we consider the statically equivalent system of forces $\boldsymbol{X}$ and moments $\boldsymbol{Y}$ measured per unit area and distributed over the middle surface. It is assumed that such a redistribution of the external load does not significantly distort the real state of stress of the shell, i.e. it is assumed that the error due to this distortion does not exceed practically allowable limits. It is difficult however to indicate a criterion enabling us to say exactly when such an approximation is practically allowable. In making such an assumption we usually put forward an argument based mainly on direct intuition; also, in assessing such an assumption, successful results of a practical application of the theory of shells are taken into account. In fact, we can consider these assumptions to be close to reality for a fairly wide class of thin elastic shells if the external load is distributed sufficiently continuously, both along the thickness of the shell and on its surfaces.

If on some sections of the shell a strong concentration of the external load occurs (for instance there are concentrated forces applied at some points), then, in the vicinity of these sections, a calculation of a shell in accordance with the formulae (2.19) and (2.24) can give results significantly distinct from the actual state of equilibrium.

The system of forces applied to the lateral surfaces is also replaced in the theory of shells by the force vector $\boldsymbol{T}_{(l)}$ and the moment vector $\boldsymbol{M}_{(l)}$ where $\boldsymbol{l}$ is the tangential normal to the boundary of the shell.

If the vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ vanish identically, i.e. if the external body and surface forces are together equivalent to zero in every elementary volume of the shell, we say that *the shell is free*. In a free shell the state of stress is due only to the forces $\boldsymbol{T}_{(l)}$ and the moments $\boldsymbol{M}_{(l)}$ acting on the contour of the shell.

The system of equations (2.19) and (2.24) in the case of a free shell has the form

$$\frac{1}{\sqrt{a}}\frac{\partial\sqrt{a}\boldsymbol{T}^{\alpha}}{\partial x^{\alpha}}=0\,, \qquad \frac{1}{\sqrt{a}}\frac{\partial\sqrt{a}\boldsymbol{M}^{\alpha}}{\partial x^{\alpha}}+\boldsymbol{r}_{\alpha}\times\boldsymbol{T}^{\alpha}=0\,. \tag{2.27}$$

The same equations are valid when there are concentrated external forces acting on the shell. But in the vicinity of the points of application of these forces we cannot in general regard the forces $\boldsymbol{T}_{(l)}$ and the moments $\boldsymbol{M}_{(l)}$ as bounded. Moreover, in these cases, at least in the vicinity of the strong concentration of the external load the stress state of the shell cannot in general be established to a sufficient approximation knowing only forces and moments.

As a rule in the theory of shells the moments of the external forces $\boldsymbol{Y}$ are neglected, i.e. it is assumed that they vanish. In what follows we shall always assume that

$$\boldsymbol{Y}\equiv 0\,. \tag{2.28}$$

The following reasoning can be presented in support of this assumption: Firstly, in most cases the forces applied to the "upper" and "lower" surfaces of the shell are directed normally (or almost normally) to this surface (for instance a hydrostatic pressure). Consequently, under these conditions the moments of the forces $\boldsymbol{P}_3^+$ and $\boldsymbol{P}_3^-$ vanish. The body force $\boldsymbol{F}$ especially for a thin shell, varies insignificantly along the thickness of the shell. Hence, the integral appearing in the relation (2.26) can be assumed to vanish. Thus, the basic system of equations (2.19) and (2.24) will be taken in the form

$$\frac{1}{\sqrt{a}}\frac{\partial\sqrt{a}\boldsymbol{T}^{\alpha}}{\partial x^{\alpha}}+\boldsymbol{X}=0\,, \tag{2.29}$$

$$\frac{1}{\sqrt{a}}\frac{\partial\sqrt{a}\boldsymbol{M}^{\alpha}}{\partial x^{\alpha}}+\boldsymbol{r}_{\alpha}\times\boldsymbol{T}^{\alpha}=0\,. \tag{2.30}$$

In what follows the vector $\boldsymbol{X}$ representing the resultant vector of surface and body forces acting on a shell will

be called *the surface load.* Thus, the forces acting on the lateral surfaces of a shell and statically equivalent to the force $\boldsymbol{T}_{(l)}$ and the moment $\boldsymbol{M}_{(l)}$, will not be reckoned among the external surface forces. The forces and the moments acting on the lateral surfaces of the shell will be called *the contour forces* ($\boldsymbol{T}_{(l)}$ is *the contour force* or *the contour load*, $\boldsymbol{M}_{(l)}$ is *the contour moment* or *the contour couple*).

2.5. Making use of the formulae (2.11) the system of vector equations (2.29) and (2.30) can be written in the following form:

$$\nabla_a T^{a\beta} - T^a b_a^\beta + X^\beta = 0 \qquad (\beta = 1, 2)\,, \tag{2.31}$$

$$\nabla_a T^a + b_{a\beta} T^{a\beta} + Z = 0\,, \tag{2.32}$$

$$\nabla_a M^{a\beta} - T^a c_{a\cdot}^{\cdot\beta} = 0 \qquad (\beta = 1, 2)\,, \tag{2.33}$$

$$M^{a\beta} b_{a\beta} + c_{a\beta} T^{a\beta} = 0\,. \tag{2.34}$$

In deriving these equations we employed the Gauss formulae

$$\boldsymbol{r}_{a\beta} = \Gamma_{a\beta}^{\lambda} \boldsymbol{r}_\lambda + b_{a\beta} \boldsymbol{n} \qquad (a, \beta = 1, 2)\,, \tag{2.35}$$

and the relations

$$\frac{\partial \boldsymbol{n}}{\partial x^a} \equiv \boldsymbol{n}_a = -b_{a\beta} \boldsymbol{r}^\beta \equiv -b_a^\beta \boldsymbol{r}_\beta \qquad (a = 1, 2)\,. \tag{2.36}$$

Thus, the system of vector equations (2.29) and (2.30) is equivalent to the system of six equations (2.31)–(2.34) which contains ten unknown quantities: four components of the stress tensor T^{11}, T^{12}, T^{21}, T^{22}, two components of the shear force T^1, T^2 and finally four components of the moment tensor M^{11}, M^{12}, M^{21}, M^{22}.

The body and surface forces will be regarded as known. In other words the following quantities will be prescribed: the vector field $\boldsymbol{X}$ on the middle surface of the shell, the forces $\boldsymbol{T}_{(l)}$ and the moments $\boldsymbol{M}_{(l)}$ along the boundary of the shell. Thus, the free terms of the equations (2.31), (2.32) are given, and along the edge of the shell the normal and tangential forces, shear forces, twisting and bending

moments-altogether five boundary conditions are prescribed. We therefore have a system of six equations with ten unknown functions subject to five boundary conditions. In general these conditions are insufficient for a unique determination of the unknown quantities. Consequntly, the problem of equilibrium of a shell is in general case statically indeterminate, i.e. the equations (2.29) and (2.30) completed by all five natural boundary conditions are insufficient for the determination of the quantities describing the state of stress of a shell. In order to give this problem a mathematically determinate form it is necessary to add some additional assumptions on the nature of the distribution of stresses in the shell, which establish a number of new relations between the unknowns, such that when added to the original equations (2.29) and (2.30) they make it possible to construct a complete system of equations. We have in mind a system of equations which together with some definite boundary conditions and some other conditions resulting from the geometrical and physical content of the problem make it possible to find all the unknown quantities. In the derivation of the additional relations we should base on some physical or geometrical assumptions taking into account certain properties of actual shells. One of the usual ways consists in establishing by means of certain hypotheses relations between the components of forces and moments on one hand and the components of the deformation tensor on the other hand. For instance, in the case of elastic shells, this is achieved by the linear Hook's law and some additional hypotheses of geometrical and statical nature (for example the so-called Kirchhoff–Love hypothesis). However, many considerable difficulties arise in this way, which are still not overcome. One of these difficulties consists in the fact that there is no uniformity in the problem of the choice of the above mentioned hypotheses. Various hypotheses forming the basis of the derivation of the above relations lead to structurally

distinct differential equations and boundary conditions. In the equations, besides the principal terms, secondary terms frequently appear whose presence does not considerably influence the accuracy of the computations but greatly complicates the problem and significantly decreases the practical efficiency of the equations thus obtained. The complexity of the problem consists in the fact that it is difficult to propose methods making it possible to seperate the secondary terms in the equations from the principal ones, in order to eliminate the former.

So far there is a great lack of coordination in the problem of the choice of the hypotheses forming the basis for constructing the equations of the shell theory. This results in the setting-up of various systems of equations and relations, and frequently it is very difficult to decide what are the advantages and defects of one system with respect to another one. Hence, there may be cases when problems of a similar physical content lead to significantly distinct equations. All these facts deprive the mathematical theory of shells of the completeness and internal unity which are inherent in many other branches of classical mechanics. The construction of a general theory of shells cannot be regarded as complete until a sound general scheme of basic equations is found. But a solution of this problem in such a general way is connected with obstacles not easy to overcome.

The point is that shells play various roles in engineering structures. In various structures shells are subject to various loads and hence it is impossible to find a general scheme which would contain all possible practical cases and would lead to satisfactory results from both the mathematical and physical point of view. In the state of stress in a shell under the action of external loads, in some cases the forces prevail, moments being negligible; in other cases, conversely the state can be a purely moment state, the forces being negligible. Most frequently, of course, we encounter the cases when neither moments

nor forces may be neglected; they should however be connected by relations which are to be established in order to construct a complete system of equations of the theory of shells. Moreover, if the shell is subject to non-stationary external loads the character of the state of stress can change considerably in the course of time. In some cases the shell may be regarded as an elastic body and the Hook's law may be employed. But in many cases it is necessary to apply non-linear relations and also take into account the possibility of exceeding the yield point and the occurrence of plastic zones. All these facts indicate that it is difficult to embrace all cases in one scheme if the aim is to construct a general sound mathematical theory which at the same time would be sufficiently effective for practical applications. Hence, without giving up investigations aimed at a further improvement of the general principles of the theory of shells, it is necessary at the same time to examine various typical classes of shells and construct for them appropriate mathematical theories. Such is the case for instance for the plane problem [60b] and also for prismatic shells [14r].

One such typical case of the state of stress in shells is the membrane state of stress. In this case a system of equations is obtained which when completed by appropriate boundary consitions admits a unique solution. In other words, the membrane state of stress is a statically determinate problem.

In the following section we shall present a derivation of the basic equations of the membrane theory of shells and we shall consider some other related problems.

§3. System of equations of the membrane state of stress of shells. Geometric interpretation

3.1. Let a shell be in the membrane state of stress, i.e. the following conditions are satisfied:

$$\boldsymbol{M}^{\alpha} = 0 \quad (\alpha = 1, 2), \quad \text{i.e.} \quad \boldsymbol{M}^{\alpha\beta} = 0 \quad (\alpha, \beta = 1, 2). \qquad (3.1)$$

Under these conditions the system of equations (2.31)–(2.34) takes the form

$$\nabla_\alpha T^{\alpha\beta}+X^\beta=0 \quad (\beta=1,2)\,, \qquad b_{\alpha\beta}T^{\alpha\beta}+Z=0\,, \tag{3.2}$$

$$T^{\alpha\beta}=T^{\beta\alpha} \quad (\alpha,\beta=1,2)\,, \tag{3.3}$$

$$T^{\alpha}=0 \qquad (\alpha=1,2)\,. \tag{3.4}$$

Thus, in the membrane state of stress both the moment $\boldsymbol{M}_{(l)}$ and the shear force $\boldsymbol{n}\boldsymbol{T}_{(l)}=T_l$ vanish on every lateral area Σ^l. Consequently, the force vector on every area Σ^l is given by the formula

$$\boldsymbol{T}_{(l)}=\boldsymbol{T}^\alpha l_\alpha \equiv T^{\alpha\beta}l_\alpha \boldsymbol{r}_\beta\,, \tag{3.5}$$

or

$$\boldsymbol{T}_{(l)}=N_l\boldsymbol{l}+H_l\boldsymbol{s}\,, \tag{3.6}$$

where

$$N_l=T^{\alpha\beta}l_\alpha l_\beta\,, \qquad H_l=T^{\alpha\beta}l_\alpha s_\beta\,; \tag{3.7}$$

N_l and H_l are the normal and tangential forces. It readily follows from the formula (3.7) that

$$N_{(l)}=N_{(-l)}\,, \qquad H_l=H_{(-l)}\,. \tag{3.7a}$$

Moreover, taking into account the symmetry of the tensor $T^{\alpha\beta}$ we have

$$H_l=-H_s\,. \tag{3.8}$$

Thus, the tangential forces on two perpendicular areas have the same absolute value but opposite signs.

The symmetric tensor field $T^{\alpha\beta}$ which satisfies the system of equations (3.2) will be called *the non-homogeneous statical field if the components of the surface forces X^α and Z do not identically vanish at the same time*. The field $T^{\alpha\beta}$ will be called *the homogeneous statical field if $X^\alpha\equiv Z\equiv 0$*. A homogeneous statical field satisfies the system of equations

$$\nabla_\alpha T^{\alpha\beta}=0 \quad (\beta=1,2)\,, \qquad b_{\alpha\beta}T^{\alpha\beta}=0 \qquad (T^{\alpha\beta}=T^{\beta\alpha})\,. \tag{3.9}$$

It is evident that a general non-homogeneous statical field corresponding to a given field of external surface

forces $\boldsymbol{X} = X^{a}\boldsymbol{r}_{a} + Z\boldsymbol{n}$ is a sum of a fixed non-homogeneous field $\tilde{T}^{\alpha\beta}$ and an arbitrary homogeneous statical field $\mathring{T}^{\alpha\beta}$. The non-homogeneous field $\tilde{T}^{\alpha\beta}$ can be chosen arbitrarily. In particular it can be subjected to some additional conditions. Sometimes these conditions can be such that they determine the field $\tilde{T}^{\alpha\beta}$ uniquely.

We also observe that the variety of homogeneous statical fields of a given shell, i.e. the set of solutions of the homogeneous system of equations (3.9), is a linear manifold.*

3.2. We found in the preceding chapter that the system of equations (3.9) is also satisfied by a bending field. In other words, the tensor $\hat{T}^{\alpha\beta}$ which is given in terms of the rotation vector V by the formulae

$$\hat{T}^{\alpha\beta} = c^{\alpha\lambda}\frac{\partial V}{\partial x^{\lambda}}\boldsymbol{r}^{\beta} \quad (\alpha, \beta = 1, 2)\,, \tag{3.10}$$

is a solution of the homogeneous system of equations (3.9).

Thus, every rotation field is associated with a definite homogeneous statical field of a membrane state of stress. This makes it possible to interpret the geometrical properties of infinitesimal bendings of a surface as definite mechanical properties characterizing a membrane state of stress of shells. We do not maintain however that the deformed state in a shell due to an infinitesimal bending of its middle surface induces a membrane state of stress of the shell. On the contrary, at least for elastic shells, such a deformed state corresponds to a purely moment state of stress [50] (Ch. XXIII). For instance, in a thin elastic shell the components of forces and moments

* To avoid confusion we observe that we distinguish between the concepts of homogeneous and constant field on a surface (scalar, vector or tensor field). A constant field is one whose elements take the same value at all points of the surface. We consider this remark necessary since sometimes in the vector analysis it is customary to call such a field homogeneous.

referred to the coordinate system in curvature lines is given by the formulae [50]

$$N_1 = \frac{2Eh}{1-\sigma^2}(\varepsilon_1 + \sigma\varepsilon_2)\,, \qquad N_2 = \frac{2Eh}{1-\sigma^2}(\varepsilon_2 + \sigma\varepsilon_1)\,,$$
$$H_1 = -H_2 = \frac{Eh}{1+\sigma}\omega\,, \tag{3.11}$$

$$G_1 = -\frac{2Eh^3}{3(1-\sigma^2)}(\varkappa_1 + \sigma\varkappa_2)\,, \quad G_2 = -\frac{2Eh^3}{3(1-\sigma^2)}(\varkappa_2 + \sigma\varkappa_1)\,,$$
$$M_1 = -M_2 = \frac{2Eh^3}{3(1+\sigma)}\tau\,, \tag{3.12}$$

where E is the Young modulus, σ the Poisson ratio of the material of the shell, $2h$ the thickness of the shell, $\varepsilon_1, \varepsilon_2, \omega$ extensions and shear of the middle surface, and $\varkappa_1, \varkappa_2$ and τ are the variations of the principal curvatures and the torsion of the surface. In infinitesimal bendings of the middle surface $\varepsilon_1 = \varepsilon_2 = \omega = 0$ and, consequently, $N_1 = N_2 = H_1 = -H_2 = 0$, i.e. the tangential components of the force vanish.

Thus, the state of stress of an elastic shell induced by an infinitesimal bending of the middle surface is not a membrane state. Most likely, to the first approximation which is sufficient for many practical purposes, such a stress, by (3.11) and (3.12), may be regarded as a purely moment state of stress.

In what follows the state of stress of a shell induced by an infinitesimal bending of its middle surface will be called *the purely bending state of stress*. It was found above that for a fairly wide class of elastic shells this state is a purely moment equilibrium state of stress.

The formulae (3.10) are equivalent to the following vector relation:

$$\frac{d\boldsymbol{V}}{ds} = \hat{T}^{\alpha\beta} l_\alpha \boldsymbol{r}_\beta\,. \tag{3.13}$$

This formula enables us to associate also with every homogeneous statical field $T^{\alpha\beta}$ a definite displacement

and rotation fields. These fields are given by the formulae (6.18) and (6.19) of Ch. V and, consequently, can be determined by integration. In the case of a simply-connected domain we obtain in this way single-valued fields of displacement and rotation, to within trivial bendings.

This means that in the case of a simply-connected surface every homogeneous statical field can be interpreted as a bending field, and conversely, by setting

$$T^{\alpha\beta} = \hat{T}^{\alpha\beta} \qquad (\alpha, \beta = 1, 2) . \tag{3.14}$$

In the case of multiply-connected surfaces, to a given homogeneous statical field $T^{\alpha\beta}$ there correspond, in general, multi-valued displacement and rotation fields. We found already in the preceding chapter that for the single-valuedness of displacement and rotation fields it is necessary and sufficient that the following relations be satisfied:

$$\int_{L_j} \boldsymbol{T}_{(l)} ds = 0 , \qquad \int_{L_j} \boldsymbol{T}_{(l)} \times \boldsymbol{r} ds = 0 , \qquad (j = 1, \ldots, m) \tag{3.15}$$

where

$$\boldsymbol{T}_{(l)} = T^{\alpha\beta} l_\alpha \boldsymbol{r}_\beta . \tag{3.16}$$

Below we shall elucidate the mechanical meaning of these conditions; we shall also examine the nature of the multi-valuedness of the displacement and rotation fields corresponding to an arbitrary statical field.

3.3. In §6 of Ch. V we proved the integral identity

$$\int_L \boldsymbol{U} \boldsymbol{T}_{(l)} ds \equiv \int_L U_\alpha T^{\alpha\beta} l_\beta ds = 0 , \tag{3.17}$$

where L is the boundary of the surface S, $\boldsymbol{U}$ is an arbitrary displacement field on S and $\boldsymbol{T}_{(l)} = T^{\alpha\beta} l_\alpha \boldsymbol{r}_\beta$, $T^{\alpha\beta}$ being an arbitrary solution of the system of equations (3.9), i.e. $T^{\alpha\beta}$ is an arbitrary homogeneous statical field of a membrane state of stress in a free shell the middle surface of which is the surface S. It is assumed that $\boldsymbol{U}$ and $T^{\alpha\beta}$ are con-

tinuous in $S+L$. The formula (3.17) can also be generalized to the case of a non-homogeneous statical field. If $\boldsymbol{U}$ is, as before, a displacement field on S and $T^{\alpha\beta}$ a non-homogeneous statical field, i.e. a solution of the non-homogeneous system of equations (3.2), then we have

$$\begin{aligned}\boldsymbol{U}\boldsymbol{X} &= u_\beta X^\beta + u_0 Z = -u_\beta \nabla_\alpha T^{\alpha\beta} + u_0 Z \\ &= -\nabla_\alpha(u_\beta T^{\alpha\beta}) + \tfrac{1}{2}T^{\alpha\beta}(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) + u_0 Z \\ &= -\nabla_\alpha(u_\beta T^{\alpha\beta}) + T^{\alpha\beta} b_{\alpha\beta} u_0 + u_0 Z = -\nabla_\alpha(u_\beta T^{\alpha\beta})\,. \quad (3.18)\end{aligned}$$

We have made use here of the system of equations for the displacement field

$$\tfrac{1}{2}(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) - b_{\alpha\beta} u_0 = 0 \qquad (\alpha, \beta = 1, 2)\,, \quad (3.19)$$

and also we have taken into account the symmetry of the tensor $T^{\alpha\beta}$. Besides we introduced the notation

$$\boldsymbol{X} = X^\alpha \boldsymbol{r}_\alpha + Z\boldsymbol{n}\,, \qquad \boldsymbol{U} = u_\alpha \boldsymbol{r}^\alpha + u_0 \boldsymbol{n}\,. \quad (3.20)$$

Integrating the relation (3.18) over the surface S and applying Green's identity we arrive at the required formula

$$\iint\limits_S \boldsymbol{U}\boldsymbol{X}\,dS + \int\limits_L \boldsymbol{U}\boldsymbol{T}_{(l)}\,ds = 0\,. \quad (3.21)$$

In particular, if for $\boldsymbol{U}$, we take an arbitrary trivial field

$$\boldsymbol{U} = \boldsymbol{\Omega}\times \boldsymbol{r} + \boldsymbol{C}\,, \quad (3.22)$$

the identity (3.21) is equivalent to the following relations:

$$\left.\begin{aligned}\iint\limits_S \boldsymbol{X}\,dS + \int\limits_L \boldsymbol{T}_{(l)}\,ds = 0\,, \\ \iint\limits_S \boldsymbol{r}\times\boldsymbol{X}\,dS + \int\limits_L \boldsymbol{r}\times\boldsymbol{T}_{(l)}\,ds = 0\,,\end{aligned}\right\} \quad (3.23)$$

which, as is known, constitute *the necessary and sufficient condition of equilibrium of the surface S treated as a rigid body*. In what follows in prescribing a distribution of the surface forces $\boldsymbol{X}$ and the contour forces $\boldsymbol{T}_{(l)}$ we always understand that this distribution satisfies the equilibrium conditions of this rigid body (3.23).

In the case of a free shell $\boldsymbol{X} \equiv 0$ and, consequently, we have the formula (3.17).

Since $\boldsymbol{U}\boldsymbol{X}dS$ and $\boldsymbol{U}\boldsymbol{T}_{(l)}ds$ are elementary works of the surface force $\boldsymbol{X}dS$ and the contour force $\boldsymbol{T}_{(l)}ds$ on the displacement $\boldsymbol{U}$, the formula (3.21) expresses the following fact:

The sum of the elementary work of external surface and contour forces applied to a shell in a membrane equilibrium state of stress, with displacement corresponding to an arbitrary infinitesimal bending of the middle surface, is equal to zero.

In other words, *infinitesimal bendings of the middle surface of a shell are possible (virtual) displacements with respect to an arbitrary system of external and contour forces creating a membrane state of stress in the shell.*

Thus, the external surface and contour forces acting on a shell and resulting in a non-zero membrane state of stress should be orthogonal to an arbitrary displacement field. In other words, the statical conditions which create in the shell an equilibrium membrane state of stress should be compatible with arbitrary constraints which admit infinitesimal bendings, and conversely. Hence, the statical conditions creating a membrane state of stress in the shell can always be completed by constraints (in particular rigid constraints) compatible with an (in particular trivial) infinitesimal bending of the middle surface of the shell. These constraints (including rigid constraints) must necessarily be compatible with an arbitrary statical field.

The equilibrium stress state of a shell which is composed of a membrane and purely bending states of stress will hereafter be called briefly the state $(\boldsymbol{T}, \boldsymbol{M})$. It is evident that such a state is intermiediate between the general moment state of stress and the purely membrane state of stress. In what follows a membrane state of stress will for the sake of brevity, be called the state $(\boldsymbol{T})$ and a purely bending state of stress the state $(\boldsymbol{M})$. The state $(\boldsymbol{T}, \boldsymbol{M})$ is a sum of the states $(\boldsymbol{T})$ and $(\boldsymbol{M})$.

In the preceding section we indicated a number of geometrical and kinematic conditions determining the state (**M**). In the following section we shall examine conditions of the type (**T**). In the state of stress (**T**, **M**) the deformation of the middle surface S is composed of two deformations, namely of an infinitesimal bending corresponding to the state (**M**) and deformations of extension (or compression) and shear corresponding to the state (**T**). The first deformation is determined directly by solving the system of equations of infinitesimal bending the conditions of type (**M**) being taken into account; the second deformation in general can be determined in an explicit form only by introducing a hypothesis on the relation between deformation and stress. In this case the statical field (**T**) is determined directly; the deformation can for instance be found in the case of a thin elastic shell by employing the approximate formulae (3.11). The formulae (3.12) make it possible to determine the components of the moments of the state (**M**).

The relations (3.15) express the fact that the contour forces applied to an arbitrary closed boundary curve of a free shell are statically equivalent to zero (the resultant force and moment vectors of these forces vanish). These relations imply that the condition under consideration will then be satisfied for an arbitrary closed curve belonging to the middle surface of the shell.

Thus, according to the formula (3.13), to a homogeneous statical field there correspond single-valued displacement and rotation fields (to within a trivial bending) if and only if the contour forces applied to every boundary contour are statically equivalent to zero. For a simply-connected surface this condition is always satisfied in view of the relation (3.17). Hence, in this case, as was already noted above, an arbitrary homogeneous statical field can be interpreted as a bending field of the middle surface.

3.4. If the surface is multiply-connected and the conditions (3.15) are not satisfied, the fields of displacement

and rotation, $\boldsymbol{U}$ and $\boldsymbol{V}$, determined in accordance with the formulae (6.18) and (6.19) of Ch.V are multi-valued functions of position on the surface. We now proceed to an examination of the nature of this multi-valuedness (see also [33a], §26).

Suppose that for a fixed coordinate system the surface is mapped homeomorphically onto a multiply-connected domain G of the plane $z = x^1 + ix^2$. Let the complement of G with respect to the entire plane consist of $m+1$ continue $G_0, \ldots, G_m$, G_0 being either absent or containing the point at infinity; $G_1, \ldots, G_m$ are bounded sets. Let

$$\vartheta_j(z) = \frac{1}{2\pi}\arg(z - z_j)\,, \tag{3.24}$$

where z_j is a fixed point of the continuum $G_j (j = 1, \ldots, m)$. Evidently, $\vartheta_j(z)$ is a multi-valued function taking an increment equal to unity when the point z describes once the contour of the continuum G_j counter-clockwise. Then (6.18) and (6.19) of Chapter 5 imply immediately that the vectors $\boldsymbol{V}$ and $\boldsymbol{U}$ have the form

$$\boldsymbol{V}(M) = \sum_{j=1}^{m} \boldsymbol{V}_j \vartheta_j(M) + \boldsymbol{V}_*(M)\,, \tag{3.25}$$

$$\boldsymbol{U}(M) = \sum_{j=1}^{m} (\boldsymbol{V}_j \times \boldsymbol{r} + \boldsymbol{V}'_j)\vartheta_j(M) + \boldsymbol{U}_*(M)\,, \tag{3.26}$$

where $\boldsymbol{U}_*$ and $\boldsymbol{V}_*$ are single-valued vector-functions of point M of the surface S, and $\boldsymbol{V}_j$ and $\boldsymbol{V}'_j$ are constant vectors

$$\boldsymbol{V}_j = \int_{L_j} \boldsymbol{T}_{(l)}\,ds\,, \qquad \boldsymbol{V}'_j = \int_{L_j} \boldsymbol{r} \times \boldsymbol{T}_{(l)}\,ds\,. \tag{3.27}$$

It is now easy to prove that to multi-valued fields of displacement and rotation, $\boldsymbol{U}$ and $\boldsymbol{V}$, of the form (3.25) and (3.26) where $\boldsymbol{V}_j$ and $\boldsymbol{V}'_j (j = 1, \ldots, m)$ are arbitrarily given constant vectors, by (3.10) there always corresponds a definite homogeneous statical field of a membrane state of stress of the shell.

The following geometrical interpretation can be given to the formulae (3.25) and (3.26): On the surface S let there be m cuts along simple smooth arcs $L'_1, \ldots, L'_m$, made in such a way that a simply-connected surface $S' = S-(L'_1+\ldots+L'_m)$ results. To this surface there corresponds on the plane $z = x^1+ix^2$ a simply-connected domain $G' = G-(\Gamma'_1+\ldots+\Gamma'_m)$ where Γ'_j are the images of the arcs L'_j $(j = 1, \ldots, m)$.

Let $L_j'^+$ and $L_j'^-$ be the left and the right edges of the cuts L'_j $(j = 1, \ldots, m)$. Then, by (3.25) and (3.26), we have

$$\boldsymbol{V}|_{L_j'^+}-\boldsymbol{V}|_{L_j'^-} = \boldsymbol{V}_j \qquad (j = 1, \ldots, m)\,, \quad (3.28)$$

$$\boldsymbol{U}|_{L_j'^+}-\boldsymbol{U}|_{L_j'^-} = \boldsymbol{V}_j\times\boldsymbol{r}+\boldsymbol{V}'_j \qquad (j = 1, \ldots, m)\,. \quad (3.29)$$

These formulae indicate that every edge of an arbitrary cut L'_j is displaced with respect to the other edge as a rigid body, these displacements being uniquely determined by the given homogeneous statical field $T^{\alpha\beta}$.

Thus, also in the case of a multiply-connected surface, to every membrane state of stress of a shell there corresponds a definite infinitesimal bending of its middle surface if the surface is converted into a simply-connected surface by making arbitrary cuts $L'_1, \ldots, L'_m$; the edges of every cut move relatively to each other as rigid bodies. These movements are uniquely determined by the given statical field.

It follows that *the absence of a non-trivial solution of the system of equations* (3.9) *satisfying the constraint conditions present on the surface and the conditions* (3.15), *is a necessary and sufficient criterion for the rigidity of a multiply-connected surface.*

In other words, if there is non-zero homogeneous statical field $T^{\alpha\beta}$ satisfying the geometrical constraints and the conditions (3.15) ensuring the single-valuedness of the displacement and rotation fields, the surface is rigid. This condition is both necessary and sufficient condition of (geometrical) rigidity.

§4. New derivation of the characteristic equation

In this section we shall present another way of deriving the characteristic equation (5.30) obtained in the preceding chapter on the basis of a purely geometrical reasoning.

Suppose that the pair of vectors $\boldsymbol{T}_*^1$ and $\boldsymbol{T}_*^2$ is a particular solution of the equation (3.29). Then the general solution of this equation has the form

$$\boldsymbol{T}^\alpha = \boldsymbol{T}_*^\alpha + \boldsymbol{T}_0^\alpha \quad (\alpha = 1, 2), \tag{4.1}$$

where $\boldsymbol{T}_0^\alpha$ is an arbitrary solution of the homogeneous equation

$$\frac{1}{\sqrt{a}} \frac{\partial \sqrt{a}\, \boldsymbol{T}_0^\alpha}{\partial x^\alpha} = 0 . \tag{4.2}$$

It is obvious that this equation is satisfied by vectors of the form

$$\boldsymbol{T}_0^\alpha = c^{\alpha\lambda} \frac{\partial \boldsymbol{W}}{\partial x^\lambda} \quad (\alpha = 1, 2), \tag{4.3}$$

where $\boldsymbol{W} = \boldsymbol{W}(x^1, x^2)$ is an arbitrary differentiable vector-function of a point of the surface. We have

$$T_0^{\alpha\beta} = \boldsymbol{T}_0^\alpha \boldsymbol{r}^\beta = c^{\alpha\lambda} \frac{\partial \boldsymbol{W} \boldsymbol{r}^\beta}{\partial x^\lambda} - c^{\alpha\lambda} \boldsymbol{W} \frac{\partial \boldsymbol{r}^\beta}{\partial x^\lambda} .$$

But

$$\frac{\partial \boldsymbol{r}^\beta}{\partial x^\lambda} = -\Gamma_{\lambda\gamma}^\beta \boldsymbol{r}^\gamma + b_\lambda^\beta \boldsymbol{n} .$$

Hence

$$T_0^{\alpha\beta} = c^{\alpha\lambda} \frac{\partial w^\beta}{\partial x^\lambda} + c^{\alpha\lambda} \Gamma_{\lambda\gamma}^\beta w^\gamma - c^{\alpha\lambda} b_\lambda^\beta w_0 = c^{\alpha\lambda} \nabla_\lambda w^\beta - c^{\alpha\lambda} b_\lambda^\beta w_0 , \tag{4.4}$$

where $w^\beta = \boldsymbol{W} \boldsymbol{r}^\beta$, $w_0 = \boldsymbol{W} \boldsymbol{n}$. Since $T_0^{\alpha\beta} = T_0^{\beta\alpha}$ we have

$$\nabla_\alpha w^\alpha - 2\mathrm{H} w_0 = 0 . \tag{4.5}$$

Further,

$$0 = \boldsymbol{n} \boldsymbol{T}_0^\alpha = c^{\alpha\lambda} \frac{\partial \boldsymbol{W} \boldsymbol{n}}{\partial x^\lambda} - c^{\alpha\lambda} \boldsymbol{W} \frac{\partial \boldsymbol{n}}{\partial x^\lambda} = c^{\alpha\lambda} \frac{\partial w_0}{\partial x^\lambda} + c^{\alpha\lambda} b_{\lambda\gamma} w^\gamma ,$$

i.e.

$$\frac{\partial w_0}{\partial x^a} + b_{a\beta} w^\beta = 0 \qquad (a = 1, 2)\,. \tag{4.6}$$

Let $\mathrm{K} \neq 0$. Then from (4.6)

$$w^\beta = -\bar{d}^{\beta a} \frac{\partial w_0}{\partial x^a}, \qquad \bar{d}^{a\beta} = \frac{1}{\mathrm{K}} c^{a\lambda} c^{\beta\lambda} b_{\lambda\gamma}\,. \tag{4.7}$$

Substituting (4.7) into (4.5) we obtain the characteristic equation

$$\nabla_a(\bar{d}^{a\beta}\nabla_\beta w_0) + 2\mathrm{H} w_0 = 0\,. \tag{4.8}$$

If w_0 is a characteristic function the vector $\boldsymbol{W}$ is given by the formula

$$\boldsymbol{W} = -\bar{d}^{a\beta}\frac{\partial w_0}{\partial x^\beta}\boldsymbol{r}_a + w_0 \boldsymbol{n}\,, \tag{4.9}$$

For the vector $\boldsymbol{T}_0^a$ in accordance with (4.3) we have

$$\boldsymbol{T}_0^a = -c^{a\beta}(\nabla_\beta \bar{d}^{\lambda\gamma}\nabla_\gamma w_0 + b_\beta^\lambda w_0)\boldsymbol{r}_\lambda \qquad (a = 1, 2)\,. \tag{4.10}$$

Thus we have obtained in a different way the formulae equivalent to the formulae (6.27) derived in the preceding paragraph.

§5. Conditions of existence of the state (*T*). Boundary value problems

5.1. The existence of the membrane state of stress of a shell denoted above by (***T***), which is a very special case of the general state of stress $\tilde{\nu}$ is possible only under very special distributions of external loads. Let us first of all enumerate the necessary conditions of existence of the state (***T***).

First, the totality of the surface and body forces is statically equivalent to a force field $\boldsymbol{X}$ distributed in the middle surface of the shell. The moments of these forces vanish identically.

Second, the forces acting on the lateral areas of the shell are statically equivalent to forces of the form $\boldsymbol{T} =$

$=\boldsymbol{N}\boldsymbol{l}+\boldsymbol{H}\boldsymbol{s}$, i.e. moments and shear forces along the boundary of the shell are equal to zero.

Third, the surface forces $\boldsymbol{X}$ and the contour forces $\boldsymbol{T}$ should satisfy the identity (3.21) in which an arbitrary displacement field $\boldsymbol{U}$ of an infinitesimal bending of the middle surface appears. If $\boldsymbol{U}$ is a trivial displacement field the condition (3.21) constitutes the usual condition (3.15) of equilibrium of a rigid body which, obviously, always should be satisfied.

The conditions enumerated above, do not yet ensure the existence of the state ($\boldsymbol{T}$). In general, establishing general laws of distribution of external loads which would ensure the existence of such states is a difficult problem, hardly soluble by purely mathematical means. The problem is also complicated by the fact that the state under consideration is of an unstable nature. In a purely mathematical consideration this fact follows from the condition (3.21) which in general is not satisfied if the fields $\boldsymbol{X}$ and $\boldsymbol{T}$ are given in an arbitrary way. Hence, putting aside the problem of physical conditions of existence of the state ($\boldsymbol{T}$) we shall deal below mainly with the purely mathematical problem of determining conditions ensuring the existence and uniqueness of a statical force field $\boldsymbol{T}_{(l)}$ satisfying the system of equations (3.2) and (3.3). As we found above, for this it is necessary that the conditions (3.21) be satisfied. Below we shall prove the sufficiency of this condition. Obviously, this does not yet mean that under these conditions the state of stress in a shell is really rigorously a membrane state. Nevertheless, many examples of engineering practice indicate that under these conditions numerous actual shells are practically in a membrane state. The point is that actual shells almost always possess a considerable rigidity with respect to bending deformations and, consequently, the presence of a small component of moment cannot in such cases significantly distort the membrane nature of the stress field. (This circumstance, evidently, takes

place also in many cases in which the conditions (3.21) are manifestly violated; see below, §5.8 and the following). It is known that convex shells have a considerably greater rigidity with respect to bending than a shell of negative curvature. Hence, convex shells are more frequently applied in practice owing to their greater stability with respect to a violation of the conditions for the existence of the membrane state. In what follows we shall understand by the state (***T***) the presence of such a distribution of surface forces ***X*** and contour forces ***T*** which ensures the existence of a statical field $T^{\alpha\beta}$ satisfying the system of equations (3.2).

We shall investigate conditions for the existence of the state (***T***) only for shells of positive curvature.

In this case the problem is reduced to a system of equations of elliptic type and we can use the results of the preceding chapters. But before proceeding to an investigation of this special case we consider it expedient to make some preliminary remarks of general nature.

We have already mentioned the unstable character of the state (***T***). But the nature of this instability depends to a great extent on the type of the shell; it is different for shells of elliptic and hyperbolic type. This fact is readily verified if it is taken into account that the type of the differential equations of the problem we are interested in, is determined by the type of the corresponding surface. The problem is reduced, as was indicated above, to elliptic equations for shells of positive curvature and to hyperbolic or parabolic equations, respectively, for shells of negative of zero curvature. But the nature of the dependence of the solutions of these equations on the data which determine them uniquely, is not the same. To verify this statement it suffices to analyse this problem for the case of the Dirichlet problem for an elliptic equation (for instance for the Laplace equation) and the Cauchy problem for a hyperbolic equation (for instance for the equation of the vibrating string).

Under small continuous changes of the boundary data, a solution of the Dirichlet problem undergoes a small change in the whole domain (we have in mind regular domains for which the Dirichlet problem is correct). In other words a solution of the Dirichlet problem is a continuous function of the boundary data. Moreover, if the coefficients and the free term of an elliptic equation are continuous and have a number of continuous derivatives, then also the derivatives of the solution of the Dirichlet problem are continuous and depend continuously on the boundary data, inside the domain. If it is taken into account that the derivatives of the solution of the Dirichlet problem have frequently a definite physical meaning (for instance the gradient of the solution can be treated as a force field or velocity field of the steady motion of a perfect fluid), then the above circumstance means that the corresponding physical problem is in fact stable (at least inside the domain, i.e. in the whole domain except for a narrow strip adjacent to the boundary; in general, the gradient of the solution can strongly increase on approaching the boundary). In general the analytic nature of a solution of an elliptic equation inside the domain is determined only by the analytic nature of the coefficients; singularities of the boundary and the boundary data are localized near the boundary and do not enter the interior of the domain.

In the case of a hyperbolic equation a Dirichlet problem is in general very little investigated, owing to its strong instability. But also the nature of stability of a solution of the Cauchy problem is considerably different from the nature of stability of a solution of the Dirichlet problem. This is readily demonstrated in the case of the equation of a vibrating string

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0\,, \tag{5.1}$$

which is also encountered in the theory of shells and, as we have seen in the preceding chapter (§11), also in

problems of infinitesimal bendings of surfaces of negative curvature. It follows from the d'Alambert formula

$$u(x, y) = \tfrac{1}{2}[\varphi(x+y)+\varphi(x-y)]+\tfrac{1}{2}\int\limits_{x-y}^{x+y}\psi(\xi)d\xi\,, \quad (5.2)$$

giving the solution of the Cauchy problem with the initial conditions

$$u|_{y=0} = \varphi(x)\,, \qquad \frac{\partial u}{\partial y}\bigg|_{y=0} = \psi(x) - \infty < x < \infty\,, \quad (5.3)$$

that for sufficiently small continuous changes of the initial data the solution undergoes a small change (if x and y change in a bounded domain), but the same assertion with respect to the derivatives is not true; this depends on differential properties of the initial data. Thus, the nature of dependence of a solution of the Cauchy problem on the initial data is more capricious and it is much more difficult to preserve the stability of a solution of a physical problem in this case than in the case of a problem governed by an elliptic equation. If we want to ensure the stability of a physical problem in the hyperbolic case we should show that perturbations of the initial data belong to a narrower class of differentiable functions. It is, however, very difficult in practical problems to ensure the fulfilment of this condition, since perturbations are often due to random causes which are difficult to control.

On the basis of the properties of equations of elliptic and hyperbolic type a number of conclusions can be drawn concerning the nature of distribution of stresses inside shells of the corresponding types. For definitness we shall consider a shell loaded only by contour forces $\boldsymbol{T}_{(l)}$. In an elliptic shell a stress due to arbitrary external loads is taken up by all its parts, and if the external load is distributed more or less uniformly (i.e. concentrated forces are excluded) no great stress concentrations arise inside the shell. Hence, under some random but

small increments of the external load it is unlikely that the additional stresses will lead to dangerous stress concentrations. The situation is different in the case of a hyperbolic shell. Then perturbations in the distribution of the contour forces are propagated inside the shell along certain zones and it may turn out that the corresponding stresses are taken up not by all parts of the shell but are localized in some regions which can also be situated far from the region of application of the contour forces. Hence, in the case of a hyperbolic shell, the probability of the occurence of dangerous stress concentrations which can have unfortunate consequences (appearance of fissures, foldings, destruction of constraints etc.) is greater. In view of this, hyperbolic shells are comparatively seldom used in practice. Shells of parabolic type are intermediate between shells of elliptic and hyperbolic type. They are more stable than hyperbolic shells and their computation is simpler then that for shells of elliptic type. Hence, parabolic shells are more frequently applied in technical structures (floors of various kinds, etc.).

We shall say a few words more about shells of mixed type; we have in mind shells whose curvature changes its sign. The presence in such a shell of regions of positive curvature can, under certain circumstances, ensure sufficient stability of the whole shell. Such a shell can preserve stability in the large in some cases even when because of great stress concentrations in the hyperbolic zones the yield point is exceeded and the material is in a plastic state. Computation of such shells leads to differential equations of mixed type. The theory of such equations has in recent years been extensively elaborated and many interesting results have been obtained, [9]; they can also be employed in problems of the theory of shells (see also Ch. V, §11, p. 528).

5.2. Let us consider an isometric–conjugate coordinate system on the middle surface of a shell of positive curva-

ture; introducing the complex stress function in accordance with the formula

$$w' = u' + iv' \equiv \sqrt{a}(T^{11} - iT^{12}) + \frac{Z}{2\sqrt{\mathrm{K}}}, \tag{5.4}$$

the system of equations (3.2) can be written in the complex form

$$\partial_{\bar{z}} w' - A w' - \bar{B} \bar{w}' = F, \tag{5.5}$$

where

$$A = -\partial_{\bar{z}} \ln \sqrt{a\sqrt{\mathrm{K}}}, \quad B = -\left(\operatorname{arch} \frac{\mathrm{H}}{\sqrt{\mathrm{K}}}\right)_{\bar{z}} e^{i\varphi} - \frac{\mathrm{K}_z}{4\mathrm{K}} e^{2i\varphi}, \tag{5.6}$$

$$F = \frac{1}{2}\sqrt{\mathrm{K}}\left(\frac{\mathrm{Z}}{\mathrm{K}}\right)_z - \frac{\sqrt{a}}{2}(X^1 - iX^2). \tag{5.7}$$

If $F \equiv 0$, i.e.

$$X^1 = \frac{1}{2}\sqrt{\frac{\mathrm{K}}{a}}\left(\frac{\mathrm{Z}}{\mathrm{K}}\right)_x, \quad X^2 = \frac{1}{2}\sqrt{\frac{\mathrm{K}}{a}}\left(\frac{\mathrm{Z}}{\mathrm{K}}\right)_y, \tag{5.8}$$

we obtain the homogeneous equation

$$\partial_{\bar{z}} w' - A w' - \bar{B} \bar{w}' = 0, \tag{5.9}$$

which we encountered in Ch. V (§6.6). The condition (5.8) indicates that the field of surface forces $\boldsymbol{X}$ has the potential $V = \mathrm{Z}/\mathrm{K}$. A potential field in an arbitrary coordinate system can be written in the form

$$\boldsymbol{X} = \frac{1}{2}\mathrm{K} d^{\alpha\beta} \frac{\partial V}{\partial x^\alpha} \boldsymbol{r}_\beta + \mathrm{K} V \boldsymbol{n}, \quad Z = \mathrm{K} V. \tag{5.10}$$

This fact follows immediately from the formulae (5.8) if it is taken into account that in the isometric-conjugate coordinate system (Ch. V, §5.4)

$$d^{11} = d^{22} = \frac{1}{\sqrt{a\mathrm{K}}}, \quad d^{12} = d^{21} = 0. \tag{5.11}$$

We now indicate one more way of deriving the equation (5.5), which makes it possible to obtain the latter

directly from the equation of the membrane state in the vector form

$$\frac{1}{\sqrt{a}}\frac{\partial\sqrt{a}\,\boldsymbol{T}^{\alpha}}{\partial x^{\alpha}}+\boldsymbol{X}=0\,,\qquad \boldsymbol{r}_{\alpha}\times\boldsymbol{T}^{\alpha}=0 \tag{5.12}$$

This equation can also be written in the form

$$\partial_{\bar{z}}\big[\sqrt{a}(\boldsymbol{T}^{1}-i\boldsymbol{T}^{2})\big]+\partial_{z}\big[\sqrt{a}(\boldsymbol{T}^{1}+i\boldsymbol{T}^{2})\big]+\sqrt{a}\boldsymbol{X}=0\,. \tag{5.13}$$

Since

$$\boldsymbol{T}^{\alpha}=T^{\alpha\beta}\boldsymbol{r}_{\beta}=(T^{\alpha1}+iT^{\alpha2})\boldsymbol{r}_{z}+(T^{\alpha1}-iT^{\alpha2})\boldsymbol{r}_{\bar{z}}\,,$$

in view of symmetry of the tensor $T^{\alpha\beta}$ we have

$$\boldsymbol{T}^{1}-i\boldsymbol{T}^{2}=(T^{11}+T^{22})\boldsymbol{r}_{z}+(T^{11}-T^{22}-2iT^{12})\boldsymbol{r}_{\bar{z}}\,.$$

Taking now the isometric–conjugate coordinate system we obtain from the relation (5.13) the relation

$$T^{11}+T^{22}=-\frac{\mathrm{Z}}{\sqrt{a\mathrm{K}}}\,. \tag{5.14}$$

Bearing in mind the notation (5.4) introduced above we obtain

$$\frac{1}{2}\sqrt{a}(\boldsymbol{T}^{1}-i\boldsymbol{T}^{2})=w'\boldsymbol{r}_{\bar{z}}-\frac{1}{2}\frac{\mathrm{Z}}{\sqrt{\mathrm{K}}}\boldsymbol{r}_{z}\,. \tag{5.15}$$

Substituting the last result into (5.13) and using the relations (Ch. II, §6.5)

$$\boldsymbol{r}_{\bar{z}\bar{z}}+A\boldsymbol{r}_{\bar{z}}+B\boldsymbol{r}_{z}=0\,, \tag{5.16}$$

$$\boldsymbol{r}_{z\bar{z}}+\frac{\mathrm{K}_{\bar{z}}}{4\mathrm{K}}\boldsymbol{r}_{z}+\frac{\mathrm{K}_{z}}{4\mathrm{K}}\boldsymbol{r}_{\bar{z}}-\frac{1}{2}\sqrt{a\mathrm{K}}\,\boldsymbol{n}=0\,, \tag{5.17}$$

we readily obtain the equation (5.5).

Multiplying the relation (5.15) throughout by $\dfrac{1}{i}\dfrac{dz}{ds}$ and taking into account that

$$\frac{\sqrt{a}}{i}\frac{dz}{ds}=l_{1}+il_{2}\,,\qquad l_{\alpha}=\boldsymbol{l}\boldsymbol{r}_{\alpha}\,, \tag{5.18}$$

we have

$$\boldsymbol{T}_{(l)} = 2\,\mathrm{Re}\left[\frac{1}{i}\,w'(z)\,\frac{dz}{ds}\,\boldsymbol{r}_{\bar{z}}\right] - \frac{\mathrm{Z}}{\sqrt{K}}\,\mathrm{Re}\left[\frac{1}{i}\,\frac{dz}{ds}\,\boldsymbol{r}_{z}\right]. \quad (5.19)$$

Hence, we easily derive the formula

$$w'(z) = -\frac{2i}{z'\sqrt{a\mathrm{K}}}\,\boldsymbol{T}_{(l)}\boldsymbol{n}_z - \frac{1}{2}\,\frac{\mathrm{Z}}{\sqrt{\mathrm{K}}}\,\frac{\bar{z}'}{z'} \qquad \left(z' = \frac{dz}{ds}\right). \quad (5.20)$$

In deriving this formula we used the relation

$$\frac{1}{2}(\boldsymbol{r}^1 - i\boldsymbol{r}^2) = -\frac{1}{\sqrt{a\mathrm{K}}}\,\boldsymbol{n}_z\,. \quad (5.21)$$

The formulae (5.19) and (5.20) imply the relations

$$w'(z) = -\frac{\boldsymbol{n}_{\bar{z}}}{\sqrt{\mathrm{K}}}\left[\boldsymbol{T}_{(l)}\frac{d\bar{z}}{dl} + \boldsymbol{T}_{(s)}\frac{d\bar{z}}{ds}\right], \quad (5.22)$$

$$Z = 2\boldsymbol{n}_z\left[\boldsymbol{T}_{(l)}\frac{dz}{dl} + \boldsymbol{T}_{(s)}\frac{dz}{ds}\right]. \quad (5.23)$$

Let φ be the angle of inclination of the unit vector $\boldsymbol{l}$ to the principal direction s_1. Then we have

$$N_l = \frac{1}{2}(N_1 + N_2) + \frac{N_1 - N_2}{2}\cos 2\varphi + H\sin 2\varphi\,, \quad (5.24)$$

$$H_l = -\frac{N_1 - N_2}{2}\sin 2\varphi + H_1\cos 2\varphi \qquad (H_1 = -H_2)\,, \quad (5.25)$$

where N_1 and N_2 are the normal and $H_1 = -H_2$ the tangential forces on the principal normal cross-sections. It follows from the formulae (5.24) and (5.25) that

$$N_1 + N_2 = N_l + N_s\,, \quad (5.26)$$

$$\frac{dN_l}{d\varphi} = 2H_l\,, \qquad \frac{dH_l}{d\varphi} + 2N_l = N_1 + N_2\,, \quad (5.27)$$

$$\frac{d\boldsymbol{T}^2_{(l)}}{d\varphi} = 2H_l(N_1 + N_2)\,. \quad (5.28)$$

The relation (5.26) means that *the sum of normal forces acting on perpendicular normal cross-sections is independent of the orientation of the cross-sections and, consequently, is a scalar function of position of the middle surface (this assertion as well as the formulae* (5.26)–(5.28) *is valid for shells of any type).*

It follows from the relation (5.28) that $|\boldsymbol{T}_{(l)}|$ attains the extremum values on the areas on which the tangential forces vanish (we assume that $N_1+N_2 \neq 0$). It is seen from (5.25) that this takes place on two perpendicular areas Σ' and Σ'' with normals $\boldsymbol{l}'$ and $\boldsymbol{l}''$ the directions of which are defined by the angles $\varphi, \varphi+\frac{\pi}{2}$ where φ is the solution of the equation

$$\operatorname{tg} 2\varphi = \frac{2H_1}{N_1-N_2}. \tag{5.29}$$

We shall call these directions *the principal axes of the force field* $\boldsymbol{T}_{(l)}$, and the curves lying on the middle surface and tangent at all points to one of the principal axes, *the principal lines of the force field.* Let us denote the normal forces on Σ' and Σ'' by N' and N''. They are termed *the principal forces* of the field $\boldsymbol{T}_{(l)}$. Since

$$\boldsymbol{T}_{(l')} = N'\boldsymbol{l}', \qquad \boldsymbol{T}_{(l'')} = N''\boldsymbol{l}'', \tag{5.30}$$

we have, in view of the formulae (5.22) and (5.23),

$$w'(z) = \sqrt{a}\left[N'\left(\frac{d\bar{z}}{dl'}\right)^2 + N''\left(\frac{d\bar{z}}{dl''}\right)^2\right], \tag{5.31}$$

$$k'N'+k''N''+Z = 0\,, \tag{5.32}$$

where k' and k'' are the normal curvatures of the surface in the directions of the principal axes $\boldsymbol{l}'$ and $\boldsymbol{l}''$. It follows from (5.31) and the formula

$$k = \sqrt{a\mathrm{K}}\left|\frac{dz}{ds}\right|^2, \tag{5.33}$$

that

$$|w'(z)| \leqslant \frac{k'|N'|+k''|N''|}{\sqrt{\mathrm{K}}}. \tag{5.34}$$

Hence, for any bounded force field $\boldsymbol{T}_{(l)}$ the corresponding complex stress function w' is bounded.

We also introduce the scalar complex stress function

$$w'_* = \frac{\mathrm{K}(N'+N'')-\mathrm{ZH}}{\sqrt{\mathrm{KE}}}+i\frac{(N''-N')\tau'}{\sqrt{\mathrm{E}}}\,, \qquad (5.35)$$

which is connected with the complex stress function w' by the relation (see p. 457)

$$w' = e^{-i\varphi}w'_*\,, \qquad \text{i.e.} \qquad w'_* = e^{i\varphi}w'\,. \qquad (5.36)$$

In the formula (5.35) τ' denotes the geodesic torsion of the surface in the direction of the principal axis of the forces $\boldsymbol{l}'$. Using the relations

$$N'+N'' = N_1+N_2\,, \qquad k_1N_1+k_1N_2+Z = 0\,,$$
$$k'N'+k''N''+Z = 0\,, \qquad (5.37)$$

we easily find that at an umbilical point ($E = 0$) the function w'_* is bounded.

If the field $X \equiv 0$ we have, by formulae (5.37), (see also §6.4 and §7.8 of Ch. V), that

$$w'_* = \frac{k_2N_2}{\sqrt{\mathrm{K}}}+iH_1 \equiv \sqrt{\mathrm{K}}p+iq\,, \qquad (5.38)$$

i.e.

$$p = \frac{N_2}{k_1} = -\frac{N_1}{k_2}\,, \qquad q = H_1 \equiv -H_2\,. \qquad (5.39)$$

These relations make it possible to give a mechanical interpretation of the quantities p and q introduced in Ch. V where we also showed their geometrical meaning. In particular, according to the formula (7.65) of Ch. V, we have

$$\tfrac{1}{2}(N_1+N_2) \equiv \tfrac{1}{2}(N'+N'') = \delta\mathrm{H}\,, \qquad (5.40)$$

i.e. *for shells free of surface forces half of the sum of principal forces can be interpreted as the variation of the mean curvature under the corresponding infinitesimal bending of the surface.*

5.3. We now proceed to the problem of determining conditions ensuring the existence and uniqueness of the force field $\boldsymbol{T}_{(l)}$ defined by the relation (5.5) and the formula (5.19) [14n, o]. Sometimes, as was agreed above, such a field will also be called the state $(\boldsymbol{T})$. Let us call the set of fields—the surface forces $\boldsymbol{X}$ distributed on S and the contour forces $\boldsymbol{T}$ prescribed on L—the field $(\boldsymbol{X}, \boldsymbol{T})$, or the external load $(\boldsymbol{X}, \boldsymbol{T})$. This field is defined by five functions: three functions of position on the surface defining the components of the field $\boldsymbol{X}$ and two functions on the contour L defining the components of the field $\boldsymbol{T}$. We proved above the necessity of the conditions (3.21) in which an arbitrary displacement field $\boldsymbol{U}$ appears; this field can be represented as follows (Ch V, §3.5):

$$U = \operatorname{Re}\left\{-\frac{2}{\sqrt{a\mathrm{K}}} w\boldsymbol{n}_z + \frac{1}{\mathrm{K}\sqrt{a}}\left(\sqrt{\mathrm{K}}\, w\right)_z \boldsymbol{n}\right\}, \qquad (5.41)$$

where w is the complex displacement function which is a solution of the homogeneous equation

$$d_{\bar{z}} w + Aw + Bw = 0\,, \qquad w = 2Ur_{\bar{z}}\,, \qquad (5.42)$$

adjoint to the equation (5.5). Introducing the expression (5.41) into the formula (3.21) and transforming the double integral by means of Green's identity we readily obtain the following identity equivalent to (3.21):

$$\operatorname{Re}\left\{\iint\limits_G wF\,dx\,dy - \frac{1}{2i}\int\limits_\Gamma ww'\,dz\right\} = 0\,. \qquad (5.43)$$

Here w' and w are arbitrary solutions of the equations (5.6) and (5.42), respectively; they will hereafter be considered as continuous in $G+\Gamma$. The last identity can also be derived directly from the equations (5.5) and (5.42). We shall below prove also the sufficiency of the identity (5.43) and, consequently, (3.21) for the existence of the force field $\boldsymbol{T}_{(l)}$ corresponding to the given field $(\boldsymbol{X}, \boldsymbol{T})$. For the sake of simplicity we shall confine ourselves to a simply-connected domain.

Suppose that given field $(\boldsymbol{X}, \boldsymbol{T})$ satisfies the condition (3.21) where $\boldsymbol{U}$ is an arbitrary displacement field. Then the function $F(z)$ given in G by (5.7), and the function $w' = f_1 + if_2$ calculated by means of (5.20) on the boundary Γ, satisfy the identity (5.43) for an arbitrary solution w of the equation (5.42) continuous in $\bar{G}$. We shall assume that the given field $(\boldsymbol{X}, \boldsymbol{T})$ is such that the following condition is satisfied:

$$(1) \quad F(z) \in L_p(G+\Gamma)\ , \ p > 2\ , \text{ and}$$
$$(2) \quad f_1 f_2 \in C(\Gamma)\ , \ 0 < \nu \leqslant 1\ .$$

The first condition means that

$$Z \in D_{1,p}(S+L)\ , \qquad X^\alpha = L_p(S+L)\ , \qquad p > 2\ . \quad (5.44)$$

Consequently, $Z \in C_\alpha(S+L)$, $\alpha = \dfrac{p-2}{p}$.

We now construct a solution $\tilde{w}'$ of the equation (5.5) in accordance with the boundary condition

$$\operatorname{Re}[\tilde{w}'(z)] = f_1 \quad (\text{on } \Gamma; \quad \tilde{w}' \in C(G\ +\Gamma))\ . \quad (5.45)$$

Such a solution exists by Theorem 4.11. Since the index of the problem is zero its general solution has (by Theorem 4.12) the form $\tilde{w}' = \tilde{w}_0' + c_1 w_1'$ (c_1 is an arbitrary real constant) where $\tilde{w}_0'$ is the particular solution of the problem and w_1' is a non-trivial solution of the homogeneous problem. Since $w_1' = i\chi_1$ on Γ (χ_1 is a real function) we have

$$\tilde{w}' = f_1 + i\tilde{f}_2 + ic_1\chi_1 \quad (\text{on } \Gamma)\ .$$

Hence the difference of the boundary values of the given function $w' = f_1 + if_2$ and the function $\tilde{w}'$ found above is

$$w' - \tilde{w}' = i(f_2 - \tilde{f}_2 - c_1\chi_1) \quad (\text{on } \Gamma)\ . \quad (5.46)$$

Since w' and $\tilde{w}'$ satisfy the identity (5.43) we have for the difference between them

$$\int_\Gamma (f_2 - \tilde{f}_2 - c_1\chi_1) \operatorname{Re}\left[w(z)\frac{dz}{ds}\right] ds = 0\ , \quad (5.47)$$

where w is an arbitrary solution of the equation (5.42). Let us now subject w to the boundary condition

$$\operatorname{Re}\left[w(z)\frac{dz}{ds}\right] = f_2 - \tilde{f}_2 - c_1\chi_1 \qquad (\text{on } \Gamma).$$

Since the index of this problem is -1 it is necessary and sufficient for its solubility that the constant c_1 be given by the relation (Theorem 4.12)

$$c_1 = \frac{\int\limits_\Gamma (f_2 - \tilde{f}_2)\chi_1 ds}{\int\limits_\Gamma \chi_1^2 ds}.$$

Under these conditions we obtain from (5.47) that, on Γ, $\tilde{f}_2 = f_2 - c_1\chi_1$. Consequently,

$$\tilde{w}' = f_1 + i\tilde{f}_2 + ic_1\chi_1 = f_1 + if_2 = w' \qquad (\text{on } \Gamma).$$

It has therefore been proved that if the field $(\boldsymbol{X}, \boldsymbol{T})$ satisfies the condition (3.21) and the function w' is given on the boundary by the formula (5.20), then it can be continuously continued inside the domain G so that it satisfies the equation (5.5); it is obvious that this continuation is unique. This means that a given field $(\boldsymbol{X}, \boldsymbol{T})$ satisfying the condition (3.21) uniquely determines the force field $\boldsymbol{T}_{(l)}$. This field is given by the formula (5.19). We call the field $(\boldsymbol{X}, \boldsymbol{T})$ satisfying the condition (3.21) *the membrane field of external load.*

5.4. In view of the first formula (5.7) the equations (5.6) and (5.42) can be written in the form

$$\partial_{\bar z} U' - \overline{B(z)}\,\overline{U'} = F', \tag{5.48}$$

$$\partial_z U + B(z)\overline{U} = 0, \tag{5.49}$$

where

$$U' = \sqrt{a\sqrt{K}}\, w'(z) = aK^{1/4}(T^{11} - iT^{12}) + \frac{1}{2} Z\sqrt{\frac{a}{K}}, \tag{5.50}$$

$$U = \frac{w(z)}{\sqrt{a\sqrt{K}}}, \qquad w = 2\,\boldsymbol{U}\boldsymbol{r}_{\bar z}, \tag{5.51}$$

$$F' = \sqrt{a\sqrt{K}}\, F(z), \tag{5.52}$$

Evidently the formula (5.43) can be written in the form

$$\mathrm{Re}\left\{\iint_G UF'\,dx\,dy - \frac{1}{2i}\int_\Gamma UU'\,dz\right\} = 0\,. \qquad (5.53)$$

It is expedient to use equations (5.48) and (5.49) when dealing with an infinite domain. This is due to the fact that in the case of an unbounded domain the function $A(z) = -\partial_{\bar z}\ln\sqrt{a\sqrt{K}}$ does not belong to the class $L_{p,2}(E)$, $p > 2$, and therefore in such cases it is not allowable to apply directly to the equations (5.5) and (5.42) some theorems proved in Chapter III. For instance, for this case the generalized Liouville theorem is not preserved in its usual form. This is seen in the case of a spherical surface when

$$B = 0\,, \qquad A = \frac{2z}{1+z\bar z}\,. \qquad (5.54)$$

It is readily seen that A does not belong to the class $L_{p,2}(E)$, $p > 2$. The equation $\partial_{\bar z}w + Aw = 0$ in this case admits six linearly independent solutions

$$\frac{z^k}{(1+\bar z z)^2}\,, \qquad \frac{iz^k}{(1+\bar z z)^2}\,, \qquad (5.55)$$

which are continuous and bounded on the entire plane and vanish at infinity. Thus, in the case under consideration the Liouville theorem is not valid for the equation $\partial_{\bar z}w + Aw = 0$. To the solutions (5.55) there correspond trivial bendings of the spherical surface.

5.5. An actual verification of the fulfilment of the condition (3.21) in the general case is hampered by the presence in this condition of an arbitrary displacement field U. Nevertheless, this is possible in a number of particular cases. For instance, in the case of a closed convex shell, the relation (3.21) represents the usual equilibrium conditions of a rigid body.

In fact, since, in this case, the contour L is absent, the relation (3 21) takes the form

$$\iint_S UX dS = 0 . \tag{5.56}$$

But here the displacement field is reduced to a trivial field, owing to the rigidity of a closed convex surface. Hence, the relation (5.56) is equivalent to the relations

$$\iint_S X ds = 0 , \qquad \iint_S r \times X ds = 0 , \tag{5.57}$$

expressing the statical equilibrium conditions for a rigid body. Since naturally we assume that the field X does satisfy these conditions, we arrive at the following result:

Under an arbitrary distribution of surface forces X a closed convex shell admits a force field $T_{(I)}$.

The conditions (5.57) are equivalent to the relations

$$\operatorname{Re} \iint_E r_{\bar z} r F \, dx \, dy = 0 , \qquad \operatorname{Re} \iint_E r \times r_{\bar z} F \, dx \, dy = 0 . \tag{5.58}$$

Consequently, in presence of a potential field of surface forces these conditions are always satisfied, for then $F \equiv 0$.

We now propose a method for actual calculating a closed convex shell, according to the membrane theory. This problem is reduced in the general case to an integral equation of Fredholm type the solution of which can be constructed by means of the method of successive approximations. In particular, for convex surfaces of the second order the problem is solved explicitly by integration.

For the justification of this method it is necessary to have an *a priori* estimate of the behaviour near infinity of the required solution and of some other functions.

We found in Ch. 2 (§8.8) that

$$r_1, r_2 \quad \text{or} \quad r_z, \; r_{\bar z} = O(|z|^{-2}) \quad (\text{near } z = \infty) . \tag{5.59}$$

It follows that

$$r_{\bar{z}}^2 = \frac{1}{4}(a_{11} - a_{22} + 2ia_{12}) = \frac{1}{2}\sqrt{\frac{a\mathrm{E}}{\mathrm{K}}}\, e^{i\varphi} = O(|z|^{-4}) , \tag{5.60}$$

$$r_z r_{\bar{z}} = \frac{1}{4}(a_{11} + a_{22}) = \frac{1}{2}\sqrt{\frac{\mathrm{K}}{a}}\,\mathrm{H} = O(|z|^{-4}) .$$

Hence

$$a_{\alpha\beta} \, , \quad \sqrt{a} = O(|z|^{-4}) \qquad (\text{near } z = \infty) . \tag{5.61}$$

In general if P is a continuously differentiable (scalar, vector, tensor) field on an ovaloid S, then

$$P_z \, , \; P_{\bar{z}} = O(|z|^{-2}) \qquad (\text{near } z = \infty) . \tag{5.62}$$

In view of (5.59) and (5.61) we have from (5.34), (5.50) and (5.51) that

$$w'(z) = O(1) , \qquad w(z) = O(|z|^{-2}) , \tag{5.63}$$

$$U'(z) = O(|z|^{-4}) , \qquad U(z) = O(|z|^{2}) . \tag{5.64}$$

It follows from (5.6) in view of (5.62) that

$$B(z) = O(|z|^{-2}) \qquad (\text{near } z = \infty) . \tag{5.65}$$

We have therefore

$$B \in L_{p,2}(E) , \tag{5.66}$$

p being an arbitrary number greater than 2. Hence, we may apply the results of Ch. III and Ch. IV to the equations (5.48) and (5 49).

By virtue of (5.59) and (5.61) we have

$$a^{\alpha\beta} = O(|z|^{4}) , \qquad r^{\alpha} = a^{\alpha\beta} r_{\beta} = O(|z|^{2}) . \tag{5.67}$$

Hence

$$X^{\alpha} \equiv X r^{\alpha} = O(|z|^{2}) \qquad (\text{near } z = \infty) . \tag{5.68}$$

In view of (5.62), (5.61) and (5.68) we find from (5.7) and (5.52) that

$$F(z) = O(|z|^{-2}) , \qquad F'(z) = O(|z|^{-6}) . \tag{5.69}$$

The problem of determining a force field $T_{(l)}$ in terms of a given field X on a closed convex surface is reduced

to the proof of existence of a solution U' of the equation (5.48) on the infinite plane E, which near the point $z = \infty$ satisfies the first condition (5.64). This problem leads to the integral equation

$$U'(z) + \frac{1}{\pi} \iint_E \frac{\overline{B(\zeta)}\,\overline{U'(\zeta)}}{\zeta - z} \, d\xi\, d\eta + \\ + \frac{1}{\pi} \iint_E \frac{F'(\zeta)}{\zeta - z} \, d\xi\, d\eta = \Phi(z) , \qquad (5.70)$$

where $\Phi(z)$ is an entire function in z. Since near $z = \infty$

$$U' = O(|z|^{-4}) , \qquad B = O(|z|^{-2}) \quad \text{and} \quad F' = O(|z|^{-6}) ,$$

the left-hand side of this equation vanishes at infinity (this fact follows at once form Theorem 1.23). It follows therefore that $\Phi(\infty) = 0$. Consequently, $\Phi(z) \equiv 0$ and our problem is reduced to the integral equation

$$U'(z) + \frac{1}{\pi} \iint_E \frac{\overline{B(\zeta)}\,\overline{U'(\zeta)}}{\zeta - z} \, d\xi\, d\eta = -\frac{1}{\pi} \iint_E \frac{F'(\zeta)}{\zeta - z} \, d\xi\, d\eta . \quad (5.71)$$

Since $B = O(|z|^{-2})$ near $z = \infty$ and the right-hand side of the equation (5.71) vanishes at infinity behaving there as $|z|^{-1}$ (this fact follows from Theorem 1.23), we can consider this equation in an arbitrary $L_p(E)$, $p > 1$. Since the homogeneous equation has no non-trivial solutions in any class $L_p(E)$, $p > 1$, the non-homogeneous equation (5.71) has always a solution and, moreover, a unique solution (Ch. III, §5).

In the case of a closed convex surface of the second order $B \equiv 0$ (Ch. V, §4) and, consequently, the required solution is given by the formula

$$U'(z) = -\frac{1}{\pi} \iint_E \frac{F'(\zeta)}{\zeta - z} \, d\xi\, d\eta . \qquad (5.72)$$

It should be observed that in the general case when $B \neq 0$ we may apply to the equation (5.71) the method

of successive approximations, every approximation being reduced to the computation of an integral of the form (5.72). As we indicated in Ch. I (§4.4) this can be done explicitly in many cases by integration.

5.6. The solutions of the equation (5.71) can be represented in the following form (Ch. III, §13,3):

$$U'(z) = \frac{1}{\pi} \iint\limits_E \left(\Omega_1(\zeta, z) F'(\zeta) + \overline{\Omega_2(\zeta, z)}\,\overline{F'(\zeta)}\right) d\xi\, d\eta\,, \qquad (5.72\text{a})$$

where $\Omega_1(z, \zeta)$ and $\Omega_2(z, \zeta)$ are the fundamental kernels of the equation (5.49) (Ch. III, §8.2). In view of (5.7) and (5.52)

$$F'(z) = \frac{1}{2} \sqrt{a}\, \mathrm{K}^{\frac{3}{4}} \partial_z \left(\frac{Z}{\mathrm{K}}\right) - \frac{1}{2} a \mathrm{K}^{\frac{1}{4}} (X^1 - iX^2)\,. \qquad (5.72\text{b})$$

Applying Green's identity and using (5.69) and (8.16) and (8.17) of Ch. III it is easy to prove the relations

$$\begin{aligned} &\iint\limits_E \Omega_1(\zeta, z) \sqrt{a}\, \mathrm{K}^{\frac{3}{4}} \partial_\zeta \left(\frac{Z}{\mathrm{K}}\right) d\xi\, d\eta \\ &\qquad = - \iint\limits_E \frac{Z}{\mathrm{K}} \frac{\partial}{\partial \zeta} [\Omega_1(\zeta, z) \sqrt{a}\, \mathrm{K}^{\frac{3}{4}}]\, d\xi\, d\eta\,, \\ &\iint\limits_E \overline{\Omega_2(\zeta, z)} \sqrt{a}\, \mathrm{K}^{\frac{3}{4}} \partial_{\bar\zeta} \left(\frac{Z}{\mathrm{K}}\right) d\xi\, d\eta \\ &\qquad = - \iint\limits_E \frac{Z}{\mathrm{K}} \frac{\partial}{\partial \bar\zeta} \overline{[\Omega_2(\zeta, z)} \sqrt{a}\, \mathrm{K}^{\frac{3}{4}}]\, d\xi\, d\eta\,. \end{aligned} \qquad (5.72\text{c})$$

The integral on the right-hand side of the first relation should be understood as the Cauchy principal value.

Inserting (5.72b) into the right-hand side of the relation (5.72a) and taking into account (5.72c) we arrive at the following formula:

$$U'(z) = \iint\limits_S \mathfrak{A}(\zeta, z) \mathbf{X}(\zeta)\, dS_\zeta\,, \qquad (5.72\text{d})$$

where $\boldsymbol{X}$ is a given field of surface forces and $\mathfrak{A}(z,\zeta)$ a complex vector-function of the form

$$\mathfrak{A}(\zeta, z) = -\frac{1}{2\pi}\frac{\boldsymbol{n}(\zeta)}{\mathrm{K}(\zeta)\sqrt{a(\zeta)}}\left(\frac{\partial\Omega_1(\zeta, z)\sqrt{a}\,\mathrm{K}^{\frac{3}{4}}}{\partial\zeta}+\frac{\partial\overline{\Omega_2(\zeta, z)}\sqrt{a}\,\mathrm{K}^{\frac{3}{4}}}{\partial\bar{\zeta}}\right)+ + \frac{\mathrm{K}^{-\frac{1}{4}}}{\pi}\left(\Omega_1(\zeta, z)\partial_\zeta \boldsymbol{n}+\overline{\Omega_2(\zeta, z)}\partial_{\bar{\zeta}}\boldsymbol{n}\right). \qquad (5.72\text{e})$$

Since (Ch. III, §8.2)

$$\Omega_1(\zeta, z) = X_1(\zeta, z)+iX_2(\zeta, z)\,,$$
$$\Omega_2(\zeta, z) = X_1(\zeta, z)-iX_2(\zeta, z)\,,$$

where X_1 and X_2 are the fundamental solutions of the equation $\partial_{\bar{\zeta}}w+B(\zeta)\overline{w} = 0$ with a pole at the point z, we have

$$\mathfrak{A}(\zeta, z) = \mathfrak{A}_1(M, N)+i\mathfrak{A}_2(M, N)\,, \qquad (5.72\text{f})$$

where

$$\mathfrak{A}_j(M, N) = \frac{1}{\pi}\operatorname{Re}\left\{2[\mathrm{K}(M)]^{-}X^{\frac{1}{4}}{}_j(\zeta, z)\partial_\zeta\boldsymbol{n}(M) - \right.$$
$$\left. -\frac{\boldsymbol{n}(M)}{\mathrm{K}(M)\sqrt{a(M)}}\partial_\zeta\!\left[X_j(\zeta, z)\sqrt{a(M)}\left(\mathrm{K}(M)\right)^{\frac{3}{4}}\right]\right\}. \qquad (5.72\text{g})$$
$$(j = 1, 2)\,.$$

Thus, the formulae (5.72g) give, on a closed surface S, displacement fields $\mathfrak{A}_1(M, N)$ and $\mathfrak{A}_2(M, N)$ with respect to a variable point M, having singularities of the pole type at a point N (ζ and z are isometric–conjugate complex coordinates of the points M and N, respectively).

In view of (5.50) and (5.19) we may write

$$\boldsymbol{T}_{(l)}(N) = \operatorname{Re}\left[\frac{2}{i\sqrt{a}\sqrt{\mathrm{K}}}\,U'(z)\frac{dz}{ds}\boldsymbol{r}_{\bar{z}}\right]+Z\boldsymbol{R}_{(l)}(N)\,, \qquad (5.72\text{h})$$

where

$$\boldsymbol{R}_{(l)}(N) = -\frac{1}{\sqrt{\mathrm{K}}}\operatorname{Re}\left[\frac{1}{i}\frac{dz}{ds}\boldsymbol{r}_z\right]. \qquad (5.72\text{i})$$

The formula (5.72h) shows that the vector $\boldsymbol{R}_l(M)$ is independent of the choice of the coordinate system and the force field; it depends only on the direction $\boldsymbol{l}$ and the shape of the surface. Simple computations prove that this vector is perpendicular to the Rodrigues vector $\boldsymbol{n}_s$

$$\boldsymbol{R}_{(l)}\frac{d\boldsymbol{n}}{ds}=0\,,$$

i.e. $\boldsymbol{R}_{(l)}$ has the direction $\boldsymbol{s}_*$ conjugate to $\boldsymbol{s}$; its direction and length are given by the relations

$$\boldsymbol{l}\boldsymbol{R}_{(l)}=-\frac{k_s}{2\mathrm{K}}\,,\qquad \boldsymbol{R}^2_{(l)}=\frac{2k_s\mathrm{H}-\mathrm{K}}{4\mathrm{K}}\,.$$

Introducing into the right-hand side of the relation (5.72h) the expression (5.72d) and using (5.72f) we obtain

$$\boldsymbol{T}_{(l(}N=\boldsymbol{T}_{a(l)}(N)\boldsymbol{P}^a_{(l)}(N)+Z\boldsymbol{R}_{(l)}(N)\,, \tag{5.73}$$

where

$$\boldsymbol{T}_{a(l)}(N)=\iint\limits_S \hat{\mathfrak{A}}_a(M,N)\boldsymbol{X}(M)dS_M \qquad (a=1,2) \tag{5.73a}$$

$$\left.\begin{aligned}\boldsymbol{P}^1_{(l)}(N)&=\mathrm{Re}\left[\frac{2}{i}e^{-i\varphi}z'\boldsymbol{r}_{\bar z}\right],\\ \boldsymbol{P}^2_{(l)}(N)&=\mathrm{Re}[2e^{-i\varphi}z'\boldsymbol{r}_{\bar z}]\,,\qquad z'=\frac{dz}{ds}\,,\end{aligned}\right. \tag{5.73b}$$

and

$$\left.\begin{aligned}&\hat{\mathfrak{A}}_1(M,N)\\ &=\frac{1}{\sqrt{a(N)}\sqrt{\mathrm{K}(N)}}\{\mathfrak{A}_1(M,N)\cos\psi(N)-\mathfrak{A}_2(M,N)\sin\psi(N)\}\,,\\ &\hat{\mathfrak{A}}_2(M,N)\\ &=\frac{1}{\sqrt{a(N)}\sqrt{\mathrm{K}(N)}}\{\mathfrak{A}_1(M,N)\sin\psi(N)+\mathfrak{A}_2(M,N)\cos\psi(N)\}.\end{aligned}\right| \tag{5.73c}$$

The vectors $\boldsymbol{P}^1_{(l)}$ and $\boldsymbol{P}^2_{(l)}$ depend only on a point of the surface and the direction $\boldsymbol{l}$ tangential to it. Hence, the vector fields $\hat{\mathfrak{A}}_a(M,N)$ $(a=1,2)$ are also invariant with

respect to the choice of the coordinate system, and for every fixed point N they are displacement fields on a closed convex surface. We shall call these fields *the fundamental fields* (with a pole at the point N) of a closed convex surface. Making use of the formula (Ch. II, §6.6)

$$\boldsymbol{n}_z = -\mathrm{H}\boldsymbol{r}_z + \sqrt{\mathrm{E}}\, e^{-i\varphi}\boldsymbol{r}_{\bar{z}},$$

we can write

$$\boldsymbol{n}_s = \boldsymbol{n}_z z' + \boldsymbol{n}_{\bar{z}}\bar{z}' = -\mathrm{H}(\boldsymbol{r}_z z' + \boldsymbol{r}_{\bar{z}}\bar{z}') + \\ + \sqrt{\mathrm{E}}(e^{-i\varphi}\boldsymbol{r}_{\bar{z}} z' + e^{-i\varphi}\boldsymbol{r}_z \bar{z}') = -\mathrm{H}\boldsymbol{s} + \sqrt{\mathrm{E}}\boldsymbol{P}^2_{(l)},$$

i.e.

$$\boldsymbol{P}^2_{(l)} = \frac{\boldsymbol{n}_s + \mathrm{H}\boldsymbol{s}}{\sqrt{\mathrm{E}}}, \tag{5.73d}$$

where $\boldsymbol{n}_s$ is the Rodrigues vector corresponding to the tangential direction $\boldsymbol{s}$ perpendicular to $\boldsymbol{l}$ ($\boldsymbol{l}\times\boldsymbol{s} = \boldsymbol{n}$). We can now easily derive the following formulae (see p. 461):

$$\boldsymbol{P}^1_{(l)} = -\sqrt{\frac{k_s}{k_{s*}\mathrm{E}}}(\boldsymbol{n}_s + \mathrm{H}\boldsymbol{s}_*), \tag{5.73e}$$

where $\boldsymbol{s}_*$ is the tangential direction conjugate to $\boldsymbol{s}$.

Introducing the complex *dyadic product (dyadic field)*

$$\Omega_{(l)}(M, N) = \hat{\mathfrak{A}}_\alpha(M, N)\boldsymbol{P}^\alpha_{(l)}(N), \tag{5.74}$$

the formula (5.73) can be written in the form

$$\boldsymbol{T}_{(l)}(N) = \iint_S \Omega_{(l)}(M, N)\boldsymbol{X}(M)\,dS_M + Z\boldsymbol{R}_{(l)}(N), \tag{5.74a}$$

In the integrand we have the complex vector

$$\Omega_{(l)}(M, N)\boldsymbol{X}(M) = [\hat{\mathfrak{A}}_\alpha(M, N)\boldsymbol{X}(M)]\boldsymbol{P}^\alpha_{(l)}(N). \tag{5.74b}$$

It should be observed that the dyads $\Omega_{(l)}(M, N)$ are invariant with respect to the choice of the coordinate system on the surface. We shall call the dyads $\Omega_{(l)}(M, N)$ *the dyadic kernel* of the closed convex shell.

The formula (5.74a) expresses by means of the dyadic kernel the force field $\boldsymbol{T}_{(l)}$ of the state ($\boldsymbol{T}$) corresponding

to an arbitrary distribution of the surface forces $\boldsymbol{X}$. Of course, the force field $\boldsymbol{X}$ satisfies the conditions (5.57).

We note that if the force field $\boldsymbol{X}$ has a potential, i.e. if $\boldsymbol{X}$ is given by a formula of the form (5.10) we have

$$\iint_S \Omega_{(l)}(M, N)\boldsymbol{X}(M)\,dS_M = 0\,. \tag{5.74c}$$

Consequently, if the external load is potential the force field of a closed convex shell is given by the formula

$$\boldsymbol{T}_{(l)}(N) = Z\boldsymbol{R}_{(l)}(N)\,. \tag{5.74d}$$

5.7. The formulae (5.72d) and (5.74a) can be generalized to the case of convex shells with openings. We assume that such a shell constitutes a part of a closed regular shell, the dyadic kernel and the fundamental displacement fields of which will be denoted as before by $\Omega_{(l)}(M, N)$ and $\hat{\mathfrak{U}}_a(M, N)$, respectively.

Making use of the formula (3.21) and taking into account the singularities of the vectors $\hat{\mathfrak{U}}_a(M, N)$ at the point N we readily obtain the following formula:

$$\boldsymbol{T}_{(l)}(N) = \iint_S \Omega_{(l)}(M, N)\boldsymbol{X}(M)\,dS_M + Z\boldsymbol{R}_{(l)}(N) + \\ + \int_L \Omega_{(l)}(M, N)\boldsymbol{T}_{(l')}(M)\,dS_M\,, \tag{5.75}$$

where $\boldsymbol{l}$ is a tangential direction at an interior point N and $\boldsymbol{l}'$ is the tangential normal to L at the point of integration M.

It should be observed that if N lies outside $S+L$ the right-hand side of the formula (5.75) vanishes, and if $N \in L$ the right-hand side is equal to $\frac{1}{2}\boldsymbol{T}_{(l)}(N)$; it should be borne in mind that in the latter case $\boldsymbol{l}$ is a given tangential direction at the point N, not the tangential normal to L.

If the force field $\boldsymbol{X}$ has a potential, in view of (5.10) and (5.41) we have

$$\boldsymbol{U}\boldsymbol{X} = \mathrm{Re}\left[u_0 Z + \sqrt{\frac{\mathrm{K}}{a}}\left(\frac{\mathrm{Z}}{\mathrm{K}}\right)_z w\right] = \frac{1}{\sqrt{a}}\,\mathrm{Re}\left(\frac{\mathrm{Z}}{\sqrt{\mathrm{K}}}w\right)_z .$$

Applying Green's identity we obtain

$$\iint_S \boldsymbol{U}\boldsymbol{X}dS = \mathrm{Re}\iint_G \frac{\partial}{\partial z}\left(\frac{\mathrm{Z}}{\sqrt{\mathrm{K}}}w\right)dx\,dy$$

$$= -\mathrm{Re}\left\{\frac{1}{2i}\int_\Gamma \frac{\mathrm{Z}}{\sqrt{\mathrm{K}}}w\,d\bar{z}\right\}$$

$$= -\mathrm{Re}\left\{\frac{1}{i}\int_L \frac{\mathrm{Z}}{\sqrt{\mathrm{K}}}(\boldsymbol{U}\boldsymbol{r}_{\bar{z}})\bar{z}'\,ds\right\} = -\int_L Z\boldsymbol{U}\boldsymbol{R}_{(l)}ds\,.$$

Thus, if the field $\boldsymbol{X}$ is potential the formula (3.21) takes the form

$$\int_L \boldsymbol{U}(\boldsymbol{T}-Z\boldsymbol{R}_{(l)})\,ds = 0\,. \tag{5.76}$$

This formula is equivalent to the relation

$$\mathrm{Re}\left\{\frac{1}{2i}\int_\Gamma ww'\,dz\right\} = 0\,, \tag{5.76a}$$

where w' is of the form (5.20) and w is an arbitrary solution of the equation (5.42).

Thus, *in presence of a potential field of surface forces on a convex shell with openings, the state* ($\boldsymbol{T}$) *exists if and only if the vector* $\boldsymbol{T}-Z\boldsymbol{R}_{(l)}$ *is orthogonal along the boundary of the shell to any displacement field* $\boldsymbol{U}$ *of the middle surface.*

In particular this condition is always satisfied if

$$\boldsymbol{T} = Z\boldsymbol{R}_{(l)} \quad (\text{on } L)\,. \tag{5.76b}$$

Now it is easy to derive from the formula (5.76) the following important relation:

$$\boldsymbol{T}_{(l)}(N) = \int_L \Omega_{(l)}(M, N)[\boldsymbol{T}(M)-Z(M)\boldsymbol{R}_{(l')}(M)]\,ds_M + Z(N)\boldsymbol{R}_{(l)}(N)\,, \tag{5.77}$$

representing the force field $\boldsymbol{T}_{(l)}$ of the state ($\boldsymbol{T}$) in presence of the following potential load on the convex surface:

$$\boldsymbol{X} = \frac{1}{2}\mathrm{K}d^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\left(\frac{\mathrm{Z}}{\mathrm{K}}\right)\boldsymbol{r}_\beta + Z\boldsymbol{n}\,. \tag{5.78}$$

It is of course assumed that the relation (5.76) is satisfied. In the formula (5.77) $\boldsymbol{l}'$ in the integrand denotes the tangential normal to L at the point M.

In the case of a non-closed convex shell or according to our terminology, a shell with edges or openings, a verification of the conditions (3.21) and (5.76) is hampered by the fact that the field $\boldsymbol{U}$ in such a case is not reduced to a trivial displacement field. Nevertheless, in a number of particular cases it is possible to verify whether these conditions are satisfied or not.

For instance let us consider the case of presence of a potential field of external loads and assume that the contour forces are absent, i.e. $\boldsymbol{T} \equiv 0$ (on L). Such being the case the relation (5.76) takes the form

$$\int_L Z\boldsymbol{U}\boldsymbol{R}_{(l)}\,ds = 0\ .$$

This relation in view of the formulae (5.51) and (5.72i) takes the form

$$\int_L \frac{Z}{\sqrt{K}}\operatorname{Re}\left[\frac{1}{i}\,w(z)\,\bar{z}'\right]ds = 0 \qquad \left(z' = \frac{dz}{ds}\right). \tag{5.79}$$

Let us consider the boundary value problem

$$\begin{aligned} \partial_{\bar z} w + Aw + B\overline{w} &= 0 \qquad (\text{in } G)\,, \\ R_l[iw\bar{z}'] &= f \qquad (\text{on } \Gamma)\,, \end{aligned} \tag{5.80}$$

If this boundary value problem has for an arbitrary right-hand side a solution, it is obvious that the condition (5.79) is not satisfied. But the index of the problem is $n = 1 - m$. Consequently, in the case of a simply-connected domain ($m = 0$) $n = 1$ and the problem, according to Theorem 4.11, always has a solution.

Thus, *in the presence of a potential load on a convex shell with one opening the state* ($\boldsymbol{T}$) *cannot occur if* $Z \neq 0$ *and* $\boldsymbol{T} \equiv 0$ *on* L.

It should be observed that, under the above conditions, with few exceptions, the state ($\boldsymbol{T}$) cannot occur on a convex shell with many openings.

5.8. The necessary and sufficient condition of existence of the state $(\boldsymbol{T})$ can also be stated in a different way. To this end let us consider the formula (5.20). We observe that a field $(\boldsymbol{X}, \boldsymbol{T})$ compatible with the state $(\boldsymbol{T})$ satisfies the following condition: *the right-hand side of the relation* (5.20) *containing contour forces* $\boldsymbol{T}$ *and the normal component* Z *of the surface forces, is the limiting value of a solution* w' *of the equation* (5.5) *continuous in* $G+\Gamma$. It can readily be proved that the fulfilment of this condition is not only necessary but also sufficient for the existence of the state $(\boldsymbol{T})$.

If we now superpose upon the field $\boldsymbol{X}$ a potential field $\mathring{\boldsymbol{X}}$, the potential being $V_0 = \frac{Z_0}{\mathrm{K}}$, the right-hand side of the equation (5.5) remains unaltered and the formula (5.20) takes the form

$$w' = \frac{2}{iz'\sqrt{a\mathrm{K}}}\boldsymbol{T}_{(l)}\boldsymbol{n}_z - \frac{Z+Z_0}{2\sqrt{\mathrm{K}}}\frac{\bar{z}'}{z'}. \tag{5.81}$$

Thus, for the new field $(\boldsymbol{X}+\mathring{\boldsymbol{X}}, \boldsymbol{T})$, the equation for the function w' remains unaltered, the only change taking place in the boundary condition. The new boundary condition contains in place of Z the sum $Z+Z_0$. We emphasize once more that the function Z_0 does not enter the right-hand side of the differential equation.

In this connection naturally the following problem, called hereafter Problem **E**, arises

Problem **E**. *Let there be given a field* $(\boldsymbol{X}, \boldsymbol{T})$. *It is required to find a potential field* $\mathring{\boldsymbol{X}}$ *such that for the field* $(\boldsymbol{X}+\mathring{\boldsymbol{X}}, \boldsymbol{T})$ *the state* $(\boldsymbol{T})$ *occurs.*

It is evident that *this problem is reduced to the determination of a function* Z_0 *of position on the surface in accordance with the condition: the right-hand side of the* relation (5.81) *should represent on the boundary* Γ *of the domain* G *the limiting boundary value of solution* w' *of the equation* (5.5), *continuous in* $G+\Gamma$. The right-hand

side of this equation has the form (5.7) and, consequently, it depends only on the field $\boldsymbol{X}$ and is independent of the unknown potential field $\mathring{\boldsymbol{X}}$.

It is readily observed that this problem is not fully determined from the mathematical point of view. The point is that if the problem has at least one solution $\mathring{\boldsymbol{X}}$ then it has infinitely many solutions. In fact, besides the field $(\boldsymbol{X}+\mathring{\boldsymbol{X}}, \boldsymbol{T})$ every field of the form $(\boldsymbol{X}+\mathring{\boldsymbol{X}}+\hat{\boldsymbol{X}}, \boldsymbol{T})$ satisfies the conditions of the problem if $\hat{\boldsymbol{X}}$ is a potential field, the potential of which vanishes everywhere on the boundary of the domain.

We now proceed to the investigation of Problem **E** putting aside for the time being the problem of conditions ensuring the uniqueness of its solution; the latter problem will be examined later.

It follows from the formula (5.81) that the function w' corresponding to the field $(\boldsymbol{X}+\mathring{\boldsymbol{X}}, \boldsymbol{T})$ is a solution of the boundary value problem

$$\begin{gathered} \partial_{\bar{z}} w' - A w' - \bar{B}\bar{w}' = F \quad (\text{in } G)\,, \\ \operatorname{Re}[i w' z'^2] = f \quad (\text{on } \Gamma)\,, \end{gathered} \tag{5.82}$$

where

$$f = \frac{2}{\sqrt{a\mathrm{K}}} \operatorname{Re}\left[\boldsymbol{T}\boldsymbol{n}_z \frac{dz}{ds}\right] = \frac{2}{\sqrt{a\mathrm{K}}} \boldsymbol{T}\frac{d\boldsymbol{n}}{ds} \quad (\text{on } \Gamma)\,. \tag{5.83}$$

Consequently, the function f is uniquely determined by the given field $\boldsymbol{T}$ of the contour forces.

Thus, the right-hand sides F and f of the problem obtained above and constituting a particular case of the generalized Riemann–Hilbert boundary value problem, depend only on the given field $(\boldsymbol{X}, \boldsymbol{T})$. If this problem, which will hereafter be called Problem $\mathbf{E}_*$, has no solution, the original Problem **E**, obviously, also has no solution. If on the other hand Problem $\mathbf{E}_*$ has solutions (even one) then Problem **E** has an infinite number of solutions.

In fact, equating on the boundary Γ of the domain G the right-hand side of the relation (5.81) to the limiting

values of the solution w' of Problem $\mathbf{E}_*$ we obtain the limiting values of the function Z_0

$$Z_0 = \frac{4}{i\bar{z}'\sqrt{a}} \boldsymbol{T}\boldsymbol{n}_z - \frac{2\sqrt{\mathrm{K}}z'}{\bar{z}'} w' - Z \qquad (\text{on } L)\,. \tag{5.84}$$

Continuing this function from the contour L inside the surface in an arbitrary way (it is sufficient to require only that the corresponding potential field be bounded) we obtain a potential field $\overset{\circ}{\boldsymbol{X}}$ which has the property that the field $(\boldsymbol{X}+\overset{\circ}{\boldsymbol{X}}, \boldsymbol{T})$ admits a force field which, according to (5.19), is given by the formula

$$\boldsymbol{T}_{(l)} = 2\,\mathrm{Re}\left[\frac{1}{i} w'(z)\frac{dz}{ds}\boldsymbol{r}_{\bar{z}}\right] + (Z+Z_0)\boldsymbol{R}_{(l)}\,. \tag{5.85}$$

Apparently, this field leads to the sum

$$\boldsymbol{T}_{(l)} = \boldsymbol{T}'_{(l)} + \boldsymbol{T}^0_{(l)}\,, \tag{5.86}$$

where

$$\boldsymbol{T}'_{(l)} = 2\,\mathrm{Re}\left[\frac{1}{i} w'(z)\frac{dz}{ds}\boldsymbol{r}_{\bar{z}}\right] + Z\boldsymbol{R}_{(l)}\,, \tag{5.87}$$

$$\boldsymbol{T}^0_{(l)} = Z_0\boldsymbol{R}_{(l)}\,. \tag{5.88}$$

Thus, it is seen from (5.87) that the first force field $\boldsymbol{T}'_{(l)}$ is determined only by the given field $(\boldsymbol{X}, \boldsymbol{T})$, and the second force field $\boldsymbol{T}^0_{(l)}$ takes the given values only on the boundary of the shell. Inside the shell for the field $\boldsymbol{T}^0_{(l)}$ we can take any field of the form (5.88) where Z_0 is an arbitrary function of the class $D_{1,p}$, $p > 2$, (consequently, this function belongs to the class $C_{\frac{p-2}{p}}$) taking on the boundary of the shell the prescribed values (5.84). In order that there exists at least one such continuation of the function Z_0 it suffices to assume that the following conditions are satisfied: (1) the boundary L of the shell consists of piecewise smooth contours of the class $C^1_{\mu,\nu_1,\dots,\nu_k}$; (2) the components of the force vector belong to the class $C_\nu(\Gamma)$ and (3) $Z \in D_{1,p}(S+L)$, $p > 2$ (see, for instance, [44a], [61a]).

Thus, the potential Z_0 exists under very weak assumptions on the data of the problem.

We now proceed to the investigation of the solubility problem for Problem $\mathbf{E}_*$ which is a particular case of the general boundary value Problem **A** investigated in Ch. IV. The index of the problem is

$$n = 2(m-1)\,. \tag{5.89}$$

First we consider the case of a simply-connected domain (a shell with one opening). Then $m = 0$ and $n = -2$. Consequently, in view of Theorem 4.5 the homogeneous problem

$$\begin{aligned} &\partial_{\bar{z}} w' - Aw' - \bar{B}\overline{w}' = 0 \qquad (\text{in } G)\,, \\ &\qquad\qquad\qquad \operatorname{Re}[iw'z'^2] = 0 \quad (\text{on } \Gamma) \end{aligned} \tag{5.90}$$

has no solution and the non-homogeneous problem is soluble only if the following three conditions are satisfied:

$$\int\limits_L \boldsymbol{U}^{(j)}\boldsymbol{T}_{(l)}\,ds + \iint\limits_S \boldsymbol{U}^{(j)}\boldsymbol{X}\,dS = 0 \qquad (j = 1, 2, 3)\,, \tag{5.91}$$

where $\boldsymbol{U}^{(j)}$ are displacement fields corresponding to the linearly independent solutions of the adjoint homogeneous problem

$$\begin{aligned} &\partial_{\bar{z}} w + Aw + B\overline{w} = 0 \qquad (\text{in } G)\,, \\ &\qquad\qquad\qquad \operatorname{Re}[iw\bar{z}'] = 0 \quad (\text{on } \Gamma)\,. \end{aligned} \tag{5.92}$$

We have now to consider two possibilities, namely: (1) all $\boldsymbol{U}^{(j)}$ are trivial displacement fields and (2) at least one of them is non-trivial. In the first case the relations (5.91) are reduced to the statical equilibrium conditions of a rigid body and, consequently, they represent a natural requirement which should always be satisfied by the field $(\boldsymbol{X}, \boldsymbol{T})$. In this case Problem $\mathbf{E}_*$ always has a solution and, consequently, Problem **E** also always has a solution. In the second case Problem $\mathbf{E}_*$ not always has a solution and, consequently, Problem **E** can also have no solution for a given field $(\boldsymbol{X}, \boldsymbol{T})$.

In the case of a doubly-connected domain (a convex shell with two openings) $m = 1$ and $n = 0$. In view of Theorem 4.6 in this case the homogeneous boundary value Problem $\mathbf{E}^*$ either has no non-zero solution ($l = 0$) or has one linearly independent solution ($l = 1$). In the first case a solution of the non-homogeneous Problem $\mathbf{E}_*$ always exists and is determined uniquely. Consequently, Problem **E** also has a solution for an arbitrary field $(\boldsymbol{X}, \boldsymbol{T})$. In the second case ($l = 1$) the non-homogeneous Problem $\mathbf{E}_*$ has a solution only if the following conditions are satisfied:

$$\iint_S \boldsymbol{U}\boldsymbol{X}\,ds + \int_L \boldsymbol{U}\boldsymbol{T}_{(l)}\,ds = 0\,, \tag{5.93}$$

where $\boldsymbol{U}$ is the displacement field corresponding to the solution of the adjoint homogeneous problem (5.92) which in the case under consideration has one non-zero solution. It is possible that the field $\boldsymbol{U}$ is trivial. Then the condition (5.93) is, obviously, satisfied as one of the equilibrium conditions of a rigid body. If the field $\boldsymbol{U}$ is non-trivial the given field $(\boldsymbol{X}, \boldsymbol{T})$ should be subject to the additional condition (5.93). This means that in this case Problem **E** has no solution for an arbitrary field $(\boldsymbol{X}, \boldsymbol{T})$.

We now proceed to the case of an arbitrary multiply-connected domain ($m > 1$). In this case the index of Problem $\mathbf{E}_*$ is $n = 2(m-1)$ and is greater than $m-1$; hence in view of Theorem 4.10 the non-homogeneous problem is always soluble. Consequently, Problem **E** has a solution for an arbitrary field $(\boldsymbol{X}, \boldsymbol{T})$.

Thus, we have obtained the following results:

For a convex shell with more than two openings Problem **E** *has a solution for an arbitrarily given field* $(\boldsymbol{X}, \boldsymbol{T})$. *For a convex shell with one or two openings, Problem* **E** *in general has no solution for an arbitrary field* $(\boldsymbol{X}, \boldsymbol{T})$. *Nevertheless, for some shapes of shells and openings and some configurations of the latter, Problem* **E** *can possess a solution for an arbitrary field* $(\boldsymbol{X}, \boldsymbol{T})$.

In general, cases of insolubility of Problem **E** are encountered much more seldom than the cases of solubility.

The method of modifying the load $(\boldsymbol{X}, \boldsymbol{T})$ by adding a potential field $\mathring{\boldsymbol{X}}$ which has been proposed in this subsection may be called *the method of potential loading*, and $\boldsymbol{X}$ *the adjusting potential load.*

5.9. We shall now make a few remarks on the conditions of uniqueness of the solution of Problem **E**. It is evident that it is possible to indicate many such conditions. It should be borne in mind however that it is not expedient to impose strong restrictions on the choice of the adjusting potential field $\mathring{\boldsymbol{X}}$. This problem should be solved from the point of view of practical expediency. In solving a definite problem the field $\mathring{\boldsymbol{X}}$ should be constructed by most simple and well-known devices; it is very important that possibilities of an actual loading of the shell by the additional forces $\mathring{\boldsymbol{X}}$ should be taken into account. At the same time fulfilment of the stabilility conditions of the shell should be ensured. It is necessary that the total load always be considerably smaller than the critical load resulting in the loss of stability of the shell.

We now indicate however two general methods of constructing the field $\mathring{\boldsymbol{X}}$ One of them consists in the determination in accordance with a given boundary value of the potential V_0 of a potential field $\mathring{\boldsymbol{X}}$ for which the integral

$$J = \iint\limits_S p\mathring{\boldsymbol{X}}^2 dS\,, \tag{5.94}$$

is minimum. Here p is a positive weight function.

In view of the formula (5.10) we have

$$J = \iint\limits_S p\mathring{\boldsymbol{X}}^2 dS = \iint\limits_S (\mathrm{K}p^2V^2 + \tfrac{1}{4}\,p\mathrm{K}^2 h^{\alpha\beta}\nabla_\alpha V \nabla_\beta V)\,dS\,, \tag{5.95}$$

where

$$h^{\alpha\beta} = \bar{d}^{\alpha\lambda} \bar{d}^{\beta}_{\lambda}\,. \tag{5.96}$$

In the coordinate system consisting of the curvature lines

$$h^{11} = \frac{1}{A^2 k_1}, \quad h^{22} = \frac{1}{B^2 k_2}, \quad h^{12} = h^{21} = 0 . \qquad (5.97)$$

The Euler differential equation corresponding to the stated problem is

$$E(V) \equiv \nabla_\alpha(\varrho h^{\alpha\beta} \nabla_\beta V) - 4\varrho V = 0 , \qquad \varrho = \frac{pK^2}{4} . \quad (5.98)$$

This is an equation of elliptic type (in the case of a surface of positive curvature). Hence, we infer that in view of the inequality $p > 0$ the Dirichlet problem for this equation always has a unique solution. The limiting values of the unknown potential $V = \frac{Z_0}{K}$ which are to be calculated in accordance with the formula (5.84), have the following form in the case of a multiply-connected domain (for instance for $m > 1$):

$$V = V_0 + c_1 V_1 + \dots + c_{3m-3} V_{3m-3} , \qquad (5.99)$$

where $V_0, V_1, \dots, V_{3m-3}$ are definite lineraly independent functions depending only on the given field $(\boldsymbol{X}, \boldsymbol{T})$, and $c_1, \dots, c_{3m-3}$ are arbitrary real constants. The formula (5.99) is obtained if it is taken into account that the homogeneous boundary value Problem $\mathring{\mathbf{E}}_*$ has $3m-3$ linearly independent solutions. But the arbitrary constants c_k obviously are uniquely determined in the variational problem under consideration. Thus, the uniqueness of the solution of Problem **E** is always ensured.

The above method enabling us to determine uniquely the potential field $\mathring{\boldsymbol{X}}$ has the merit that this procedure ensures in a sense the minimum of the additional potential load resulting in the modified membrane field $(\boldsymbol{X} + \mathring{\boldsymbol{X}}, \boldsymbol{T})$. Knowing this field we may draw conclusions concerning the degree and nature of deviation of the given field of external loads $(\boldsymbol{X}, \boldsymbol{T})$ from the totality of membrane loads. For a characteristic of this deviation we may take

a number equal to the minimum of the integral (5.94) (for instance with the weight $p = 1$); it is evident that this number is uniquely determined by the given field $(\boldsymbol{X}, \boldsymbol{T})$. If this number turns out to be sufficiently small in comparison with the value of the integral for the given load $\boldsymbol{X}$, the last fact may be considered as a criterion of practical applicability of the membrane theory to the computation of the shell under the prescribed load $(\boldsymbol{X}, \boldsymbol{T})$.

Unfortunately, the practical efficiency of the above method is limited by the fact that, first, it is necessary to determine the limiting values of the unknown potential V_0 by solving the boundary value problem (5.82) and, second, it is necessary to solve the Dirichlet boundary value problem for a fairly complicated equation (5.98). Nevertheless, it should be indicated that in many practically important cases the above problems-especially the last one, can be solved easily by an application to the corresponding variational problem of, for instance, direct methods. If the middle surface of the shell is a convex surface of the second degree the problem (5.82) is reduced to the Riemann–Hilbert problem for analytic functions, which in many cases can explicitly be solved, [60a].

In practical problems the continuation of the function Z_0 can be carried out in the following way. Let us fix a narrow boundary strip S' of the shell and let us continuously continue the function Z_0 from the boundary inside the surface so that $Z_0 \equiv 0$ outside the boundary strip S'. Such a continuation is always possible if the boundary and the boundary values of Z_0 are sufficiently smooth; it is also possible to ensure inside the shell an arbitrarily large degree of smoothness of the function Z_0, [44a], [54]. Such being the case, the whole additional load will be applied in the vicinity of the boundaries of the shell resulting in general in a considerable distortion of the stress distribution near the boundary; far from the

boundaries however the deviation usually does not exceed the admissible error. It should be noted that the above assertions are confirmed by results of engineering practice. Computations of most engineering structures are performed in accordance with the membrane theory and, if necessary, the results are amended in accordance with the so-called *edge effect* [50]. In our case to this effect the force field $\boldsymbol{T}^0_{(l)} = Z_0 \boldsymbol{R}_{(l)}$ corresponds, acting on a narrow boundary strip (in connection with this problem see also [74]).

5.10. The essence of *the method of adjusting loads* consists in the fact that it permits to introduce into a given distribution of external loads an alteration which leads to a new distribution compatible with the state ($\boldsymbol{T}$). In §5.7 we found that this can be achieved almost always by introducing an additional potential load. It is evident, however, that this is not the only method of solution of the above problem. In this and in the following sections we shall indicate other methods, making it possible to attain the same goal in somewhat different ways.

The formula (2.20) indicates that the field $\boldsymbol{X}$ depends not only on statical quantities e.g. body and surface forces, but also on quantities characterizing geometrical mechanical and physical properties of the shell e.g. the mean and principal curvatures H and K, of the middle surface, thickness and density of the material which, obviously, enter the integral term of the right-hand side of the relations (2.20). Suitably modifying these characteristics, in many cases we can achieve that the external load $(\boldsymbol{X}, \boldsymbol{T})$ is membrane. To this end a modification of the shape of the upper and lower surfaces, and sometimes also of the middle surface, is required as well as introducing a shell of variable density. These circumstances, however, cannot create a serious obstacle for an application of the above indicated method, since our theory with no alternations can also be applied to shells of variable density and

variable thickness. This is actually a merit of the mem brane theory, when compared with the general bending theory of shells, that it is based on exact equilibrium conditions of the continuous medium and no hypotheses are employed in deriving these equations, except the fundamental laws of statics of a rigid body and the continuity hypothesis of the material of the shell. Hence, our theory may be applied to an arbitrary continuous shell, no matter whether it is composed of a homogeneous or non-homogeneous material.

It is also easy to modify the equations of the membrane theory in such a way that they apply also to shells of variable thickness. Let $z = h^+(x, y)$ and $z = h^-(x, y)$ be the equations of the upper and lower surfaces of the shell. Assume that the shape of these surfaces differs insignificantly from that of the middle surface. In other words we assume that the first derivatives of the functions h^+ and h^- with respect to the variables x and y are so small that their squares and higher powers may be neglected. Under these conditions, repeating the method of derivation of the equation (2.19) of §2.3 for a shell bounded by the surfaces $z = h^+(x, y)$ and $z = h^-(x, y)$ we arrive at an equation differing from the previous one only in that the free term has the form

$$\boldsymbol{X} = \sqrt{\frac{g^+}{a}}\boldsymbol{P}^+ - \sqrt{\frac{g^-}{a}}\boldsymbol{P}^- + \int\limits_{h^-}^{h^+} \sqrt{\frac{g}{a}}\boldsymbol{F}dz\,, \quad (5.100)$$

where $\boldsymbol{P}^+$ and $\boldsymbol{P}^-$ are forces applied to the upper and lower surfaces of the shell, respectively, $\boldsymbol{F}$ is the body force, $\sqrt{g^+/a}$ and $\sqrt{g^-/a}$ are the values of $1 - 2\mathrm{H}z + \mathrm{K}z^2$ for $z = h^+$ and $z = h^-$, respectively. If the shell is sufficiently thin we may set

$$\boldsymbol{X} = \boldsymbol{P}^+ - \boldsymbol{P}^- + \int\limits_{h^-}^{h^+} \boldsymbol{F}dz\,. \quad (5.100a)$$

Finally, if the body force is the force of gravity we have

$$\boldsymbol{X} = \boldsymbol{P}^+ - \boldsymbol{P}^- + \gamma\,, \qquad \gamma = (h^+ + h^-)\,\varrho\mathrm{e}\,, \quad (5.100b)$$

where γ is the weight of the shell per unit area, ϱ the density of the material of the shell and g the acceleration due to gravity.

The quantities γ, ϱ and $h^+ + h^-$ are sometimes not prescribed; it is expedient in some cases to regard them as unknowns and to determine them by subjecting the external load $(\boldsymbol{X}, \boldsymbol{T})$ to the condition of the membrane state. This sometimes leads not only to a membrane state, but, for a special law of distribution of the surface forces $\boldsymbol{P}^+$ and $\boldsymbol{P}^-$, it is possible to ensure existence of a definite particular form of distribution of the force field inside the shell, [84].

Consider now the following problem:

Problem **F**. *Let $(\boldsymbol{X}, \boldsymbol{T})$ be a given field of external loads. Let $\boldsymbol{e}$ be a field of directions on S, where $|\boldsymbol{e}| = 1$. This field is, in general, not constant. It is required to find a scalar field $\overset{*}{Z}$ on S, such that the field of external loads $(\overset{*}{\boldsymbol{X}}, \boldsymbol{T})$ where $\overset{*}{\boldsymbol{X}} = \boldsymbol{X} + \overset{*}{Z}\boldsymbol{e}$ statisfies the conditions of the membrane state*

$$\iint\limits_S \overset{*}{\boldsymbol{X}}\boldsymbol{U}dS + \int\limits_L \boldsymbol{T}_{(l)}\boldsymbol{U}ds = 0 , \tag{5.101}$$

where U is an arbitrary displacement field on S.

To solve this problem let us construct the complete system (with respect to S) of displacement fields $\boldsymbol{U}_i^*$ ($i = 0, 1, \ldots$) which are subject to the following conditions:

$$\iint\limits_S U_{ie}^* U_{je}^* dS = \delta_{ij} \qquad (U_{ie}^* = \boldsymbol{e}\boldsymbol{U}_i^*) . \tag{5.101a}$$

Such a system can be constructed in the following way: According to the formulae (5.41) and (5.51) the displacement field can be expressed by the formula

$$\boldsymbol{U} = \mathrm{Re}\left\{-2\mathrm{K}^{-\frac{1}{4}}U\boldsymbol{n}_z + \frac{\boldsymbol{n}}{\mathrm{K}\sqrt{a}}\partial_z\left(\sqrt{a}\,\mathrm{K}^{\frac{3}{4}}U\right)\right\}, \tag{5.102}$$

where U is an arbitrary solution (continuous in $G+\Gamma$) of the equation

$$\partial_{\bar{z}} U + B\overline{U} = 0\,. \tag{5.102a}$$

Let us now construct a system of solutions U_i $(i = 1, 2, \ldots)$ complete with respect to the domain G of the equation (5.102a). A method of construction of such a system was indicated in Ch. III (§15). Substituting U_i into the formula (5.102) we obtain a definite complete infinite system of displacement fields $\boldsymbol{U}_i$ $(i = 1, 2, \ldots)$. It is evident that now the condition (5.101) is equivalent to the infinite system of relations

$$\iint_S \overset{*}{\boldsymbol{X}} \boldsymbol{U}_i dS + \int_L \boldsymbol{T}\boldsymbol{U}_i ds = 0 \qquad (i = 1, 2, \ldots)\,. \tag{5.102b}$$

Substituting $\overset{*}{\boldsymbol{X}} = \boldsymbol{X} + \overset{*}{Z}\boldsymbol{e}$ we obtain

$$\iint_S \overset{*}{Z} U_{ie} = b_i \qquad (i = 1, 2, \ldots;\ U_{ie} = \boldsymbol{e}\boldsymbol{U}_i)\,, \tag{5,102c}$$

where

$$b_i = -\iint_S \boldsymbol{X}\boldsymbol{U}_i dS - \int_L \boldsymbol{T}\boldsymbol{U}_i ds \qquad (i = 1, 2, \ldots)\,. \tag{5.102d}$$

Evidently, the numbers b_i are known, since by hypothesis the load $(\boldsymbol{X}, \boldsymbol{T})$ is prescribed. Thus, the determination of the scalar field $\overset{*}{Z}$ has been reduced to a special problem of the theory of moments (5.102c). In order to find the solution of this problem it is expedient to subject the choice of the complete system of fields $\boldsymbol{U}_i$ to some special conditions.

It is possible that there exist linearly independent displacement fields $\boldsymbol{U}_0^*, \ldots, \boldsymbol{U}_k^*$ which are orthogonal to the field $\boldsymbol{e}$, i.e.

$$\boldsymbol{e}\boldsymbol{U}_j^* = 0 \qquad (j = 1, 2, \ldots, k)\,. \tag{5.102e}$$

For example, if $\boldsymbol{e}$ is a constant field the relations (5.102e) are satisfied by three linearly independent trivial bending fields. It can be proved that, in the general case, the

number k of solutions of the problem (5.102e) does not exceed 3. It may, in particular, happen that $k = 0$. Let us now choose the complete system of fields $\boldsymbol{U}_i^*$ so that the first k fields $\boldsymbol{U}_1^*, \ldots, \boldsymbol{U}_k^*$ are linearly independent solutions of the problem (5.102e), if such exist. The remaining fields $\boldsymbol{U}_{k+1}^*, \boldsymbol{U}_{k+2}^*, \ldots$ can be subject to the conditions (5.101a). This can always be achieved if we apply to the functions $\boldsymbol{e}\boldsymbol{U}_j^* = \varphi_j$ $(j = k+1, k+2, \ldots)$ the Schmidt orthogonalization procedure (these functions are linearly independent; otherwise the sequence of fields $\boldsymbol{U}_j^*$ $(j = 1, 2, \ldots)$ would contain linearly dependent elements). Under these conditions the system of relations (5.102c) is obviously equivalent to the following system:

$$\iint\limits_S \boldsymbol{X}\boldsymbol{U}_j^* dS + \int\limits_L \boldsymbol{T}\boldsymbol{U}_j^* ds = 0 \qquad (j = 1, \ldots, k)\,, \tag{5.102f}$$

$$\iint\limits_S \overset{*}{Z}\varphi_j dS = b_j^* \qquad (j = k+1, k+2, \ldots)\,, \tag{5.102g}$$

where

$$b_j^* = -\iint\limits_S \boldsymbol{X}\boldsymbol{U}_j^* ds - \int\limits_L \boldsymbol{T}_{(l)}\boldsymbol{U}_j^* ds\,. \tag{5.102h}$$

It is evident that a formal solution of the system of relations (5.102g) is given by the series

$$\overset{*}{Z} = \sum_{j=k+1}^{\infty} b_j^* \varphi_j + \hat{Z}\,, \tag{5.102i}$$

where $\hat{Z}$ is an arbitrary function of position on the surface, which is orthogonal to the system of functions φ_j: $(\hat{Z}, \varphi_j) = 0$ $(j = 1, 2, \ldots)$. We still have to prove that this series is convergent (it is sufficient to establish its convergence in a $D_{1,p}$, $p > 2$). We shall assume that this is true and we shall not investigate the conditions of convergence.

Thus we have obtained the following result:

Problem **F** *is soluble if and only if the given load* $(\boldsymbol{X}, \boldsymbol{T})$ *satisfies the conditions* (5.102f). *Under these conditions the problem has solutions given by the formula* (5.102i).

In particular, *if the field of directions* $\boldsymbol{e}$ *is such that the problem* (5.102e) *has no solution, i.e. there exists no displacement field orthogonal to the field of directions* $\boldsymbol{e}$, *in this case Problem* **F** *is always soluble (it would be interesting to indicate examples of occurrence of this case).*

We now apply the above results to the case when the field $\boldsymbol{e}$ is the field of vertical directions and the unknown scalar field $\overset{*}{Z}$ is equal to the weight of the shell $\gamma = (h^+ + h^-)\varrho g$. In this case $k = 3$ and the relation (5.102f) constitutes a part of the equilibrium conditions for an absolutely rigid body.

Thus we have obtained the following result:

If on the surface S a field of load $(\boldsymbol{X}, \boldsymbol{T})$ is given satisfying the conditions (5.102f) where $k = 3$, and $\boldsymbol{U}_1^*, \boldsymbol{U}_2^*, \boldsymbol{U}_3^*$ are linearly independent trivial displacement fields orthogonal to the field of vertical directions, then the shell is a membrane if its weight is given by the formula

$$\gamma \equiv (h^+ + h^-)\varrho g = \sum_{j=4}^{\infty} b_j^* \varphi_j + \hat{Z} \qquad (5.102\text{j})$$

$$(\varphi_j = \boldsymbol{e}\boldsymbol{U}_j)\,,$$

where the function $\hat{Z}$ is orthogonal to the system φ_j and should so be chosen that $\gamma \geqslant 0$. If it is impossible to satisfy the condition $\gamma \geqslant 0$, this means that it is impossible to achieve the membrane state of the shell by changing its weight.

We note that an analogous problem of ensuring the membrane state of a shell by an appropriate choice of its weight was considered before by other authors for various particular cases [84].

5.11. In the preceding sections we indicated ways of solving our problem by introducing a definite modification of the field of surface forces $\boldsymbol{X}$, regarding the contour forces $\boldsymbol{T}$ as completely prescribed in accordance with an arbitrary law. In this section we consider cases when

the problem can be solved by means of a partial assignment of the distribution of the contour forces $\boldsymbol{T}$ on the boundary. Assuming that the field $\boldsymbol{X}$ is given in accordance with an arbitrary law, we shall prescribe at every point of the contour only one component of the force $\boldsymbol{T}$; the second will then be determined by the condition that the field $(\boldsymbol{X}, \boldsymbol{T})$ constitutes a state $(\boldsymbol{T})$.

Let $\boldsymbol{t}$ be a tangential direction at a boundary point of the surface ($|t| = 1$) forming with the principal normal $\boldsymbol{l}$ an angle φ. Denoting the projection of the force $\boldsymbol{T}$ on $\boldsymbol{t}$ by $T_{(lt)}$ consider the boundary condition

$$T_{(lt)} \equiv N_l \cos\varphi + H_l \sin\varphi = f \quad (\text{on } L), \qquad (5.103)$$

where f is a given function of position on the contour L; in what follows we assume that (1) $\varphi \in C_\nu(L)$, (2) $f \in C_\nu(L)$, $0 < \nu \leqslant 1$, and the contour L consists of a finite number of smooth contours of the class C^1_μ. Making use of the formula (5.19) we can write condition (5.103) in the form

$$\mathrm{Re}\left[w'(z)\frac{dz}{ds}\frac{dz}{dt}\right] = g \quad (\text{on } L), \qquad (5.104)$$

where

$$g = -\frac{1}{\sqrt{a}} f - \frac{\mathrm{Z}}{2\sqrt{\mathrm{K}}} \mathrm{Re}\left[\frac{dz}{ds}\frac{d\bar{z}}{dt'}\right], \qquad (5.105)$$

$\boldsymbol{t}'$ is the direction tangential to S and conjugate with respect to $\boldsymbol{t}$, i.e. $\boldsymbol{t} \times \boldsymbol{t}' = \boldsymbol{n}$. The derivation of the relation (5.104) offers no difficulties if we remember that

$$t_\alpha = c_{\alpha\beta} t'^{\beta} \quad (\alpha = 1, 2), \qquad (5.106)$$

$$T_{(lt)} = -\sqrt{a}\,\mathrm{Re}\left[w'(z)\frac{dz}{ds}\frac{dz}{dt'}\right] - \frac{\mathrm{Z}\sqrt{a}}{2\sqrt{\mathrm{K}}} \mathrm{Re}\left[\frac{dz}{ds}\frac{d\bar{z}}{dt'}\right]. \qquad (5.107)$$

Thus, the boundary condition (5.103) leads to the generalized Riemann–Hilbert problem of the form (5.104). Such problems were already dealt with in the preceding chapter in connection with some geometric problems

(Ch. V, §10). Hence, we shall not deal with details of proof here; we shall only reproduce, with necessary additions, some results obtained previously.

We assume that the direction $\boldsymbol{t}$ belongs to the class $\boldsymbol{l}$ (Ch. V, §10.5). Then the index of the problem (5.104) is given by

$$n = 2(m-1)\,. \tag{5.108}$$

Therefore, in the case of a convex shell with one opening ($m = 0$) $n = -2$, and by Theorem 4.12 the homogeneous problem has no solution, and the non-homogeneous problem is soluble only if the following three conditions are satisfied:

$$\int\limits_L f u_t^{(j)} ds + \iint\limits_S \boldsymbol{U}^{(j)} \boldsymbol{X} dS = 0 \qquad (j = 1, 2, 3)\,, \tag{5.109}$$

where $u_t^{(j)}$ are the projections of the field $\boldsymbol{U}^{(j)}$ on $\boldsymbol{t}$, and the fields $\boldsymbol{U}^{(j)}$ $(j = 1, 2, 3)$ are solutions of the adjoint homogeneous kinematic problem

$$u_{t'} = 0 \qquad (\text{on } L)\,. \tag{5.110}$$

Two cases can occur, namely: (1) all fields $\boldsymbol{U}^{(j)}$ are trivial and (2) at least one field is not trivial. In the first case the relations (5.109) are the equilibrium conditions of statics of a rigid body. Consequently, in this case the boundary value problem always has a solution. In the second case it is evident that the problem is not always soluble. For a convex shell with two openings ($m = 1$) $n = 0$ and by Theorem 4.6 two cases may occur, namely (1) either the non-homogeneous problem (5.104) is always soluble and the corresponding homogeneous problem has no non-trivial solution, or (2) the homogeneous problem has one solution and the non-homogeneous problem is soluble only if the following conditions are satisfied:

$$\int\limits_L f u_t ds + \iint\limits_S \boldsymbol{U}\boldsymbol{X} dS = 0\,, \tag{5.111}$$

where $\boldsymbol{U}$ is a (non-trivial) solution of the problem (5.110). We should distinguish the case when $\boldsymbol{U}$ is trivial. Then

the relation (5.111) is identical with one of the equilibrium conditions of the statics of a rigid body and consequently, the problem always has a solution.

Finally, in the case $m > 1$ the index of the problem $n = 2(m-1)$ is greater then $m-1$ and by Theorem 4.10 the non-homogeneous problem (5.104) always has a solution and the corresponding homogeneous problem has exactly $3m-3$ linearly independent solutions. Consequently, the force field has the form

$$T_{(l)} = T^0_{(l)} + \sum_{k=1}^{3m-3} c_k T^k_{(l)} . \tag{5.112}$$

The arbitrary constants c_k can be determined by means of additional conditions of the point type, prescribing the forces $T_{(l)}$ at k interior points $M_1, \dots, M_k$ and at k' boundary points $M'_1, \dots, M'_{k'}$. The following conditions should be satisfied: (1) $2k + k' = 2n+1-m$ and (2) on each of the m arbitrarily chosen boundary contours an odd number of the fixed points M'_j should lie; apart from these conditions these points are arbitrary. At the interior points M_j we can arbitrarily fix both components of the vector force $T_{(l)}$ and at the boundary points M'_j we can arbitrarily fix only the component $T_{(u')}$, since the second component $T_{(u)}$ is, by the conditions of the problem, given beforehand.

We note that under the above additional conditions the boundary value problem (5.104) is correct if $m > 1$, i.e. in the case of convex shells with more than two openings. For shells with one or two openings the problem in general is incorrect.

5.12. It is now natural to formulate the problem of the possibility of establishing boundary conditions of the form (5.104) in practice. We shall now indicate some definite devices making it possible to establish such conditions. To this end we consider the bush constraints bonds dealt with in the preceding chapter.

Let there be rigid or elastic bodies (bushes) inserted into the openings and slits; they are bounded by conical surfaces which tightly fit the sides of the shell. As before let us denote the surfaces of these bodies by Σ. If the surfaces Σ are ideally smooth the reaction forces will evidently be directed along the normal to Σ. Consequently, along the boundary of the surface S the following condition is satisfied:

$$H_l \equiv T^{\alpha\beta} l_\alpha s_\beta = 0 \qquad (\text{on } L)\,. \tag{5.113}$$

In the case of a sufficiently thin shell it can be assumed that the reaction forces are constant along every generator of the surface Σ. It is then obvious that the moments of the reaction forces, as well as the tangential and shear forces vanish.

Another example of the occurrence of the boundary condition (5.113) is obtained if we subject the lateral surfaces of the shell to a hydrostatic of hydrodynamic pressure. If it is assumed that the lateral surfaces are ideally smooth and the fluid perfect, the pressure will be directed perpendicularly to the lateral surfaces and, consequently, the tangential forces will vanish, i.e. the boundary condition (5.113) occurs. It should be noted that the value of the pressure in this problem is not assumed to be known. The pressure is determined as a consequence of the solution of the boundary value problem (5.113), which in the presence of a surface load is reduced to the generalized Riemann–Hilbert problem of the form

$$\begin{aligned} &\partial_{\bar z} w' - Aw' - \bar{B}\bar{w}' = F \qquad (\text{in } G)\,,\\ &\operatorname{Re}\left[iw'(z)\frac{dz}{ds}\frac{dz}{dl}\right] = 0 \qquad (\text{on } \Gamma)\,. \end{aligned} \tag{5.114}$$

If the shell is subject to a potential field of surface forces we have the homogeneous problem

$$\begin{aligned} &\partial_{\bar z} w' - Aw' - \bar{B}\bar{w}' = 0 \qquad (\text{in } G)\,,\\ &\operatorname{Re}\left[iw'(z)\frac{dz}{ds}\frac{dz}{dl}\right] = 0 \qquad (\text{on } \Gamma)\,. \end{aligned} \tag{5.115}$$

In the case of a simply-connected domain the results of the preceding section imply that this homogeneous problem has only a trivial solution and the force field is determined by the formula (5.75).

It is now easy to elucidate the way in which the following boundary condition is set up on an edge of the shell:

$$N_l \equiv T^{\alpha\beta} l_\alpha l_\beta = f \,. \tag{5.116}$$

Let us imagine that a hollow bush of conical shape has been inserted into one of the openings of the shell; on the interior surface of the bush there act normal forces constant along each of the generators. Then the moments and shear forces corresponding to these forces obviously vanish. Consequently, along the middle line of the interior surface of the cone a force of the form (5.116) acts. We now assume that the surface Σ is rough. Then the normal force N_l acting on the interior surface of the cone is entirely taken up by the lateral surface of the shell, and the tangential force vanishing on the interior surface of the cone because of the roughness of its exterior surface, will not in general be equal to the tangential force on the lateral surface of the shell. Hence, on the boundary of the shell only one condition of the form (5.116) is given. The tangential force H_l, appearing because of the roughness of the surface Σ, is determined by the condition that the membrane state of stress is set up in the shell, and it is compatible with the condition (5.116).

An investigation of the boundary condition of the general form (5.103), when instead of the normal force or tangential force at every point of the contour we are given a force in an inclined direction $\boldsymbol{t}$, is also of importance for the examination of the stability of solutions of boundary value problems with conditions of the form (5.113) or (5.116). The point is that in view of the non-ideal smoothness of the contact surfaces Σ or some other causes, it is in practice very difficult to establish exactly the value of the normal or tangential force. In fact, in practice the

force is always given in an inclined direction. It is therefore of importance to find the influence on the solution of the problem with the boundary condition of the form (5.103) of a small error in the value of the angle φ. If it is found that for a small variation of this angle the solution of the problem undergoes a variation of the same order, then it means that the problem is stable with respect to a variation of the angle of inclination of the direction $\boldsymbol{t}$.

In other words, an investigation of general boundary value problem (5.103) enables us to determine the nature of correctness of the problems (5.113) and (5.116). Hence, an investigation of these problems has not only a theoretical but also a practical value (see Ch. 5, §10.5).

5.13. In the preceding sections we indicated various conditions ensuring the applicability of the membrane theory to the computation of convex shells loaded by forces of the form $(\boldsymbol{X}, \boldsymbol{T})$. These conditions impose certain restrictions on the character of the distribution of the external loads. For closed convex shells these restrictions consist in the familiar equilibrium conditions of an absolutely rigid body and, consequently, are entirely natural. However, in the case of shells with openings these conditions are of a more complicated form. They essentially narrow the class of admissible loads and, moreover, are difficult to realize in practice. Nevertheless, the membrane theory is extensively employed in computations of shells encountered in engineering structures and its results give satisfactory practical answers. In many cases it may be applied even if it is known beforehand that the external load certainly does not satisfy the conditions of the membrane state. This means that some deviations from the exact fulfilment of the conditions of the membrane state are admissible in practice. In this connection naturally the following problem arises: to find conditions enabling us to determine bounds of practical applicability of the membrane theory of shells.

It is obvious that if a load $(\boldsymbol{X}, \boldsymbol{T})$ is membrane, i.e if it satisfies the condition

$$\iint_S \boldsymbol{X}\boldsymbol{U}\,dS + \int_L \boldsymbol{T}\boldsymbol{U}\,ds = 0 \,. \tag{5.117}$$

where $\boldsymbol{U}$ is an arbitrary displacement field, then all loads close to it (in a certain sense) may in practice also be regarded as membrane. Now the problem is to give a more precise mathematical statement of the definition of the concept of closeness of two loads $(\boldsymbol{X}, \boldsymbol{T})$ and $(\boldsymbol{X}_0, \boldsymbol{T}_0)$. At first sight it may seem most natural to define closeness of two fields of external loads pointwise, i.e. to regard two loads ε- close if at all points M of the surface S and its boundary L the following inequalities are satisfied:

$$|\boldsymbol{X}(M) - \boldsymbol{X}_0(M)| < \varepsilon \qquad (\text{on } S)\,,$$

$$|\boldsymbol{T}(M) - \boldsymbol{T}^0(M)| < \varepsilon \qquad (\text{on } L)\,.$$

It can however easily be verified that this concept of closeness does not in fact essentially suit the physical nature of the phenomenon and, moreover, it represents too strong a restriction on the practical applicability of the membrane theory. In this respect it is more expedient to introduce the concept of ε- closeness by means of the integral metric

$$\|(\boldsymbol{X}, \boldsymbol{T}) - (\boldsymbol{X}_0, \boldsymbol{T}_0)\| \equiv \Big\{\iint_S |\boldsymbol{X} - \boldsymbol{X}_0|^2 ds + \int_L |\boldsymbol{T} - \boldsymbol{T}^0|^2 ds\Big\}^{\frac{1}{2}} < \varepsilon \,.$$

We therefore introduce the following Hilbert space:

Let $\mathcal{R}$ be the set of pairs $(\boldsymbol{X}, \boldsymbol{T})$ satisfying the following conditions: (1) $\boldsymbol{X}$ is a vector field given on the surface S, (2) $\boldsymbol{T}$ is a vector field given on L and satisfying the condition

$$\boldsymbol{n}\boldsymbol{T} = 0 \qquad (\text{along } L)\,, \tag{5.118}$$

and

$$\text{(3)} \qquad \|(\boldsymbol{X}, \boldsymbol{T})\| = \left\{\iint_S \boldsymbol{X}^2 dS + \int_L \boldsymbol{T}^2 ds\right\}^{\frac{1}{2}} < \infty . \qquad (5.119)$$

The non-negative number $\|(\boldsymbol{X}, \boldsymbol{T})\|$ will be called the norm of the element $(\boldsymbol{X}, \boldsymbol{T})$.

We define in $\mathcal{R}$ the operations of multiplication by a real constant c, addition, subtraction and scalar multiplication by means of the formulae

$$c(\boldsymbol{X}, \boldsymbol{T}) = (c\boldsymbol{X}, c\boldsymbol{T}) , \qquad (5.120)$$

$$(\boldsymbol{X}_1, \boldsymbol{T}_1) + (\boldsymbol{X}_2, \boldsymbol{T}_2) = (\boldsymbol{X}_1 + \boldsymbol{X}_2, \boldsymbol{T}_1 + \boldsymbol{T}_2) , \qquad (5.121)$$

$$(h_1, h_2) = \iint \boldsymbol{X}_1 \boldsymbol{X}_2 dS + \int_L \boldsymbol{T}_1 \boldsymbol{T}_2 ds \quad \big(h_i = (\boldsymbol{X}_i, \boldsymbol{T}_i)\big) , \qquad (5.122)$$

Hence, $\mathcal{R}$ is converted into a Hilbert space.

Let $\boldsymbol{U}$ be a displacement field on the surface S. Then the pair $(\boldsymbol{U}, \boldsymbol{U}_s)$ where $\boldsymbol{U}_s$ is the tangential component of the field $\boldsymbol{U}$, i.e.

$$\boldsymbol{U}_s = \boldsymbol{U} - (\boldsymbol{n}\boldsymbol{U})\boldsymbol{n} , \qquad (5.123)$$

is evidently an element of $\mathcal{R}$, and the set of such elements constitutes a subspace of the space $\mathcal{R}$, which hereafter will be denoted by $\mathcal{R}_0$. Let $\mathcal{R}_*$ be the orthogonal complement of $\mathcal{R}_0$ to the entire space $\mathcal{R}$. The condition of orthogonality of $\mathcal{R}_0$ and $\mathcal{R}_*$ is given by the relation (5.117). Consequently, all membrane external loads $(\boldsymbol{X}, \boldsymbol{T})$ are elements of the subspace $\mathcal{R}_*$. Denote now by $d(\boldsymbol{X}, \boldsymbol{T})$ the distance of the element $(\boldsymbol{X}, \boldsymbol{T})$ to the subspace $\mathcal{R}_*$. This number as we shall see below, is a certain characteristic of the degree of closeness of the stress field corresponding to the load $(\boldsymbol{X}, \boldsymbol{T})$ to the totality of the membrane states of stress.

It is known from functional analysis that this number is equal to the length of the projection of the element $(\boldsymbol{X}, \boldsymbol{T})$ on the subspace $\mathcal{R}_0$. It is now expedient to introduce in $\mathcal{R}_0$ an orthogonal base.

Constructing a complete system of displacement fields $\boldsymbol{U}_i$ $(i = 1, 2, \dots)$ (see §5.10) and considering the pairs

$$\boldsymbol{e}_n = (\boldsymbol{U}_n, \boldsymbol{U}_{ns}), \tag{5.124}$$

we have a complete sequence in $\mathcal{R}_0$. We assume that it is normalized and orthogonal, i.e.

$$(\boldsymbol{e}_n \cdot \boldsymbol{e}_m) \equiv \iint\limits_{S} \boldsymbol{U}_n \boldsymbol{U}_m dS + \int\limits_{L} U_{nS} \boldsymbol{U}_{mS} ds = \delta_{nm} . \tag{6.125}$$

This can always be achieved by an application of the Schmidt orthogonalization procedure. It can easily be verified that the sequence $\boldsymbol{e}_n$ is an orthogonal base in $\mathcal{R}_0$. The condition (5.117) therefore is equivalent to the relations

$$c_i \equiv (\boldsymbol{h}, \boldsymbol{e}_i) = \iint\limits_{S} \boldsymbol{X}\boldsymbol{U}_i dS + \int\limits_{L} \boldsymbol{T}\boldsymbol{U}_i ds = 0 \tag{5.126}$$
$$\big(i = 0, 1, \dots;\ \boldsymbol{h} \equiv (\boldsymbol{X}, \boldsymbol{T})\big) .$$

Thus, the vanishing of all the Fourier coefficients c_i of the load $(\boldsymbol{X}, \boldsymbol{T})$ with respect to the orthonormal base of the subspace $\mathcal{R}_0$ is a necessary and sufficient condition for this load to be membrane.

Nevertheless, it should be observed that in such a formulation this condition is hardly applicable in practice, since it is very difficult in definite problems to find to what extent the infinite system of relations (5.126) is satisfied. This difficulty arises especially in the cases when the load $(\boldsymbol{X}, \boldsymbol{T})$ is given graphically or in a tabular form. Hence, it is expedient to make an attempt to find another formulation of the above condition, which would be more suitable for practical applications.

The condition (5.126) means that the element $\boldsymbol{h} = (\boldsymbol{X}, \boldsymbol{T})$ of the space $\mathcal{R}$ belongs to the subspace $\mathcal{R}_*$. In the language of functional analysis this condition can be expressed by the following relation:

$$d(\boldsymbol{X}, \boldsymbol{T}) \equiv \|\boldsymbol{h} - \boldsymbol{h}_*\| = 0 , \tag{5.127}$$

where h_* is an element of the subspace $\mathcal{R}_*$, closest to h. It is given by the relation

$$h_* = h - \sum_{i=0}^{\infty} c_i e_i \,, \qquad c_i = (h, e_i) \,. \tag{5.128}$$

Hence we easily obtain

$$d(X, T) = \|h - h_*\| = \sqrt{\sum_{i=0}^{\infty} c_i^2} \,. \tag{5.129}$$

In view of this formula the condition of the membrane state (5.127) acquires a more practical meaning. Essentially it means that *the load (X, T) may in practice be regarded as membrane if the sum of the squares of its Fourier coefficients c_i with respect to an orthogonal base of the subspace $\mathcal{R}_0$ is sufficiently small.*

It should be observed that this condition can be used practically in computations of many particular shells. Such a case is for instance represented by a fairly wide class of shells described by convex surfaces of the second degree. In this case the problem is reduced to the Cauchy–Riemann system of equations and it is easy to construct the orthogonal base e_n in an explicit form.

In the general case a practical determination of the number $d(X, T)$ leads to considerable mathematical difficulties. First of all it is evident that a difficulty arises in constructing the orthogonal base e_i of the subspace $\mathcal{R}_0$. In practical problems, however, an exact value of the number $d(X, T)$ is by no means required. It is sufficient to be able only to determine its upper bounds close to the exact value. For this we can make use of the following method which does not in fact require a construction of an orthogonal base.

Let there be given a load $h = (X, T)$ belonging to the sphere $\Sigma_M: \|(X, T)\| \leqslant M$. Let $\mathcal{R}_*^M$ be the set of elements of the subspace $\mathcal{R}_*$ belonging to Σ_M. Consider now an infinite sequence of linearly independent elements $h_j =$

$= (\boldsymbol{X}_j, \boldsymbol{T}_j)$ $(j = 1, 2, \ldots)$ of the set $\mathcal{R}_*^M$ which satisfy the following condition: every element of the set $\mathcal{R}_*^M$ can be approximated to within $\varepsilon > 0$ by means of linear combinations of the form $c_1 h_1 + \ldots + c_n h_n$.

Let us determine the number d_n by the condition

$$d_n = \min_{c_k} \left\| h - \sum_{k=1}^{n} c_k h_k \right\| . \tag{5.130}$$

It is evident that this problem is reduced to the solution of the system

$$\sum_{k=1}^{n} (h_j, h_k) c_k = (h, h_j) \qquad (j = 1, \ldots, n) .$$

Let us now prove that for any fixed positive $\varepsilon > 0$ there exists a number n, such that

$$d \leqslant d_n \leqslant d + \varepsilon .$$

In fact, since the element h_* closest to $h = (\boldsymbol{X}, \boldsymbol{T})$, of the subspace $\mathcal{R}_*$ belongs to $\mathcal{R}_*^M$, a linear combination $h' = c_1' h_1 + \ldots + c_n' h_n$ can be found which satisfies the condition $\|h_* - h'\| < \varepsilon$.

Such being the case solving the system (5.130), we have for the element $h_\varepsilon = c_1 h_1 + \ldots + c_n h_n$

$$d \leqslant d_n = \|h - h_\varepsilon\| \leqslant \|h - h'\| \leqslant \|h - h_*\| + \|h' - h_*\| \leqslant d + \varepsilon .$$

This completes the proof.

If in general a finite or infinite sequence of elements $(\boldsymbol{X}_j, \boldsymbol{T}_j)$ is given which belong to the subspace $\mathcal{R}_*$, then evidently we have

$$d(\boldsymbol{X}, \boldsymbol{T}) \leqslant d_0(\boldsymbol{X}, \boldsymbol{T}) = \min \|(\boldsymbol{X}, \boldsymbol{T}) - (\boldsymbol{X}_j, \boldsymbol{T}_j)\| .$$

If it is not known beforehand whether the sequence $(\boldsymbol{X}_j, \boldsymbol{T}_j)$ is dense in $\mathcal{R}$, then, obviously, we cannot estimate the degree of closeness of d_0 to d. It may however occur that d_0 is sufficiently small. Then the load $(\boldsymbol{X}, \boldsymbol{T})$ can be practically regarded as membrane.

In conclusion we observe that the smallness of the number $d(\boldsymbol{X}, \boldsymbol{T})$ can be ensured by equating to zero a number n of the first Fourier coefficients of the load $(\boldsymbol{X}, \boldsymbol{T})$, i.e.

$$c_1 = \ldots = c_n = 0 . \tag{5.131}$$

It should now be observed that it is by no means necessary in these relations that the displacement field $\boldsymbol{U}_k$ satisfies the conditions (5.125). This remark makes easier a practical solution of the problem, since we do not have to apply an orthogonalization procedure.

The fulfilment of the conditions (5.131) can, for instance, be ensured by loading the shell by concentrated forces at n prescribed points of the surface. It is also possible to indicate a number of practical ways of ensuring the fulfilment of the relation (5.131), [140].

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